Sign Pattern of Graph Eigenvectors and Hyperplane Arrangements

Josef Leydold

Institut für Statisik, WU Wien, Austria

Josef.Leydold@statistik.wu-wien.ac.at

http://statistik.wu-wien.ac.at/staff/leydold/

Joint work with

Tomaž Pisanski

Türker Bıyıkoğlu

Win Hordijk

Graph Laplacian

Let $\mathrm{G}(\mathrm{V}, \mathrm{E})$ denote a connected, simple, undirected graph without loops with vertex set V and edge set E.
(However G might be weighted.)

The matrix

$$\boldsymbol{\Delta} = \boldsymbol{\Delta}(\mathrm{G}) = \mathbf{D}(\mathrm{G}) - \mathbf{A}(\mathrm{G})$$

is called the *Laplacian* of the graph G.

$\mathbf{A}(\mathrm{G})$ ... adjacency matrix.
$\mathbf{D}(\mathrm{G})$ ... diagonal matrix vertex degrees as entries (sum of weights) .

Nodal domains
(Sign Graphs)

For a (connected) graph $G(V, E)$ with Laplace operator $\boldsymbol{\Delta}(G)$ let

$$\lambda_1 < \lambda_2 \leq \lambda_3 \leq \ldots \leq \lambda_N$$

denote the eigenvalues (counting multiplicity) of $\boldsymbol{\Delta}$ with corresponding eigenvectors

$$\psi_1, \psi_2, \psi_3, \ldots, \psi_N$$

i.e.

$$\boldsymbol{\Delta}\psi_k = \lambda_k \psi_k$$

We define a *strong positive nodal domain* of ψ as a maximal, connected induced subgraph of G on vertices with $\psi(v) > 0$.

We define a *weak positive nodal domain* of ψ as a maximal, connected subgraph of G on vertices with $\psi(x) \geq 0$ and with at least one vertex with $\psi(x) > 0$.

Notice that weak nodal domains may have vertices in common. On these vertices ψ vanishes. (Zero vertex)

A zero vertex of ψ either is adjacent to other zero vertices, or it is adjacent to vertices of *both* strict signs.

Discrete nodal domain theorems

Theorem (Weak nodal domain theorem).
ψ_k *has at most* k weak *nodal domains.*

Theorem (Strong nodal domain theorem).
ψ_k *has at most* $k+m-1$ strong *nodal domains.*

where m is the multiplicity of eigenvalue λ_k, i.e., we have

$$\lambda_{k-1} < \lambda_k \leq \lambda_{k+1} \leq \ldots \leq \lambda_{k+m-1} < \lambda_{k+m}$$

Both bounds are sharp.
(Eigenvectors of a path.)

Do we know better bounds for special graph classes?

No! (AFAIK)
(. . . Well, Türker does for trees, cographs, . . .)

Do we know lower (non-trivial) bounds?

Do we know all possible values for the numbers of nodal domains?

No! (AFAIK)
(. . . Türker always warns us that this problem is NP-complete, i.e. from the Mafia).

Worse!
We even have no idea how to *compute* the minimum and maximum number of nodal domains for a particular eigenvalue of a graph that is given explicitly!

Basis of Eigenspace

Let λ be an eigenvalue of multiplicity $m \geq 2$.

Orthonormal basis of eigenspace ($\cong \mathbb{R}^m$)

$$u_1, \ldots, u_m$$

Every eigenvector f is given by

$$f(x) = \sum_{i=1}^{m} a_i \, u_i(x) = \langle \mathbf{a}, \mathbf{u}(x) \rangle$$

where $\mathbf{a} = (a_1, \ldots, a_m)$, and

$$\mathbf{u}(x) = (u_1(x), \ldots, u_m(x))$$

is the vector that contains the values of the basis at vertex x.

Alternatively: If $\mathbf{U}$ is the matrix containing the basis as its columns then the $\mathbf{u}(x)$ are its rows.

The convex hull of

$$\{\mathbf{u}(x) : x \in V\}$$

forms a polytope in $\mathbb{R}^m$.

It is called the *eigenpolytope* by Godsil.

Sign pattern

Obviously the number of nodal domains only depend on sign of eigenvector on each of the vertices.

The sign at vertex x is given by

$$\operatorname{sign}\langle \mathbf{a}, \mathbf{u}(x)\rangle$$

There is a 1-1 relation between the eigenvector f and coordinate vector $\mathbf{a}$.

$$f(x) = 0 \quad \Leftrightarrow \quad \langle \mathbf{a}, \mathbf{u}(x)\rangle = 0$$

Hyperplane arrangements

Set of eigenvectors that vanish on vertex x corresponds to the set

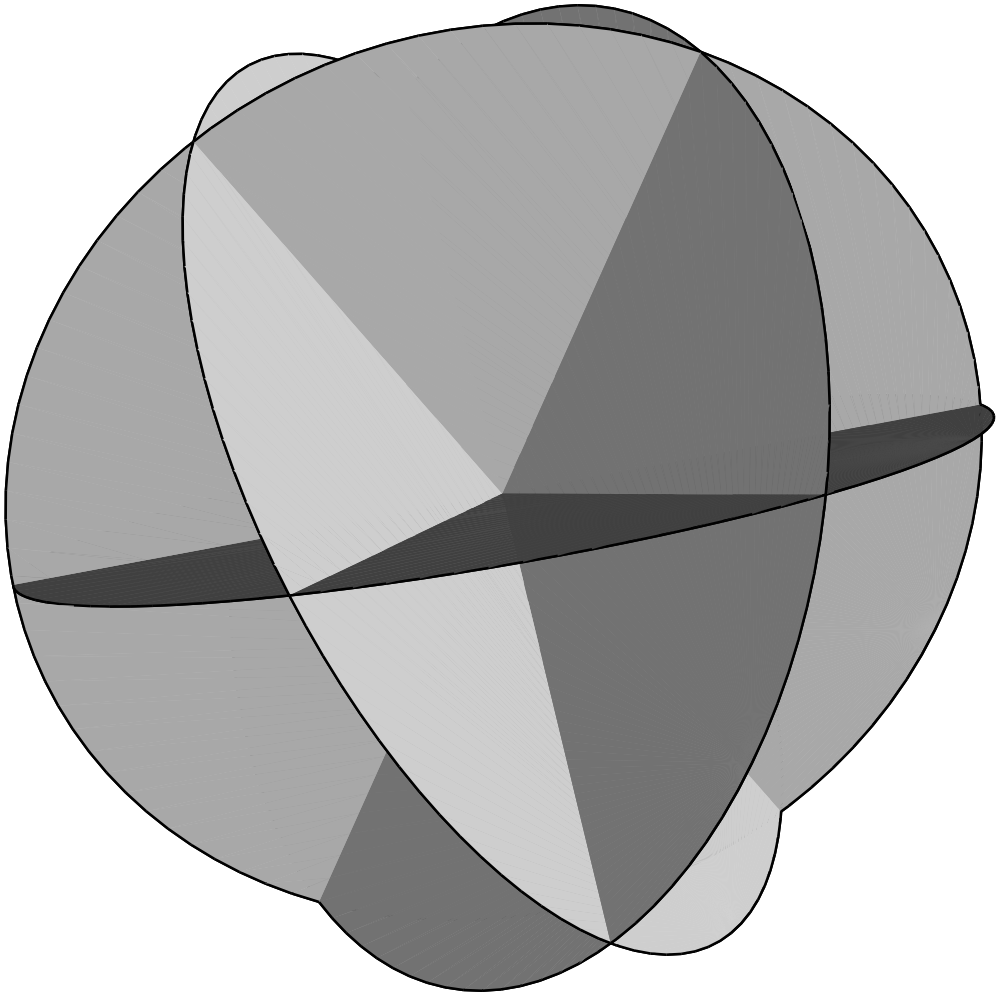
$$H(x) = \{\mathbf{a} \in \mathbb{R}^m : \langle \mathbf{a}, \mathbf{u}(x) \rangle = 0\}$$

For a vertex $x \in V$

- either $H(x) = \mathbb{R}^m$
  (and $f(x) = 0$ for all eigenvectors),
- or $H(x)$ is a hyperplane through the origin in $\mathbb{R}^m$.

The set of all proper hyperplanes $H(x)$, $x \in V$, forms a *hyperplane arrangement* $\mathcal{H}$ in $\mathbb{R}^m$.

Hyperplane arrangement



Sign vectors

Each hyperplane $H(x)$ splits $\mathbb{R}^m$ into three pieces:

- the hyperplane itself given by $\langle \mathbf{a}, \mathbf{u}(x) \rangle = 0$
- the positive part $\langle \mathbf{a}, \mathbf{u}(x) \rangle > 0$
- and the negative part $\langle \mathbf{a}, \mathbf{u}(x) \rangle < 0$

In each part $f(x)$ has the same sign.

Using $\mathcal{H}$ we can asign a *sign vector* of length $|V|$ to each $\mathbf{a}$ and hence to eigenvector f.

This sign vector corresponds to the vector of signs of the eigenvector.

Cellular complex

We can restrict the eigenspace $\mathbb{R}^m$ to the set or normalized eigenvectors S^{m-1}, i.e. the $(m-1)$-dimensional sphere.

$\mathcal{H} \cap S^{m-1}$ forms a *cellular complex*.

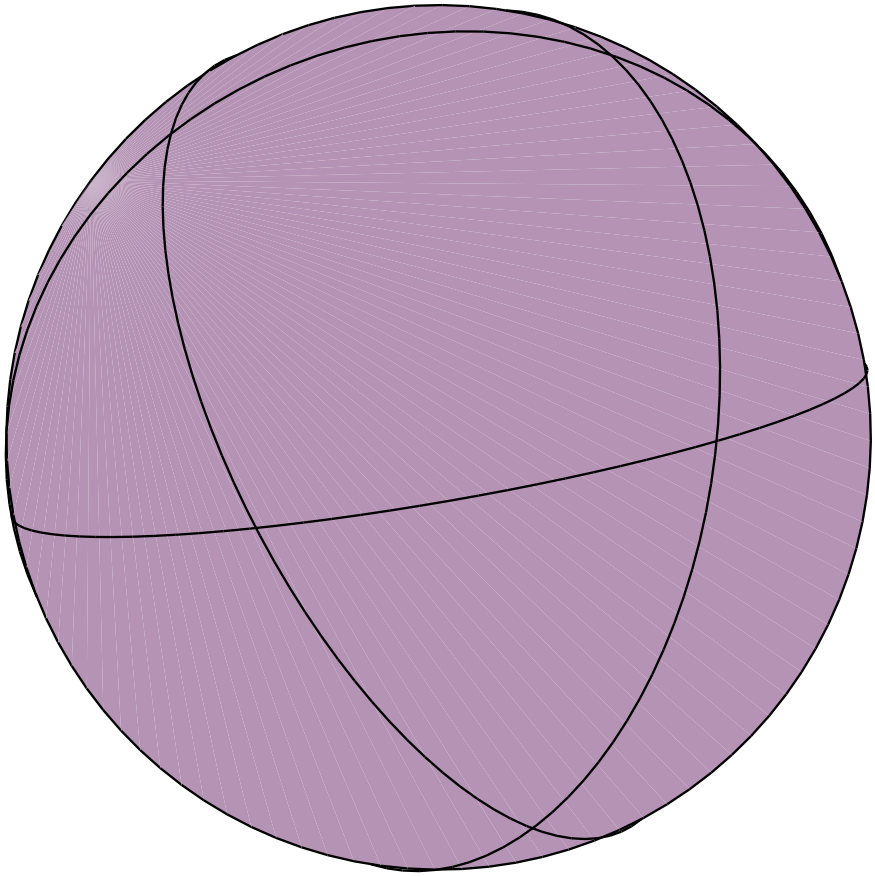
0-cell: single point.

1-cell: piece of line.

2-cell: homeomorph to disc, where boundary consists of union of 1-cells.

n-cell: homeomorph to n-dimensional ball, where boundary consists of union of $(n-1)$-cells.

Cellular complex



Cells of $\mathcal{H} \cap S^{\mathrm{m}-1}$ are given by intersections of hyperplanes.

We have a *Polytope*.

The sign vector (and hence the number of nodal domains) is constant on every cell of this complex.

Every cell is uniquely determined by its sign vector.

Finding all possible values for the number of nodal domain is equivalent with finding all cells of this complex.

(I.e., this problem is from the Mafia.)

Example: Cube

The eigenpolytope of the eigenvalue 4 is spanned by:

$(1,1,1),(1,1,-1),(1,-1,1),(1,-1,-1),$
$(-1,1,1),(-1,1,-1),(-1,-1,1),(-1,-1,-1)$

Due to symmetry we only have the cells

dim	shape	# sND	# wND
2	rectangle	4	4
2	triangle	3	3
1	edge	4	3
0	point	3	2

Example: Star with 5 nodes

The eigenpolytope of the eigenvalue 1 is spanned by:

$(\frac{\sqrt{2}}{2}, 0, \frac{1}{2}), (-\frac{\sqrt{2}}{2}, 0, \frac{1}{2}), (0, \frac{\sqrt{2}}{2}, -\frac{1}{2}),$
$(0, -\frac{\sqrt{2}}{2}, -\frac{1}{2}), (0, 0, 0)$

Due to symmetry we only have the cells

dim	shape	# sND	# wND
2	rectangle	4	2
2	triangle	4	2
1	edge	3	2
0	point	2	2