For Nodal Domains you need a visa

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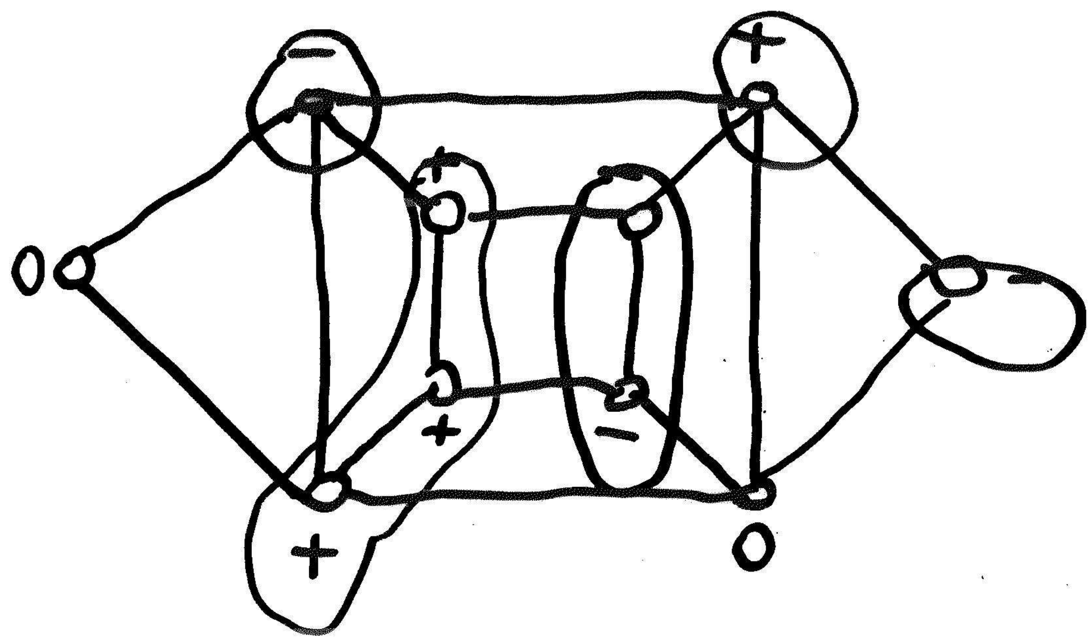
Nodal Domains

$G = (V, E)$ a connected graph
We give for each vertices a sign $+$, $-$ or 0

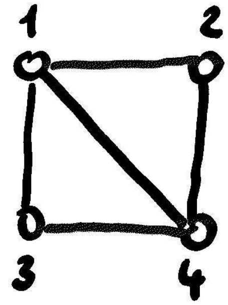


- positive nodal domains (ND_+)
- negative nodal domains (ND_-)

$\# \ ND = \# \ ND_+ + \# \ ND_-$

Example

G:

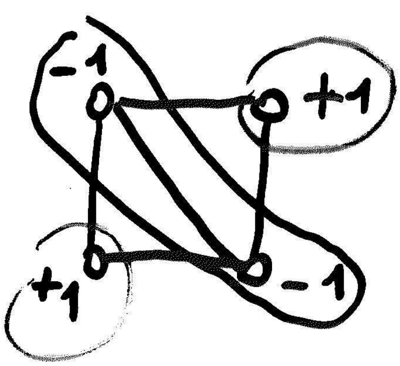


$$L = \begin{pmatrix} 3 & -1 & -1 & -1 \\ -1 & 2 & 0 & -1 \\ -1 & 0 & 2 & -1 \\ -1 & -1 & -1 & 3 \end{pmatrix}$$

Eigenvalues $\lambda = \quad 0 \quad 2 \quad 4 \quad 4$

Eigenvectors:

$$x_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} \quad x_2 = \begin{pmatrix} 0 \\ 1 \\ -1 \\ 0 \end{pmatrix} \quad x_3 = \begin{pmatrix} -1 \\ 1 \\ 1 \\ -1 \end{pmatrix} \quad x_4 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ -1 \end{pmatrix}$$



eigenvector x_4 has 3 nodal domains

L Laplacian matrix of a graph G

$$L_{uv} = \begin{cases} d_u & \text{if} \quad u = v \\ 0 & \text{if} \quad u \text{ and } v \text{ are not adjacent} \\ -1 & \text{if} \quad u \text{ and } v \text{ are adjacent} \end{cases}$$

Look at the signs of eigenvector of L

Eigenvalues of L are numbered in non-decreasing order:

$$\lambda_1 \le \cdots \le \lambda_{k-1} < \lambda_k = \lambda_{k+1} = \cdots = \lambda_{k+r-1} < \lambda_{k+r} \le \cdots \le \lambda_n$$

Discrete nodal domain theory
(Davies, Gladwell, Leydold, Stadler 2001)
Each eigenvectors of λ_k has at most $k+r-1$ nodal domains

Nodal domain theory for trees

G is a **tree**

If an eigenvector y of λ_k has no vanishing coordinate, then y has exactly k nodal domains.

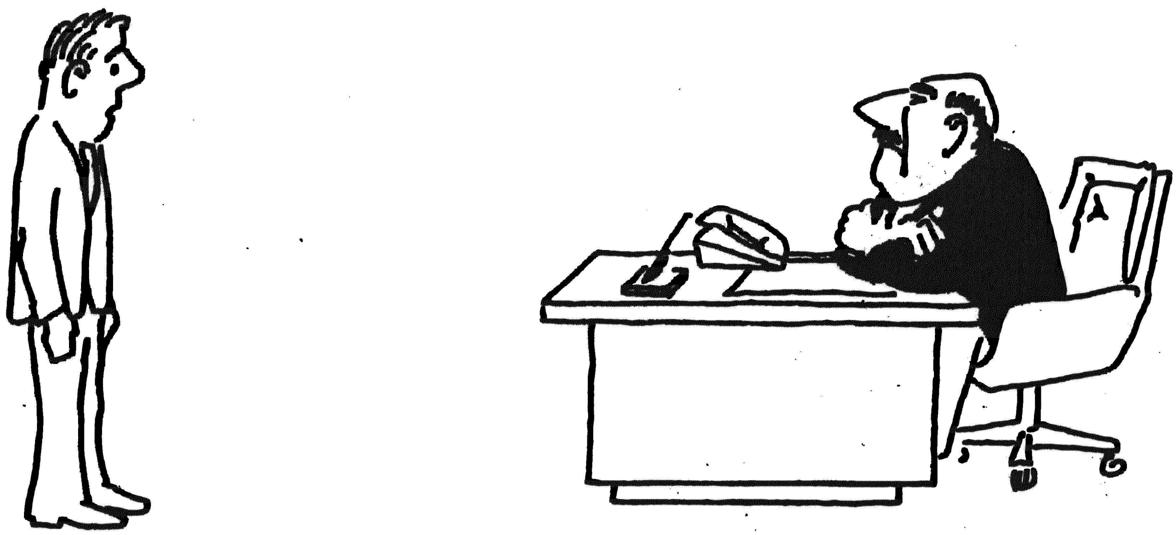
Max nodal domains for a tree
Input: Tree G, eigenbasis of λ
Output: Eigenvector z of λ with maximum number of nodal domains

We can find such an eigenvector z in $O(n^2)$ time

Min nodal domains for a tree
Input: Tree G, eigenbasis of λ
Output: Eigenvector z of λ with **minimum** number of nodal domains

This problem is **NP-complete**

NP-complete I



"I can't find an efficient algorithm, I guess I'm just too dumb."

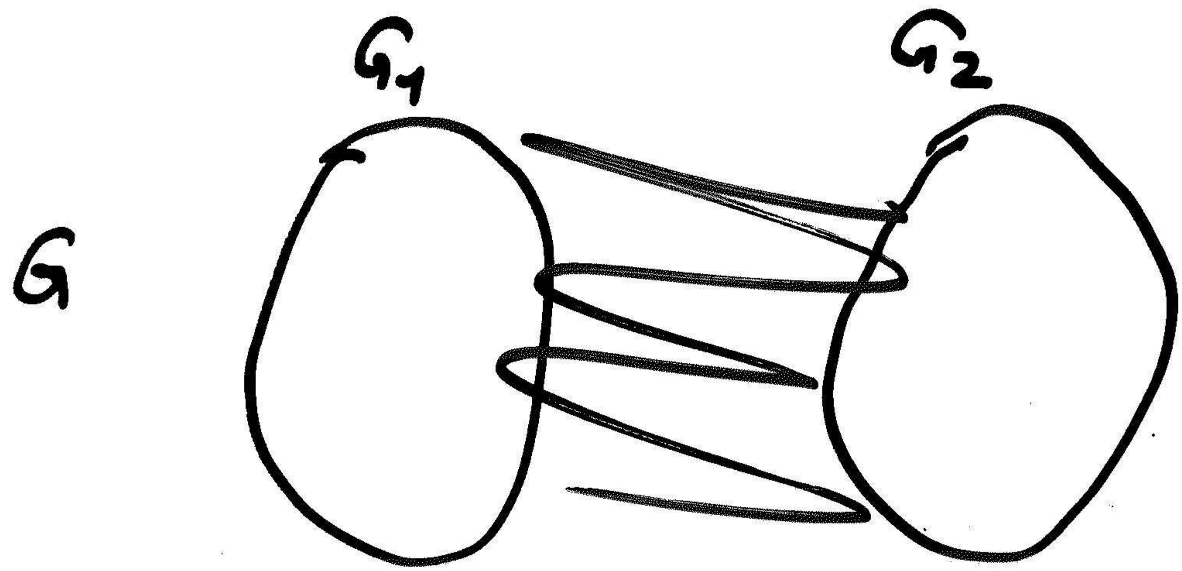
NP-complete II

"I can't find an efficient algorithm, but neither can all these famous people."

A folklore observation of Laplacian

G_1 and G_2 graphs with s and r vertices

$G = G_1 \vee G_2$ (G is join of G_1 and G_2)



(x, λ) Eigenpair of G_1
(y, μ) Eigenpair of G_2 $\Rightarrow$

$(\binom{x}{0}, \lambda + r)$ Eigenpair of $G = G_1 \vee G_2$

$(\binom{0}{y}, \mu + s)$ Eigenpair of $G = G_1 \vee G_2$

With folklore observation

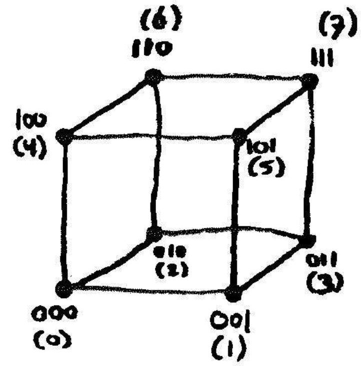
the maximum or minimum number of nodal domains is easy to find for:

- **threshold graphs**

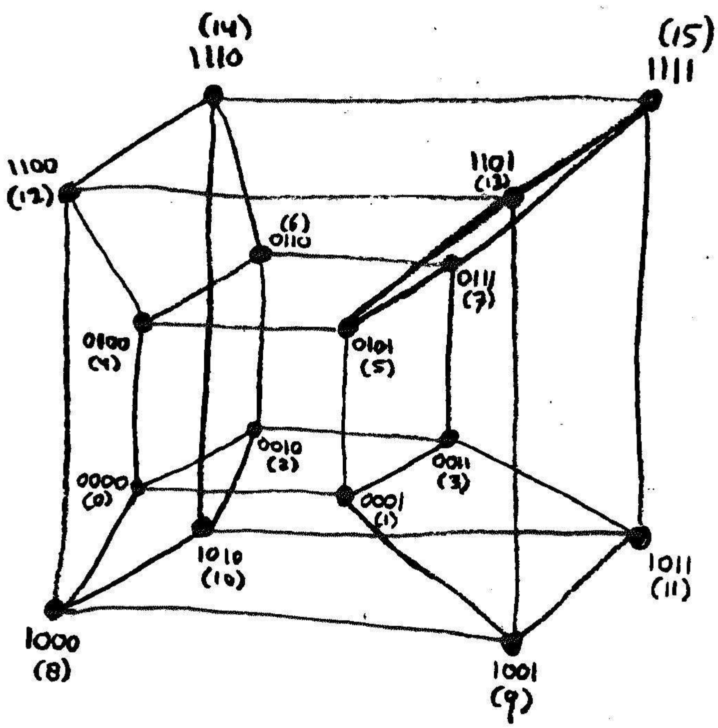
- **cographs**

Hypercubes (H_n)

H_3



H_4



min. number of nodal domains of Hypercubes

The eigenvalues λ of Hypercube H_n are:

$\lambda =$ 0, 2, 4,....., $2n$, $2n + 2$

- If the eigenvalue $\lambda \leq n \Rightarrow$
  λ has an eigenvector with **two** nodal domains

- All eigenvalues $\lambda \leq 2n$ has an eigenvector with **two weak** nodal domains

Weak Nodal Domains