A Faber-Krahn type inequality for regular trees

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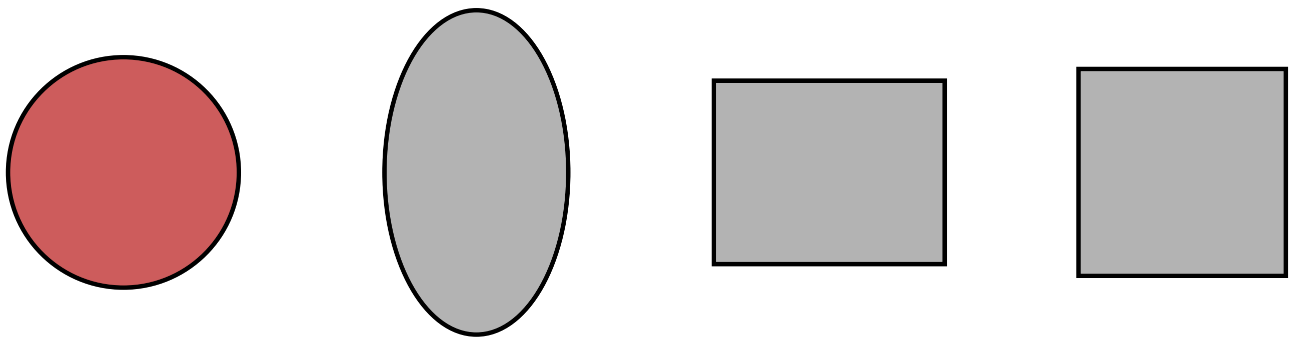
The Faber-Krahn Theorem

Among all bounded domains $D \subset \mathbb{R}^n$ with fixed volume, a ball has the lowest first Dirichlet eigenvalue.

$$-\Delta u = \lambda u, \qquad u\big|_{\partial D} = 0$$



Graph Laplacian

Let $G = (V, E)$ be a Graph with vertex set V, edge set E and weights $\frac{1}{c_e} > 0$.

Laplacian of G:

$$\Delta = \Delta(G) = D(G) - A(G)$$

$A(G) \ldots$ adjacency matrix.
$D(G) \ldots$ diagonal matrix with $D_{v,v} = \sum_{e=(v,u)\in E} \frac{1}{c_e}$

Contrary to the “classical” Laplace-Beltrami operator on manifolds, the graph Laplacian $\Delta(G)$ is defined as a positive operator.

Rayleigh Quotient

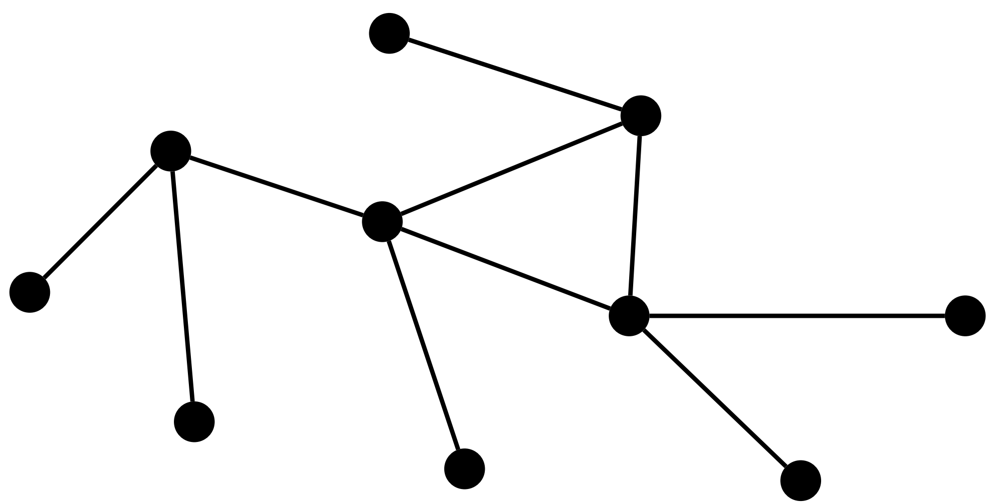
The associate **Rayleigh quotient** on a real valued function f on V is the fraction

$$\mathcal{R}_G(f) = \frac{\langle \Delta f, f \rangle}{\langle f, f \rangle} = \frac{\sum_{(u,v)\in E} \frac{1}{c_e} (f(u) - f(v))^2}{\sum_{v \in V} (f(v))^2}$$

Geometric realization

The **geometric realization** of G is a metric space $\mathcal{G}$, consisting of

- the vertices V,
- arcs of length c_e glued between u and v for every edge $e = (u, v) \in E$.



A "Continuous" Graph Laplacian

Define two measures on $\mathcal{G}$:

- $\mu_1(G) = |V| \ldots$ number of vertices
- $\mu_2(G) = \sum_{e \in E} c_e \ldots$ cumulated length of edges (i.e. the Lebesgue measure of $\mathcal{G}$)

Let $\mathcal{S}$ denote the set of all continuous functions on $\mathcal{G}$, which are differentiable on $\mathcal{G} \setminus V$.

Introduce operator $\Delta_{\mathcal{G}}$ on $\mathcal{G}$ by the Rayleigh quotient

$$\mathcal{R}_{\mathcal{G}}(f) = \frac{\int_{\mathcal{G}} |\nabla f|^2 d\mu_2}{\int_{\mathcal{G}} f^2 d\mu_1}, \qquad f \in \mathcal{S}$$

A "Continuous" Graph Laplacian

The eigenfunctions of the first eigenvalue of $\Delta_{\mathcal{G}}$ are edgewise linear functions.

$\Delta(G)$ and $\Delta_{\mathcal{G}}$ have the same eigenvalues and eigenfunctions.
(The restrictions of the eigenfunctions of $\Delta_{\mathcal{G}}$ to V are the graph Laplacian eigenvectors. Friedman, 1993)

Graph with Boundary

A **graph with boundary** is a graph

$$G(V_0 \cup \partial V, E_0 \cup \partial E)$$

V_0 . . . interior vertices

∂V . . . boundary vertices

E_0 . . . edges between interior vertices (interior edges)

∂E . . . edges between boundary and interior vertices (boundary edges)

Dirichlet Operator for Graphs

Restrict Rayleigh quotient $\mathcal{R}_{\mathcal{G}}(f)$ to those functions $f \in \mathcal{S}$ which vanish at all boundary vertices, i.e.

$$f\big|_{\partial V} = 0$$

The “discrete” version of this operator lives on interior vertices

$$\Delta_0 = D_0 - A_0$$

$A_0 \ldots$ adjacency matrix of $G(V_0, E_0)$

$D_0 \ldots$ diagonal matrix with $(D_0)_{v,v} = \sum_{e=(v,u)\in E_0 \cup \partial E} \frac{1}{c_e}$.

d-regular Trees with Boundary

We look at trees with the following properties:

- all interior vertices have degree d,
- all boundary vertices have degree 1,
- all interior edges have length 1,
- all boundary edges have lengths ≤ 1,
- there is at least one interior vertex.

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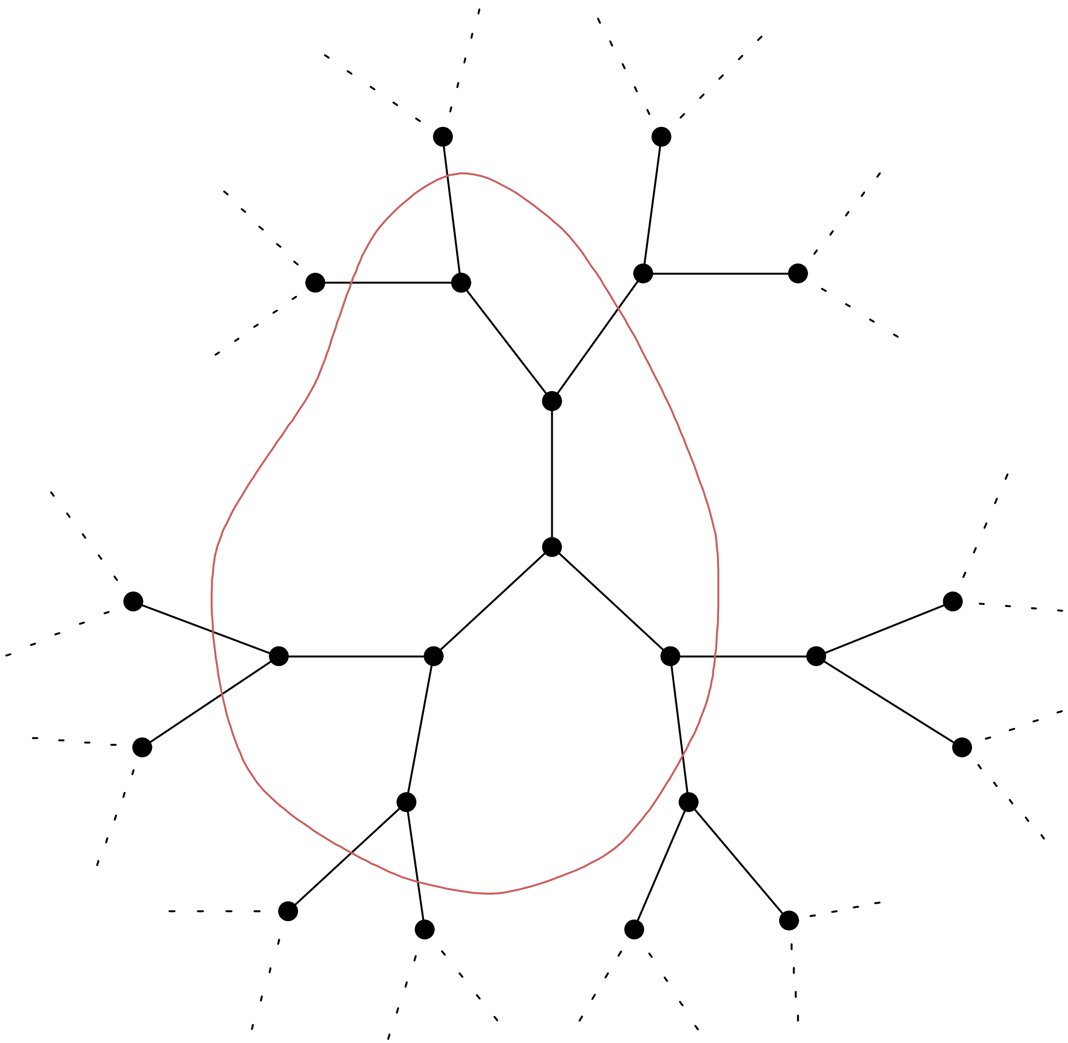
We get such a graph by cutting a region out of an infinite regular tree with edges of length 1. We insert a (new) boundary vertex where we have cut an edge.

d-regular Trees with Boundary



The Faber-Krahn Property

Let $\nu(G)$ denote the lowest (Dirichlet) eigenvalue of $\Delta_0(G)$.

We say a d-regular tree with boundary G has the **Faber-Krahn property**, if

$$\nu(G) \leq \nu(G')$$

for all d-regular trees with boundary G' with $\mu_2(G') = \mu_2(G)$.

Balls

A **ball** $B_d(c, r)$ is d-regular tree with boundary with

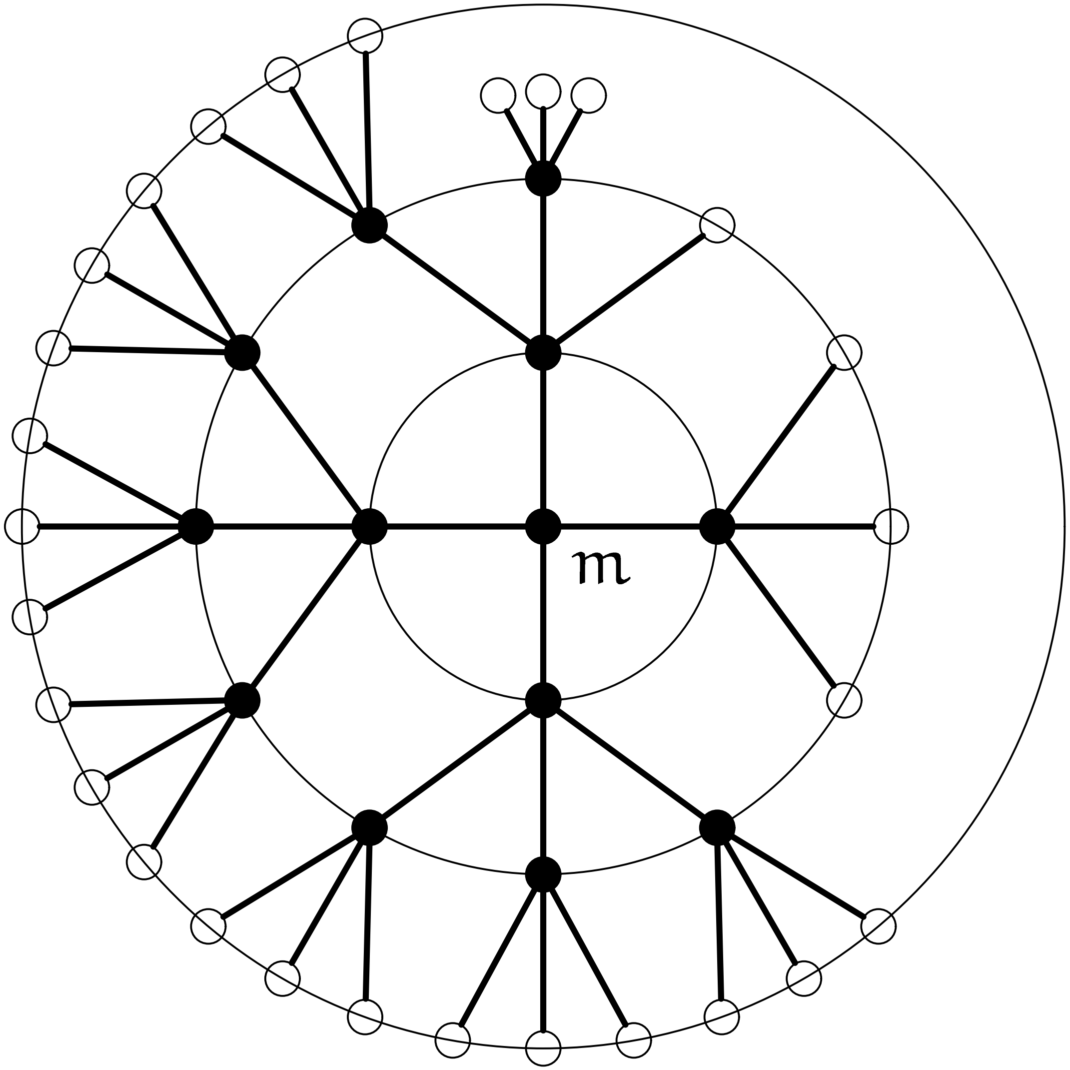
- a center $c \in \mathcal{G}$, and
- a radius $r > 0$,

such that $\text{dist}(c, v_0) = r$ for all boundary vertices $v_0 \in \partial V$.

$\text{dist}(u, v)$ denotes the geodesic distance between $u, v \in \mathcal{G}$.

Amazingly, regular trees with the Faber-Krahn property are not balls.

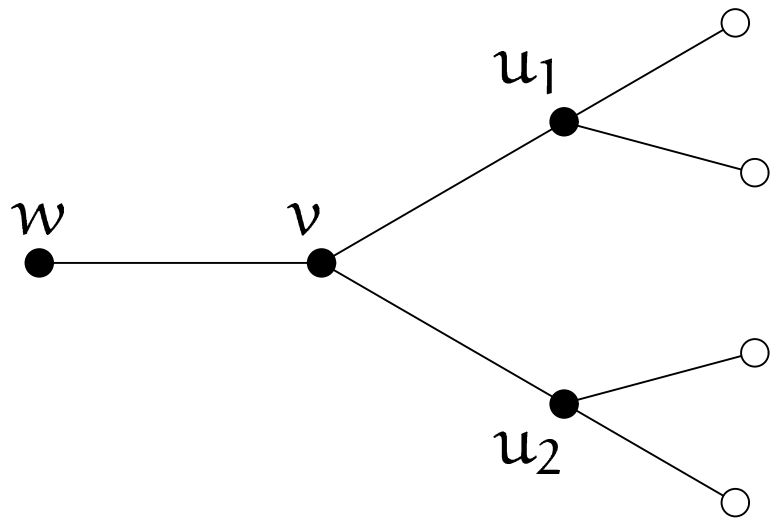
“Onion-Shaped” Tree



Branch

Let $h(v) = \text{dist}(m, v)$ denote the **height** of the vertex $v \in G$.
(m … root of tree.)

For an edge (w, v) the **branch** $\text{Br}(w, v)$ at vertex w is the maximal subgraph induced by w, v and all descendants $u \in V$ of v (i.e. the geodesic path $(w, \ldots, u)$ contains v).



Branch

The **length** $\ell(\mathrm{Br}(w, v))$ is the maximal distance between w and boundary vertices.

The branch is called **balanced** if $\mathrm{dist}(m, u_0)$ is the same for all boundary vertices $u_0 \in \partial V \cap \mathrm{Br}(w, v)$.

“Onion-Shaped” Tree // Definition

We say a d-regular tree with boundary $G(V_0 \cup \partial V, E_0 \cup \partial E)$ is **onion-shaped** if there exists a root $m \in V_0$ of the tree such that the following holds

(O1) G is connected.

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(O1) G is connected.

(O2) $B_d(m, r) \subseteq G \subseteq B_d(m, r+1)$ for an $r \in \mathbb{N}_0$ (if $|V_0| = 1$ then $r = 0$). Thus $|h(v_0) - h(u_0)| \leq 1$ for all boundary vertices $u_0, v_0 \in \partial V$.

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(O4) If two branches Br(w_1, v_1) and Br(w_2, v_2), $h(w_1) \geq h(w_2)$, are not balanced, then Br$(w_1, v_1) \subseteq$ Br(w_2, v_2).

A Faber-Krahn Theorem for Trees

A d-regular tree with boundary G, $d \geq 3$, has the Faber-Krahn property if and only if G is onion-shaped and the following holds:

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A d-regular tree with boundary G, $d \geq 3$, has the Faber-Krahn property if and only if G is onion-shaped and the following holds:

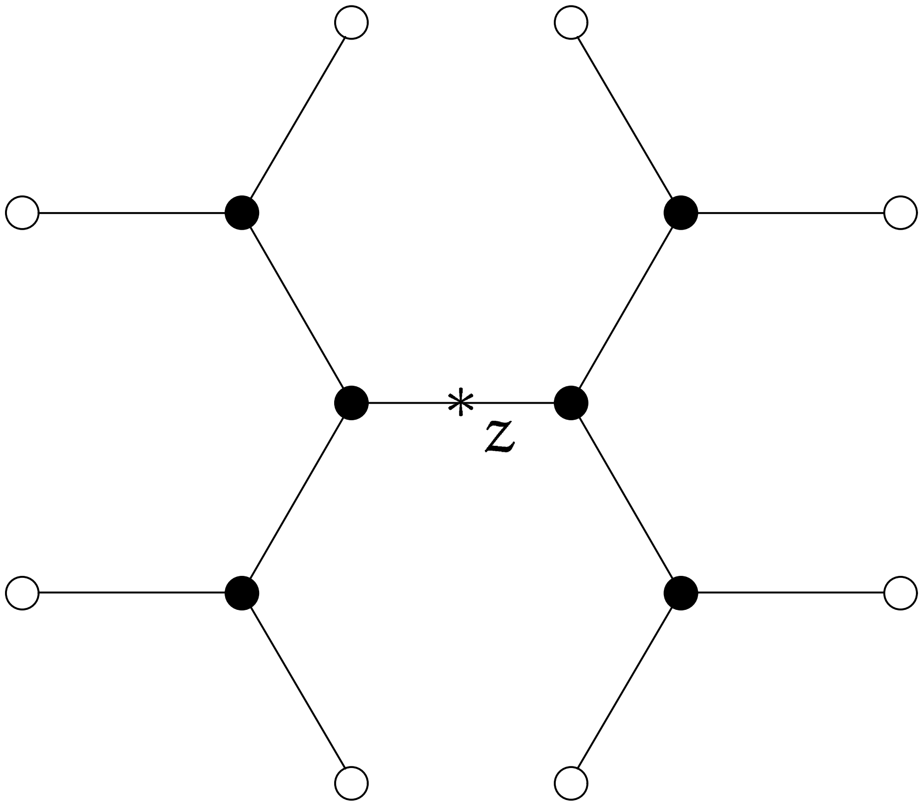
(F0) There is only one interior vertex, i.e. $|V_0| = 1$.

(F1) All branches of length $\ell \in (1, 2]$ are balanced, and there is at most one balanced branch of length $\ell \in (1, 2)$, provided that

$d \geq 5$, or

$d = 4$ and $G \subseteq B_4(z, 4.5)$, or

$d = 3$ and $G \subseteq B_3(z, 2.5)$.



The ball $B_3(z, 2.5)$

A Faber-Krahn Theorem for Trees (cont.)

(F2) All branches of length $\ell \in (2, 3]$ are balanced, and there is at most one balanced branch of length $\ell \in (2, 3)$, provided that

$d = 4$ and $B_4(z, 4.5) \subseteq G$, or

$d = 3$ and $B_3(z, 2.5) \subseteq G \subseteq B_3(z, 9.5)$.

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(F3) All branches of length $\ell \in (3,4]$ are balanced, and there is at most one balanced branch of length $\ell \in (3,4)$, provided that

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A Faber-Krahn Theorem for Trees (cont.)

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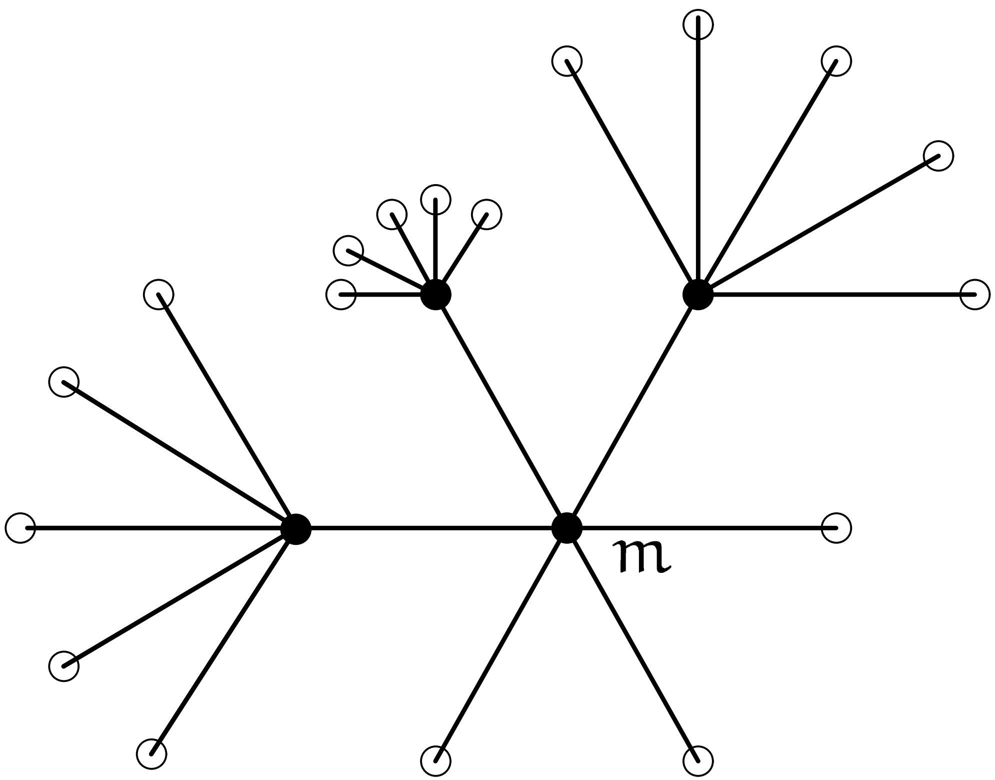
$d = 3$ and $B_3(z, 9.5) \subseteq G$.

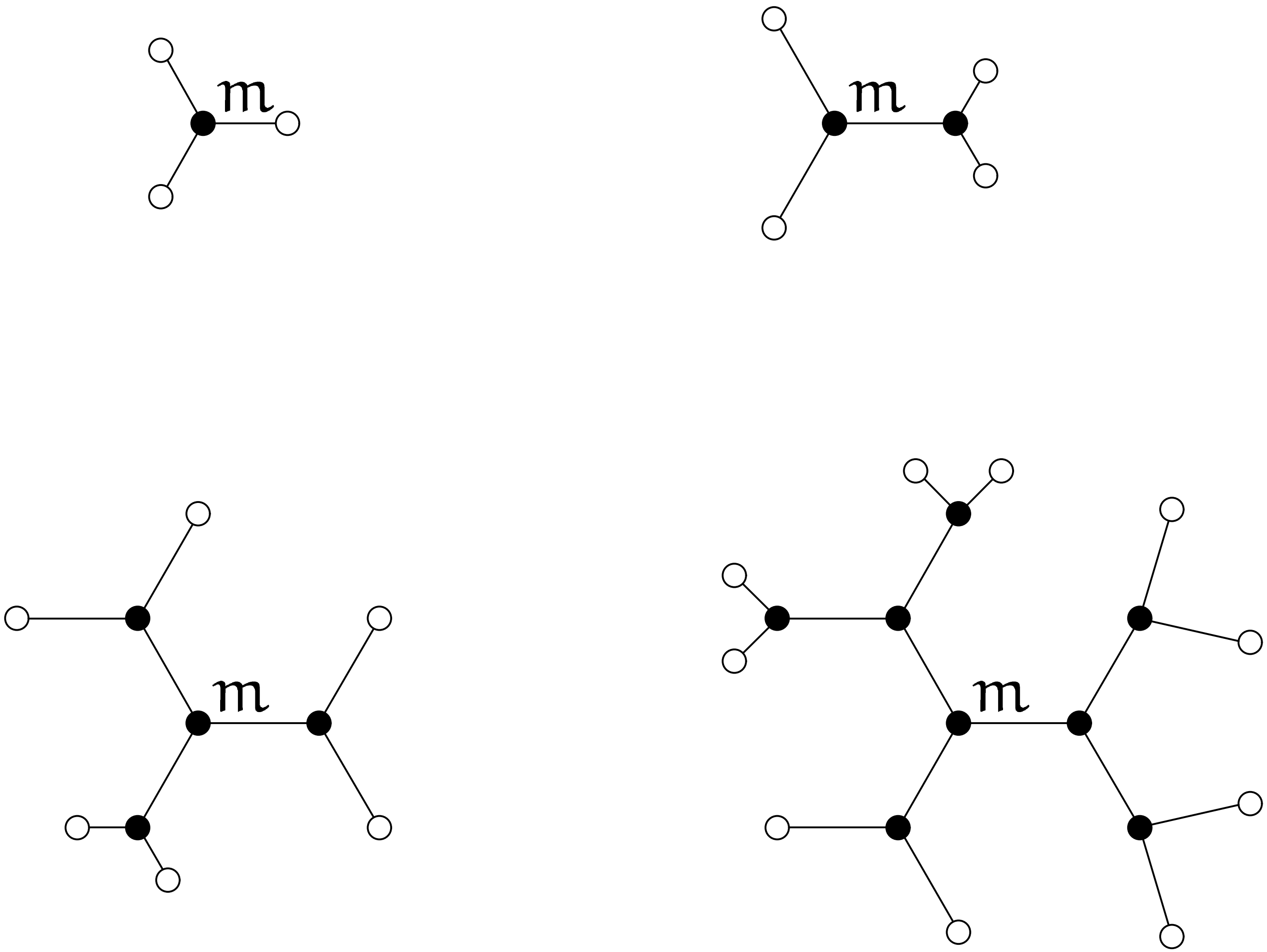
For a given volume μ_2, G is uniquely defined up to homomorhpism.

Example

A 6-regular tree with the Faber-Krahn property:
$d = 6$, $\mu_2(G) = 18$, 4 interior vertices.



Examples



Some basic properties of $\nu(G)$

- $\Delta(G)$ is a positive operator, i.e. $\nu(G) > 0$.
- An eigenfunction f to the eigenvalue $\nu(G)$ is either positive or negative on all interior vertices of G.
- $\nu(G)$ is continuous as a function of G in the metric $\rho(G, G') = \mu(G \setminus G') + \mu(G' \setminus G)$.
- $\nu(G)$ is monotone in G, i.e. if $G \subset G'$ then $\nu(G) > \nu(G')$.
- $\nu(G)$ is a simple eigenvalue.

For d-regular trees G with boundary we have

- $\nu(G) > d - 2\sqrt{d-1}$.

Spiral-like Ordering (Pruss 1998)

We say that a well-ordering $\prec$ on $G(V_0 \cup \partial V, E_0 \cup \partial E)$ is **spiral-like** providing the following conditions hold for all vertices $v, w, v_1, v_2, w_1, w_2 \in V_0$ and $u_1, u_2 \in \partial V$:

(S1) If $h(v) < h(w)$ then $v \prec w$.

(S2) If $v_1 \prec v_2$ and w_i is a child of v_i (i.e. $(v_i, w_i) \in E$ and $h(w_i) = h(v_i) + 1$), for $i = 1, 2$, then $w_1 \prec w_2$.

(S3) If (v_1, u_1) and (v_2, u_2) are boundary edges of lengths c_1 and c_2, respectively, with $h(v_1) = h(v_2)$ and $c_1 > c_2$, then $u_1 \prec u_2$.

In (S3) the ordering of some boundary vertices is reverse to their lengths.

Geometry of Eigenfunctions

Let G be a d-regular tree with boundary with the Faber-Krahn property. Then

(M1) G is connected;

(M2) $|h(v_0) - h(u_0)| \leq 1$ for all boundary vertices $u_0, v_0 \in \partial V$;

(M3) There exists a spiral-like ordering $\prec$ such that $u \prec v \Rightarrow f(u) \geq f(v)$, for all vertices $u, v \in V$.

(M4) The normal derivative of f at all boundary edges of length $c_e < 1$ is the same.

Rearrangements

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Remove two edges

$$(v_1, u_1) \quad \text{and} \quad (v_2, u_2)$$

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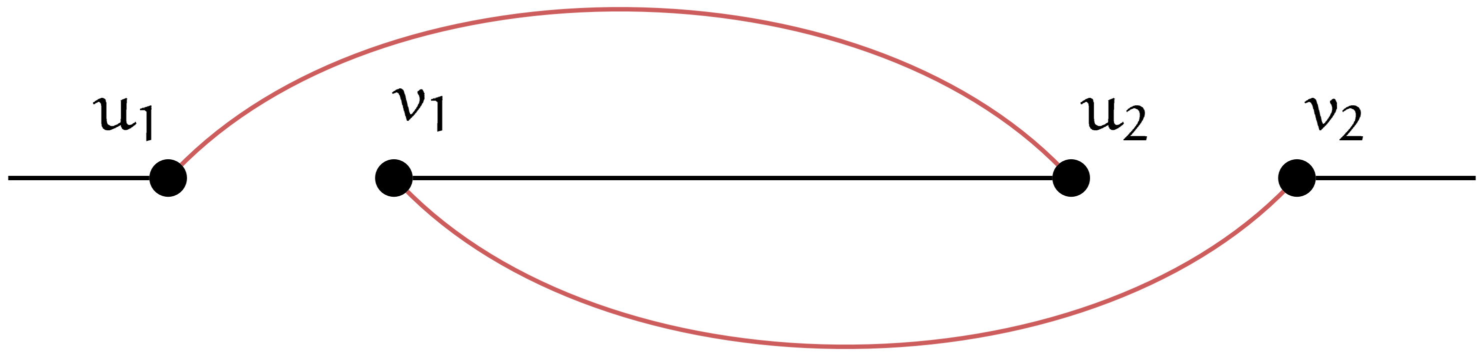

Remove two edges

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of length c_1 and c_2, respectively,
and replace these by the respective edges

$$(v_1, v_2) \quad \text{and} \quad (u_1, u_2)$$

of length c_2 and c_1.

Rearrangements

We get a new graph G' with same vertex set V and new edge set E' which again is a d-regular tree.

Lemma:

- $\mu_2(G') = \mu_2(G)$
- Furthermore, if

$$f(v_1) \geq f(u_2), \quad f(v_2) \geq f(u_1), \quad \text{and} \quad c_1 \leq c_2$$

then

$$\nu(G') \leq \nu(G)$$

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Lemma:

- $\mu_2(G') = \mu_2(G)$ (obvious)
- Furthermore, if

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Have to show: $\nu(G') \le \mathcal{R}_{G'}(f) \le \mathcal{R}_G(f) = \nu(G)$

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$\nu(G')$ is lowest eigenvalue

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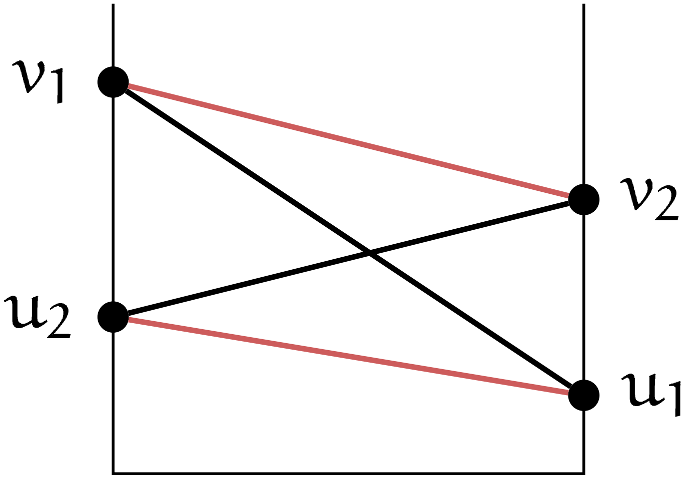
Have to show: $\nu(G') \le \mathcal{R}_{G'}(f) \le \mathcal{R}_G(f) = \nu(G)$

$$\mathcal{R}_G(f) = \frac{\langle \Delta f, f \rangle}{\langle f, f \rangle} = \frac{\sum \frac{1}{c_e}(f(u) - f(v))^2}{\sum (f(v))^2}$$

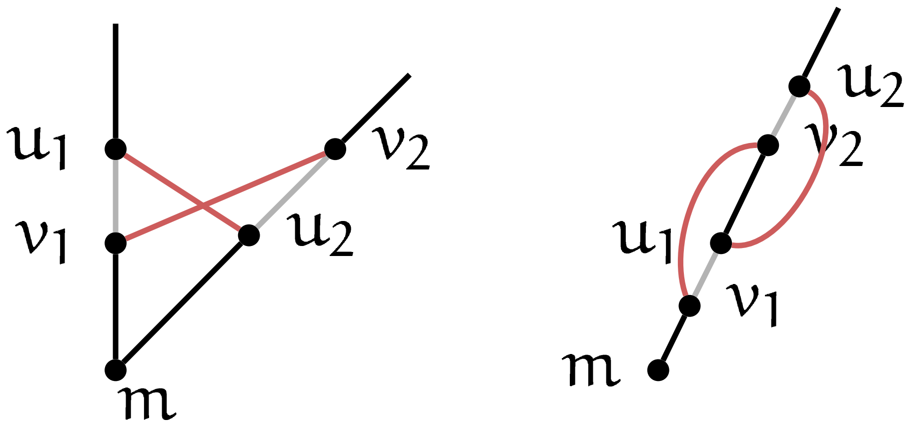


Rearrangements

Now define a ordering $\prec$ on the interior vertices of G, such that

$$v_i \prec v_j \quad \text{if} \quad f(v_i) \geq f(v_j).$$

Then rearrange the interior edges as described to make $\prec$ spiral-like on V_0 (stepwise, beginning with maximum m).



Derivative at Boundary Edges

Normal derivative of f at the boundary edge $e_j = (v_j, u_j) \in \partial E$, $v_j \in V_0$, of length $c_j = c_{e_j}$ is

$$f(v_j)/c_j$$

The "average" normal derivative of n boundary edges is given by

$$\sum_{j=1}^{n} f(v_j) \Bigg/ \sum_{j=1}^{n} c_j$$

Perturbation of Boundary Edges

Replace each of these n edges e_j by edges $\bar{e}_j$ of length $\bar{c}_j$, where

$$\bar{c}_j = f(v_j)\,\frac{\sum_{i=1}^{n} c_i}{\sum_{i=1}^{n} f(v_i)}.$$

Then the normal derivative is the same for all these boundary edges.
If such an edge $\bar{e}_j$ is longer than 1, then replace all the edges e_j by edges $e_j(\varepsilon)$ of lengths

$$c_j(\varepsilon) = (1-\varepsilon)c_j + \varepsilon\,\bar{c}_j, \quad \text{where } \varepsilon \in (0,1].$$

Make ε as great as possible, i.e. (either) one edge $e_j(\varepsilon)$ has length $c_j(\varepsilon) = 1$ or $\varepsilon = 1$.

Perturbation of Boundary Edges

Again, we get a new graph G' with same vertex set V and new edge set E' which again is a d-regular tree.

Lemma:

- $\mu_2(G') = \mu_2(G)$
- $\nu(G') \leq \nu(G)$.
  Equality holds if and only if $\bar{c}_j = c_j$ for all j.

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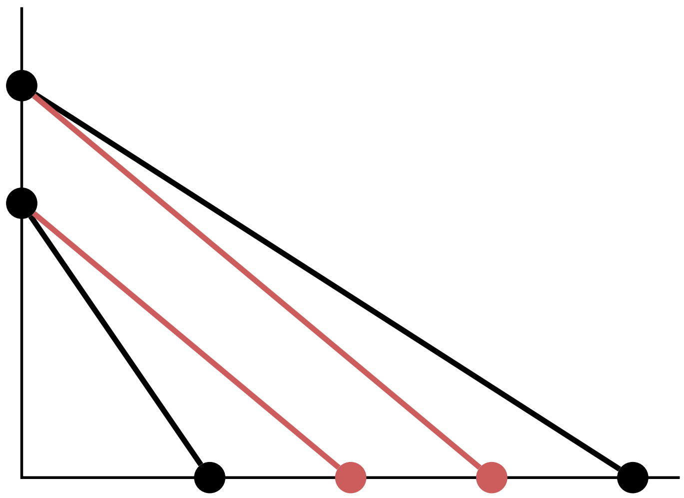

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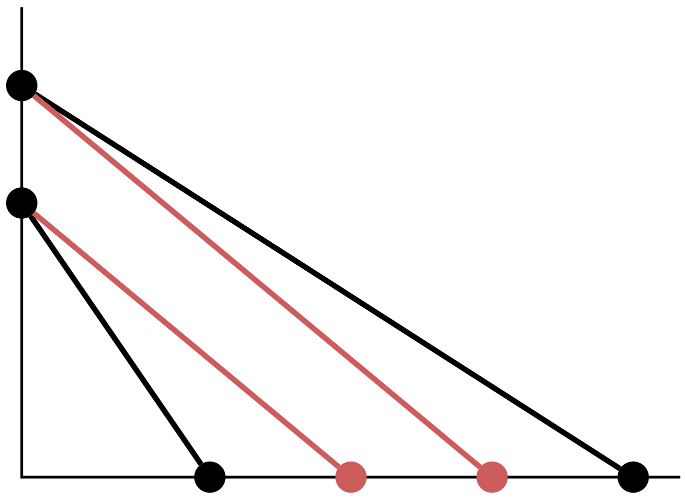

Alternatively, use perturbation theory for linear operators (matrices).

Resumé 1

Using the above lemmata and a recursive rearrangement of G results in a graph G' with decreased first Dirichlet eigenvalue and

(M2) $|h(v_0) - h(u_0)| \leq 1$ for all boundary vertices $u_0, v_0 \in \partial V$;

(M3) There exists a spiral-like ordering $\prec$ such that $u \prec v \Rightarrow f(u) \geq f(v)$, for all vertices $u, v \in V$.

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For onion-shaped property we have to show that all almost all branches are balanced.

Recursion Formula for Branches

Let $\{(v_0, v_1), (v_1, v_2), \ldots, (v_{n-1}, v_n)\}$ denote a geodesic path in G with $v_0 \in \partial V$ and $v_i \in V_0$, $i = 1, \ldots, n$.

If $\mathrm{Br}(v_j, v_{j-1})$, $j = 1, \ldots, n$, are balanced branches then

$$f(v_2) = ((d-1) + (1-\nu)c)\, f(v_1)/c$$
$$f(v_j) = (d-\nu) f(v_{j-1}) - (d-1) f(v_{j-2})$$

where c is the length of the boundary edge (v_0, v_1).

Recursion Formula for Branches

Consequently, we can express $f(v_j)$ by

$$f(v_j) = s\,(\alpha_j(d,\nu) + \beta_j(d,\nu)\,c)$$

where s denotes the normal derivative at the boundary edge (v_1, v_0), i.e. $f(v_1)/c$.

The coefficients α_j and β_j are polynomials which are given by the recursions

$$\alpha_1 = 0, \quad \alpha_2 = d-1, \quad \text{and}$$
$$\alpha_i = (d-\nu)\,\alpha_{i-1} - (d-1)\,\alpha_{j-2},$$

$$\beta_1 = 1, \quad \beta_2 = 1-\nu, \quad \text{and}$$
$$\beta_i = (d-\nu)\,\beta_{i-1} - (d-1)\,\beta_{i-2}.$$

Proof of the theorem

By means of this recursion formula, the normal derivative at the "leaves" of balanced branches at the same vertex can be compared.

For the proof it is shown that many cases that are not excluded by Resumé 1 cannot exist.

(And it is too tedious to give any details.)

$\nu(B_3(z, 2.5))$, $\nu(B_4(z, 4.5))$ and $\nu(B_3(z, 9.5))$ are the zeros of $\beta_2(d, \nu)$ and $\beta_3(d, \nu)$, respectively.

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