Local Algorithm for Strong Products

Werner Klöckl

Institute for Applied Mathematics
Montanuniversität Leoben, Austria

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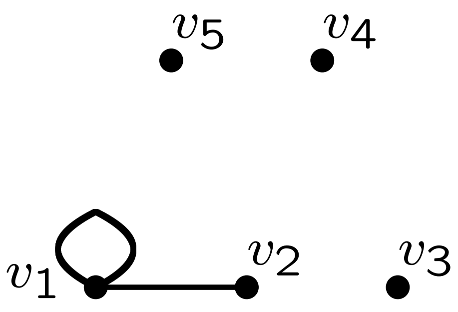
Definition *Graph* $G = (V(G), E(G))$ $(= (V, E))$.

$V(G)$ ... *vertex set*, $E(G)$ ... *edge set*.

Undirected graph: The elements of E *are subsets of* V *with* $|e| \in \{1, 2\}$. $|e| = 1$ ... *loop*

Example: $V = \{v_1, v_2, v_3, v_4, v_5\}$, $E = \{\{v_1, v_2\}, \{v_1\}\}$.

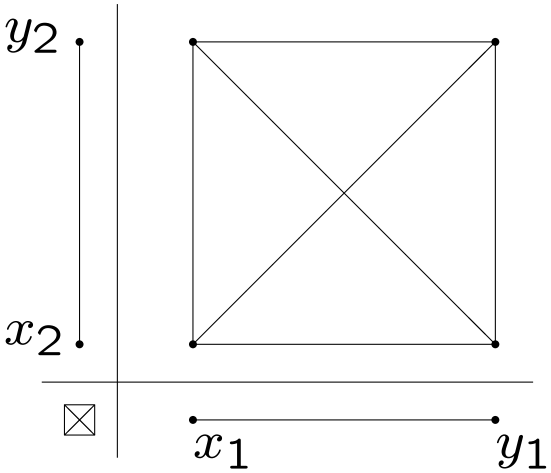
$$G = (V, E)$$



Definition Strong product $G_1 \boxtimes G_2$:

$V(G_1 \boxtimes G_2) = \{(x_1, x_2) \mid x_1 \in V(G_1) \text{ and } x_2 \in V(G_2)\}$

$E(G_1 \boxtimes G_2) = \{\{(x_1, x_2), (y_1, y_2)\} \mid (\{x_i, y_i\} \in E(G_i) \text{ for } i = 1, 2),$
$(\{x_1, y_1\} \in E(G_1) \text{ and } x_2 = y_2) \text{ or } (\{x_2, y_2\} \in E(G_2) \text{ and } x_1 = y_1)\}.$



Properties: commutative, associative, K_1 ... unit

Definition G *is prime (with respect to* $\boxtimes$ *), if* $\nexists\ A \boxtimes B = G$ *with* A, B *nontrivial, i.e.* $|V(A)|, |V(B)| > 1$.

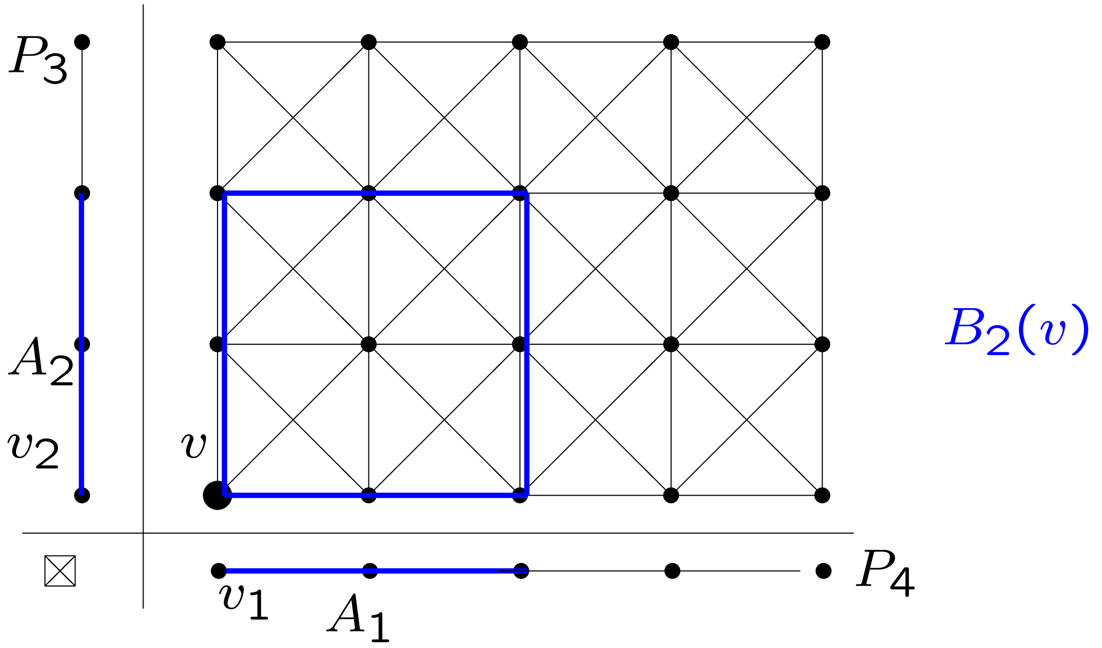
Question: Is the prime factor decomposition unique? Algorithm?

Theorem (Dörfler and Imrich, 1969) *PFD (* $\boxtimes$ *) is unique for finite, connected undirected graphs.*
There is a polynomial algorithm $(O(|V(G)|^5))$ *to compute it* (Feigenbaum and Schäffer 1992).

Theorem *There are polynomial algorithms to compute the PFD with respect to* $\Box$ (Feigenbaum, 1985) *and* $\times$ (Imrich, 1997)

Idea: G given. Cover it by subgraphs. $\longrightarrow$ Factorize subgraphs $\longrightarrow$ Suggest global factors

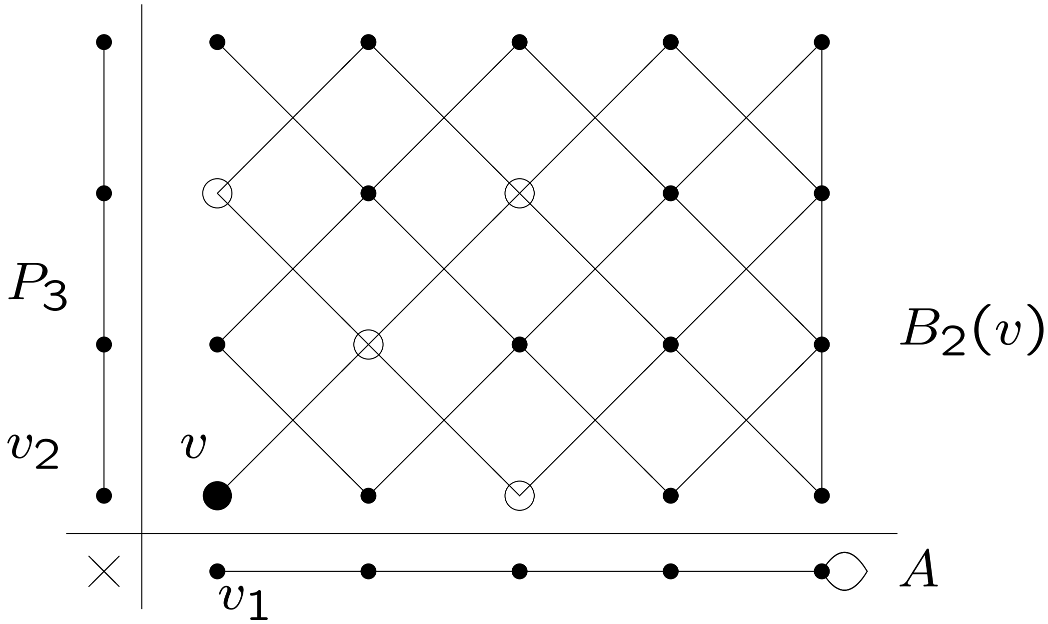
Definition $G = (V, E)$, $v \in V$. $B_n(v)$ *induced by* $\{x \in V \mid d(x, v) \leq n\}$ *... ball with radius* n *and center* v.



Remark: Balls in products are products ($\boxtimes$). Conclusion:

Prime test: $B_2(v)$ ($\subset G$) prime $\Rightarrow$ G prime.

What about cardinal products?



Algorithm

Choose $v \in V(G)$, $W = V(B_2(v))$, $B_2(v) = A_1 \boxtimes A_2 \boxtimes \ldots \boxtimes A_n$

while $(W \neq V(G))$

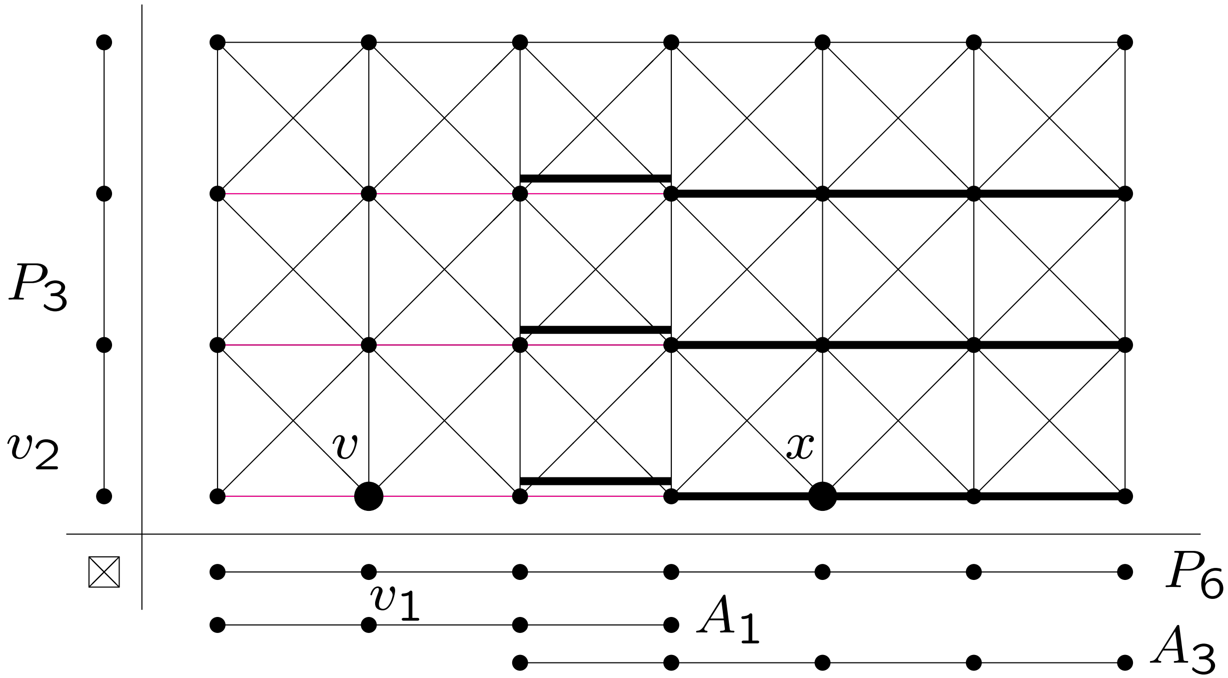
$x \in V(G) \setminus W$ with $d(x, W) = 1$, $W = W \cup V(B_2(x))$,
$B_2(x) = A_{n+1} \boxtimes A_{n+2} \boxtimes \ldots \boxtimes A_{n+m}$

for $k = 1 : m$

for $j = 1 : n$ $\{$

if $((A'_j \cap A'_{n+k}) \neq \emptyset)$ $A'_j = A'_j \cup A'_{n+k}$ $\}$

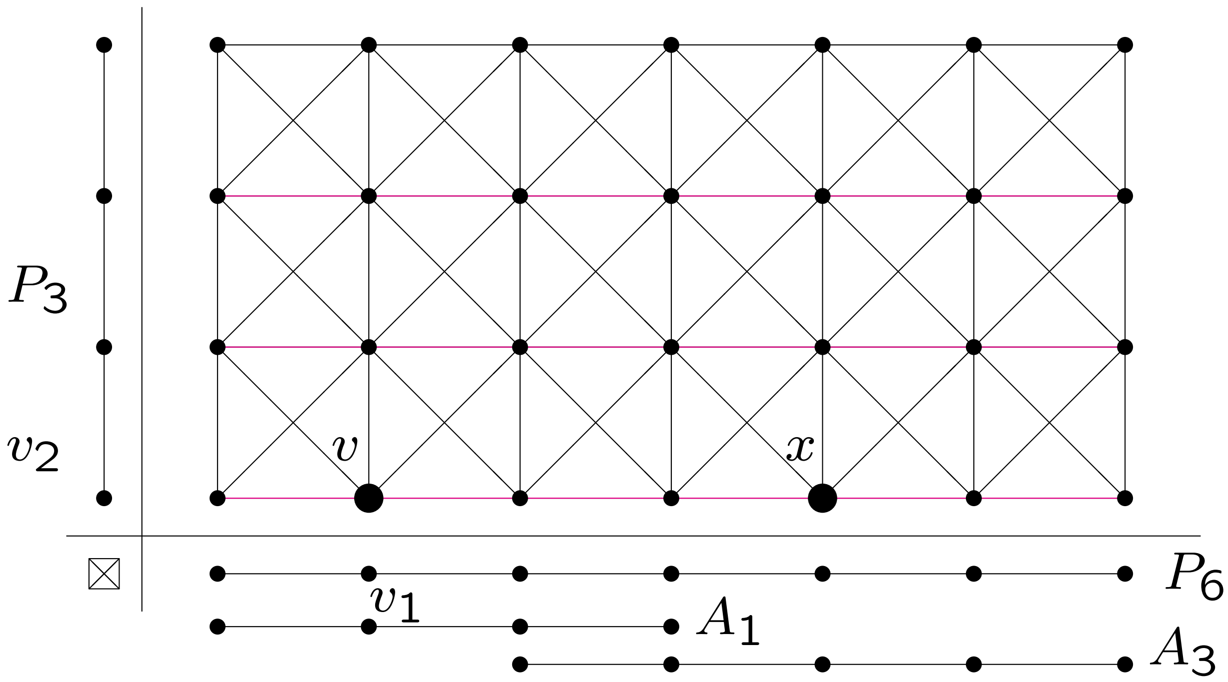
$A'_{n+k} = \emptyset$



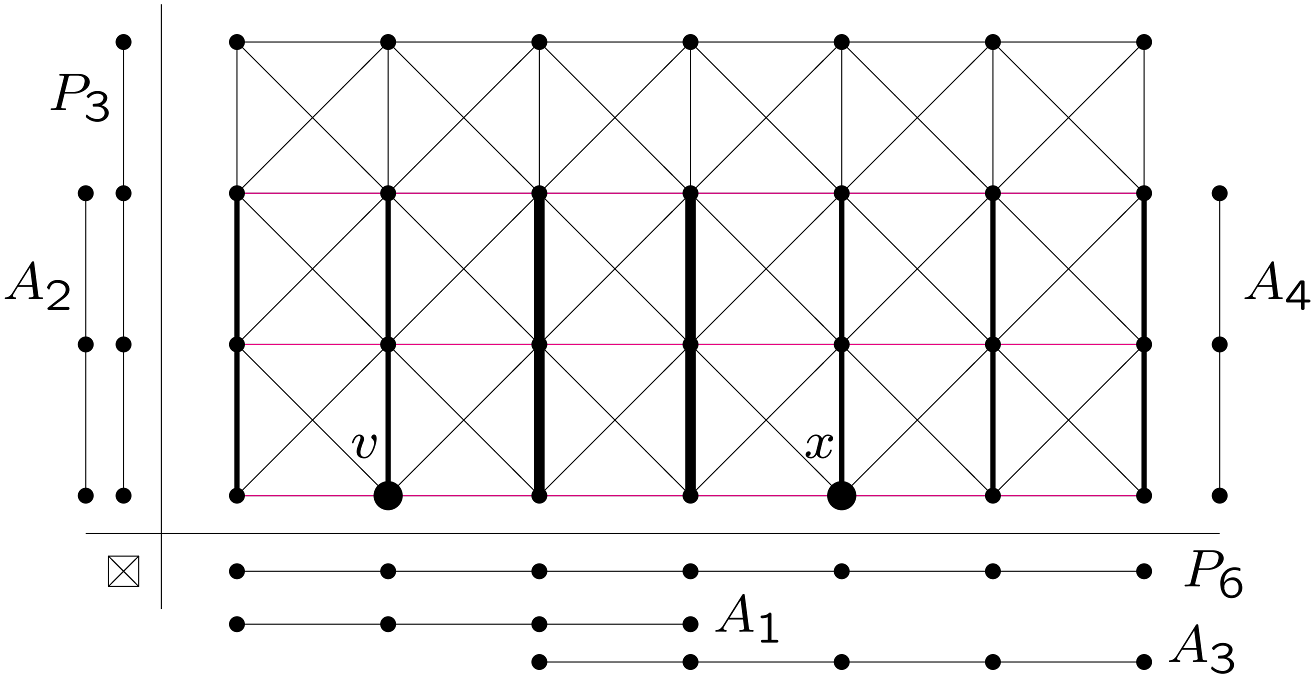
$A'_1 = \{e \in E(B_2(v)) \mid e \in$ copy of $A_1\} = \{$magenta edges$\}$

$A'_1 \cup A'_3$: magenta and **thick** edges

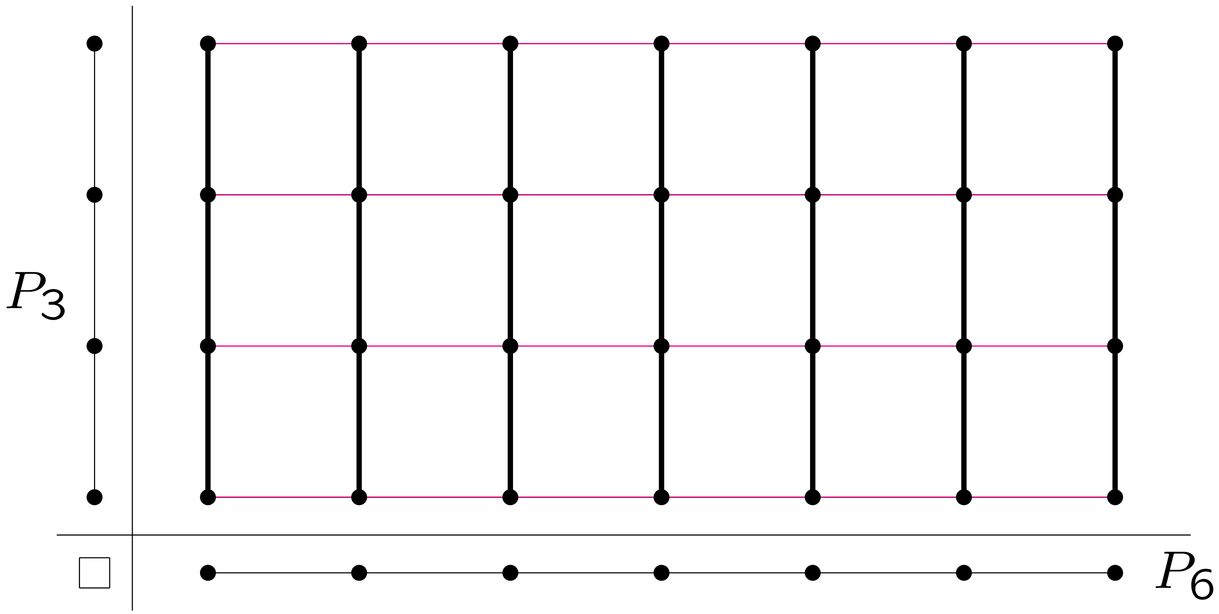
Situation after the first union:



$A'_2 \cap A'_4$: very thick edges



Output:



Approximate products?