Dynamic Programming and Automata

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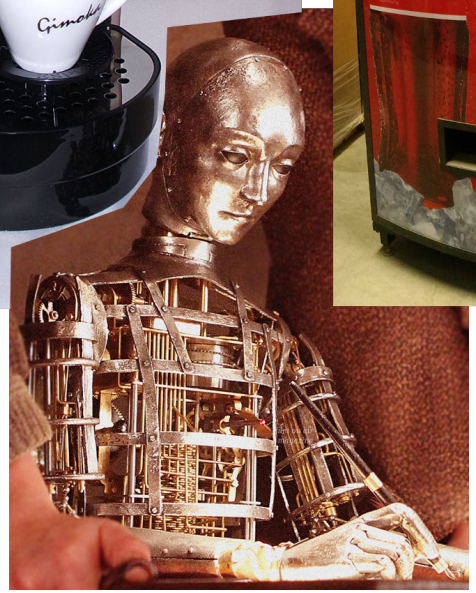
15.02.2018

Me



Me

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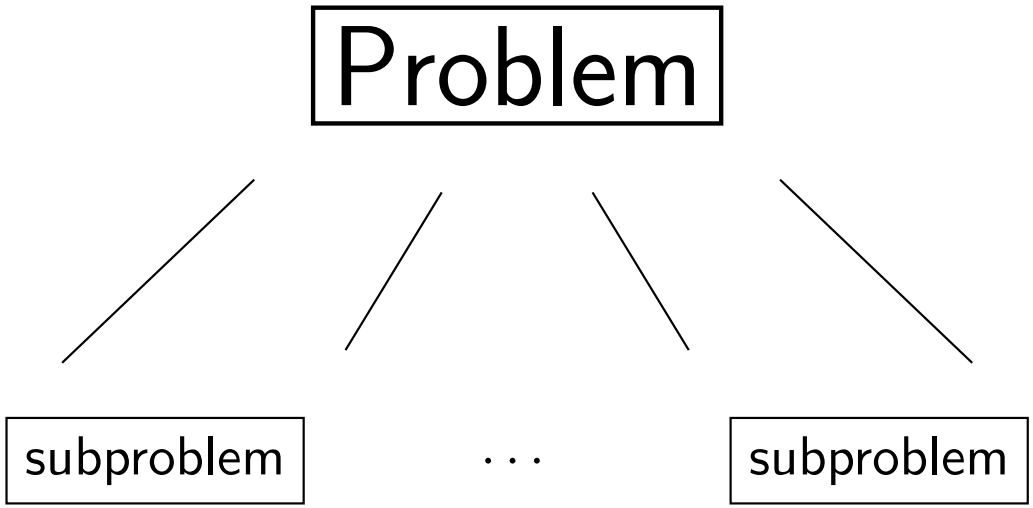
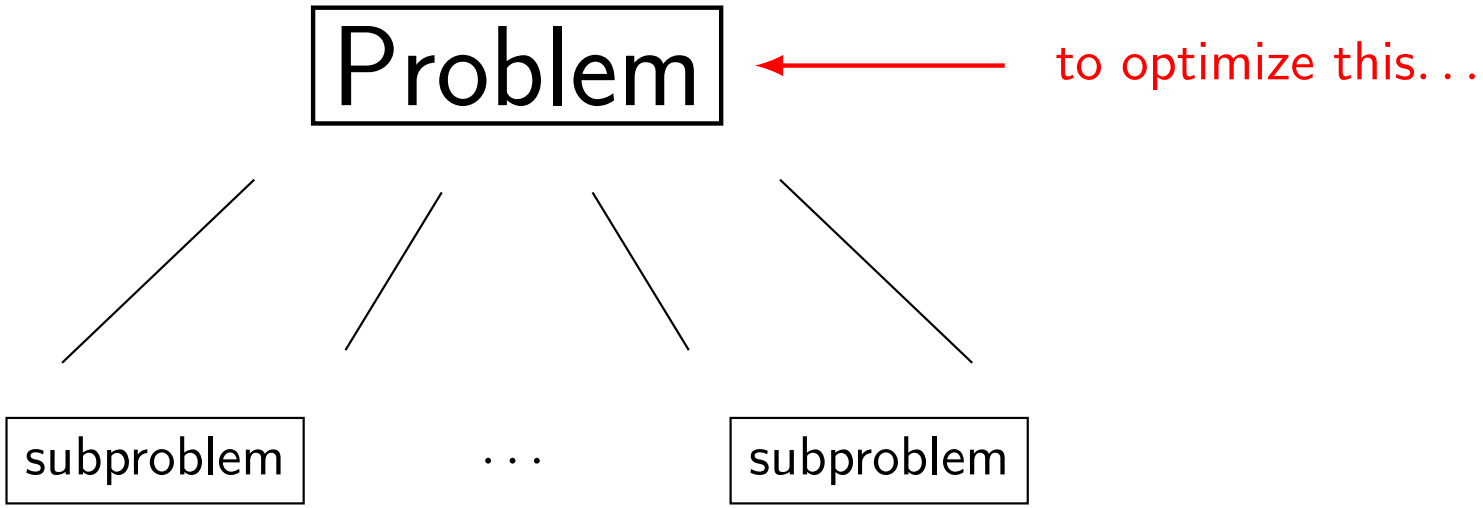
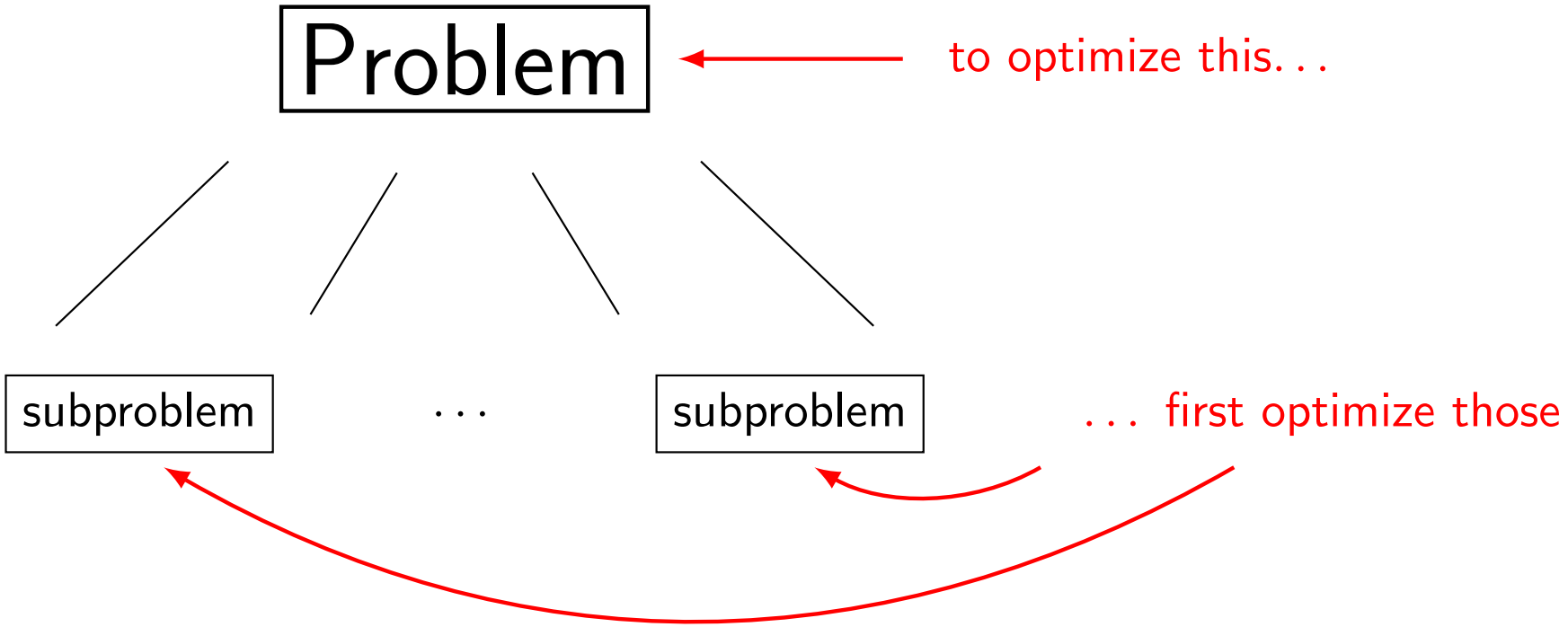
Dynamic Programming



Dynamic Programming



Dynamic Programming



Overview

Dynamic Programming

Recursive equations

Overview

Dynamic Programming

Recursive equations

Algebraic Dynamic Programming

Grammars + evaluation algebra

Overview

Dynamic Programming

Recursive equations

Algebraic Dynamic Programming

Grammars + evaluation algebra

Let's try something new:
use automata

Example 1: Needleman-Wunsch

Alignment of ATC and GAC

Alignment 1

A	T	C
G	A	C

Score: $-1 - 1 + 1 = -1$

match $= 1$
mismatch $=$ -1
insertion $=$ -1
deletion $=$ -1

Example 1: Needleman-Wunsch

Alignment of ATC and GAC

Alignment 1

A T C
G A C

Score: $-1 - 1 + 1 = -1$

Alignment 2

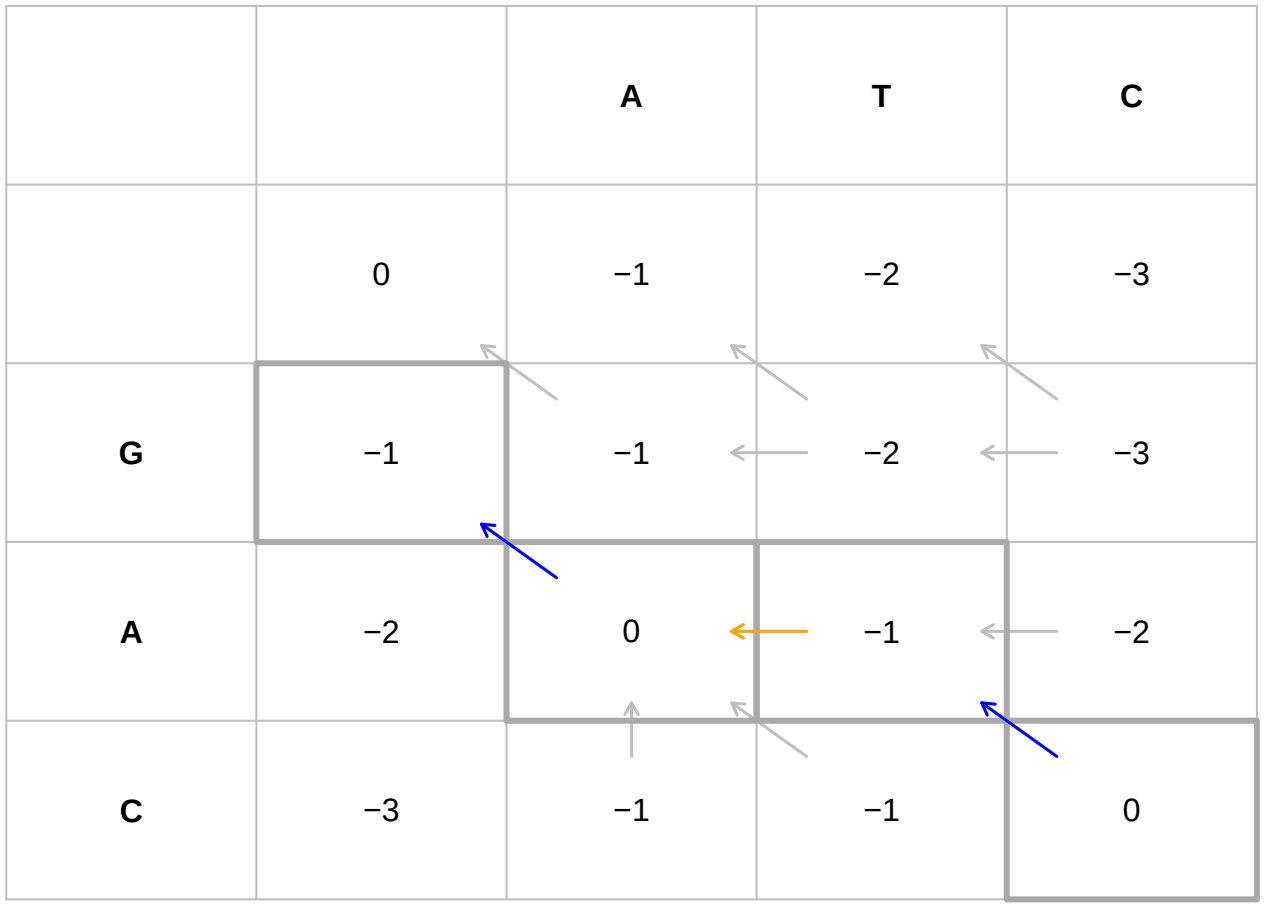
- A T C
G A - C

Score: $-1 + 1 - 1 + 1 = 0$

match $= 1$
mismatch $=$ -1
insertion $=$ -1
deletion $=$ -1

Example 1: Needleman-Wunsch

Alignment of ATC and GAC



match = 1
mismatch = -1
insertion = -1
deletion = -1

Example 1: Alignment Automaton

Grammar:

$$X \to \underset{\text{match}}{\binom{u}{u}X} \mid \underset{\text{mismatch}}{\binom{u}{v}X} \mid \underset{\text{ins}}{\binom{u}{-}X} \mid \underset{\text{del}}{\binom{-}{v}X} \mid \underset{\text{end}}{\epsilon}$$

Example 1: Alignment Automaton

Grammar:

Weight:	1	-1	-1	-1	0
$X \to$	$\binom{u}{u}X$	$\binom{u}{v}X$	$\binom{u}{-}X$	$\binom{-}{v}X$	ϵ
	match	mismatch	ins	del	end

Example 1: Alignment Automaton

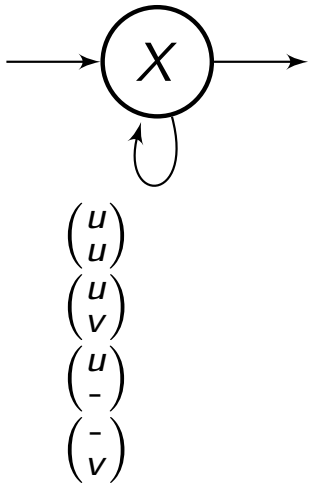
Grammar:

Weight: 1 -1 -1 -1 0

$$X \to \binom{u}{u}X \mid \binom{u}{v}X \mid \binom{u}{-}X \mid \binom{-}{v}X \mid \epsilon$$

match mismatch ins del end

Automaton $\mathcal{A}$:



Example 1: Alignment Automaton

Grammar:

Weight: 1 -1 -1 -1 0

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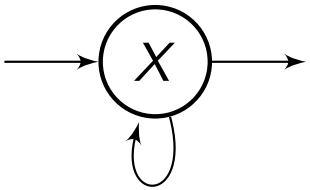
match mismatch ins del end

Automaton $\mathcal{A}$:



$\binom{u}{u} / \ 1$
$\binom{u}{v} / -1$
$\binom{u}{-} / -1$
$\binom{-}{v} / -1$

Example 1: Alignment Automaton

Grammar:

Weight: 1 -1 -1 -1 0

$$X \to \binom{u}{u}X \mid \binom{u}{v}X \mid \binom{u}{-}X \mid \binom{-}{v}X \mid \epsilon$$

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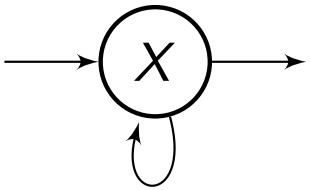
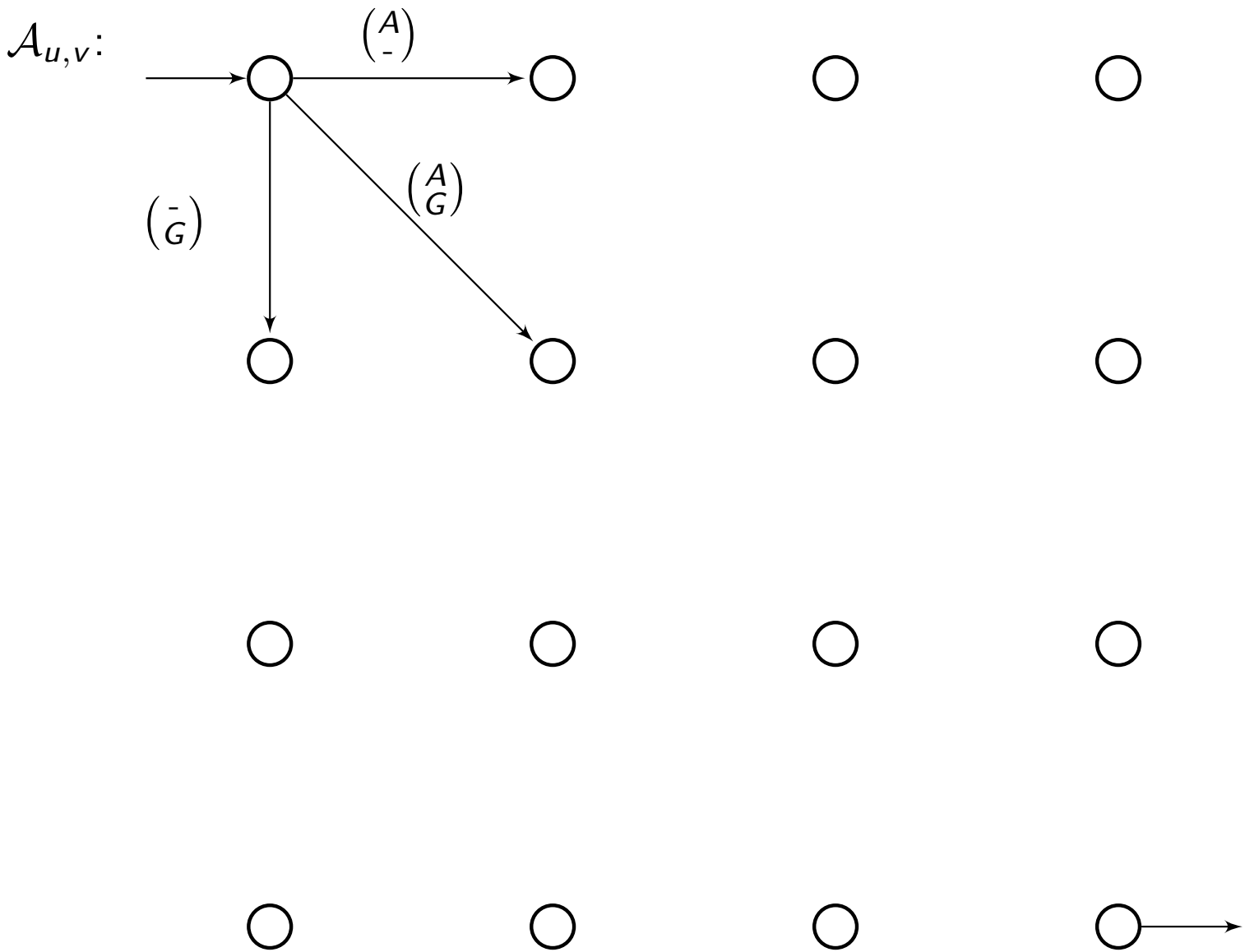


$\binom{u}{u} / \ 1$
$\binom{u}{v} / -1$
$\binom{u}{-} / -1$
$\binom{-}{v} / -1$

For experts:

Weight structure:
$(\mathbb{Z} \cup \{-\infty\}, \max, +, -\infty, 0)$

Example 1: Automaton for $u = ATC$ and $v = GAC$



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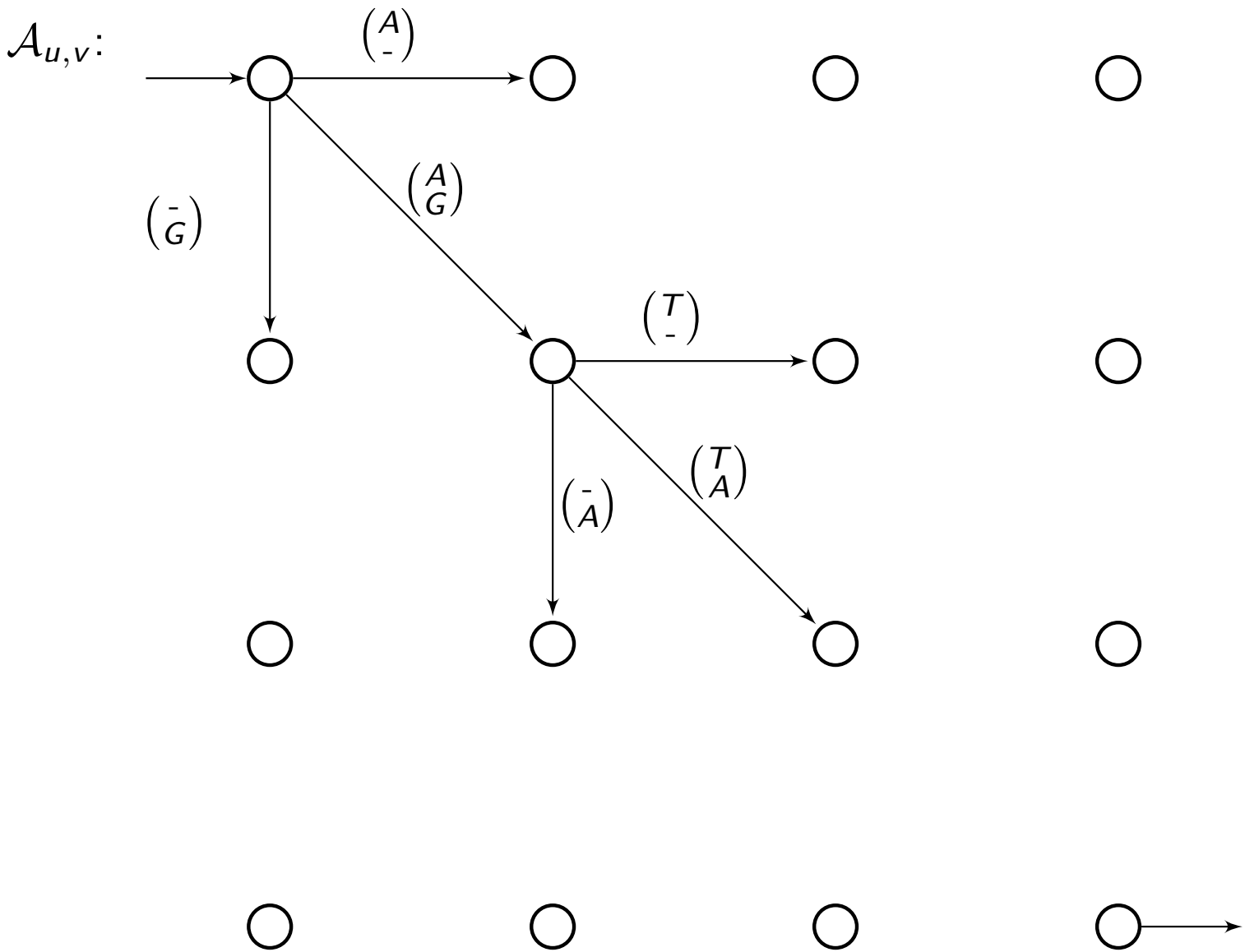
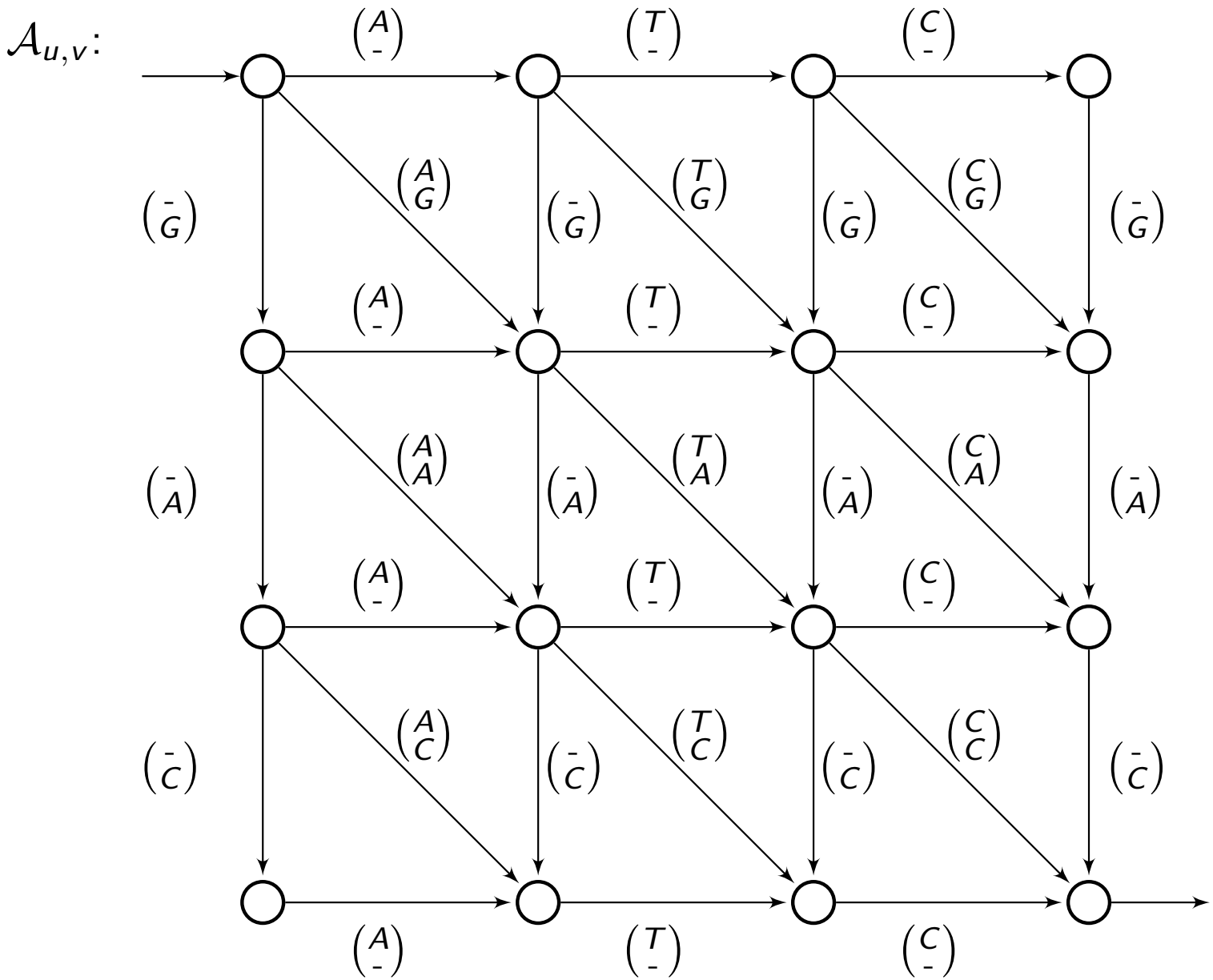
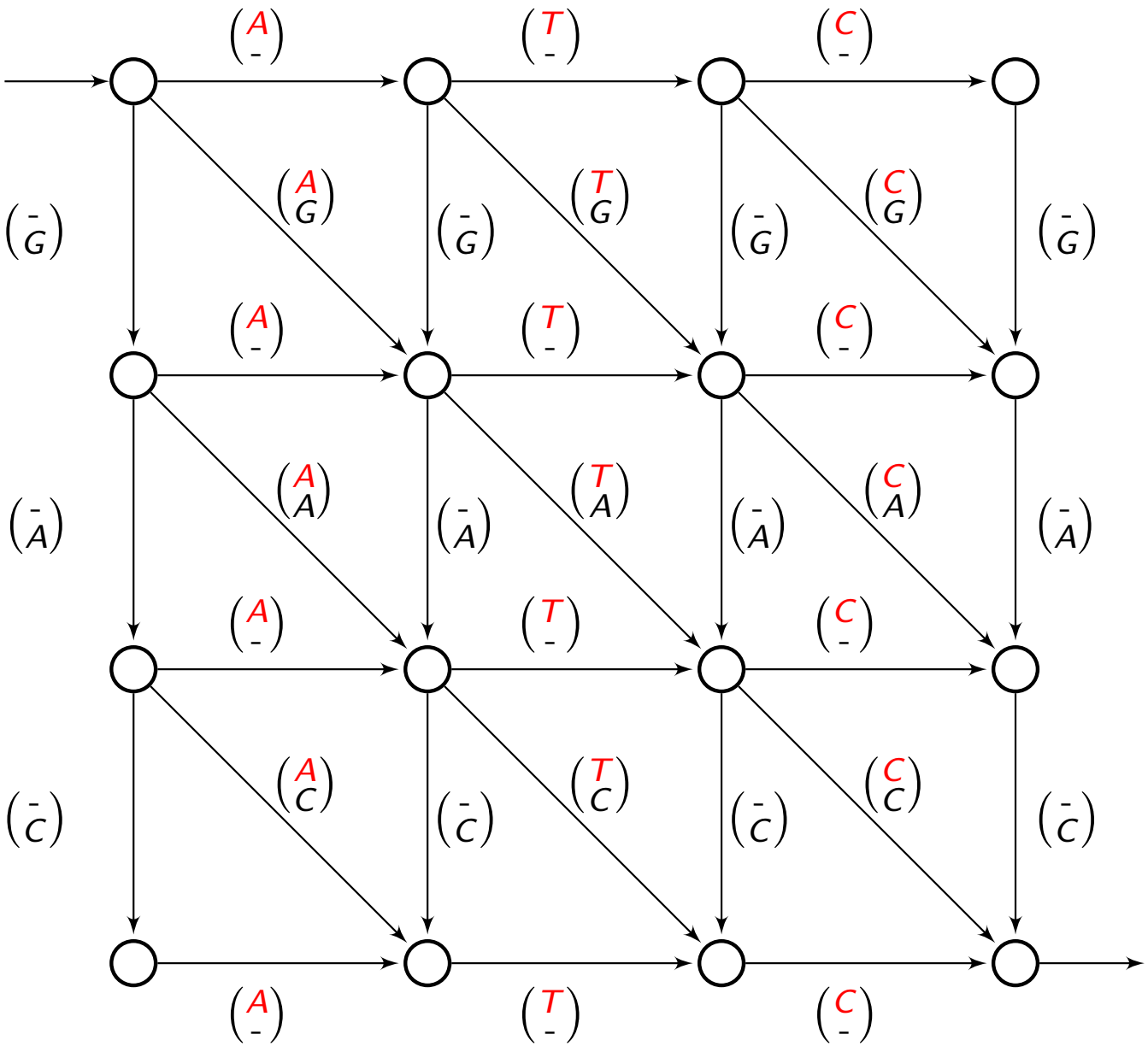


Example 1: Automaton for $u = ATC$ and $v = GAC$

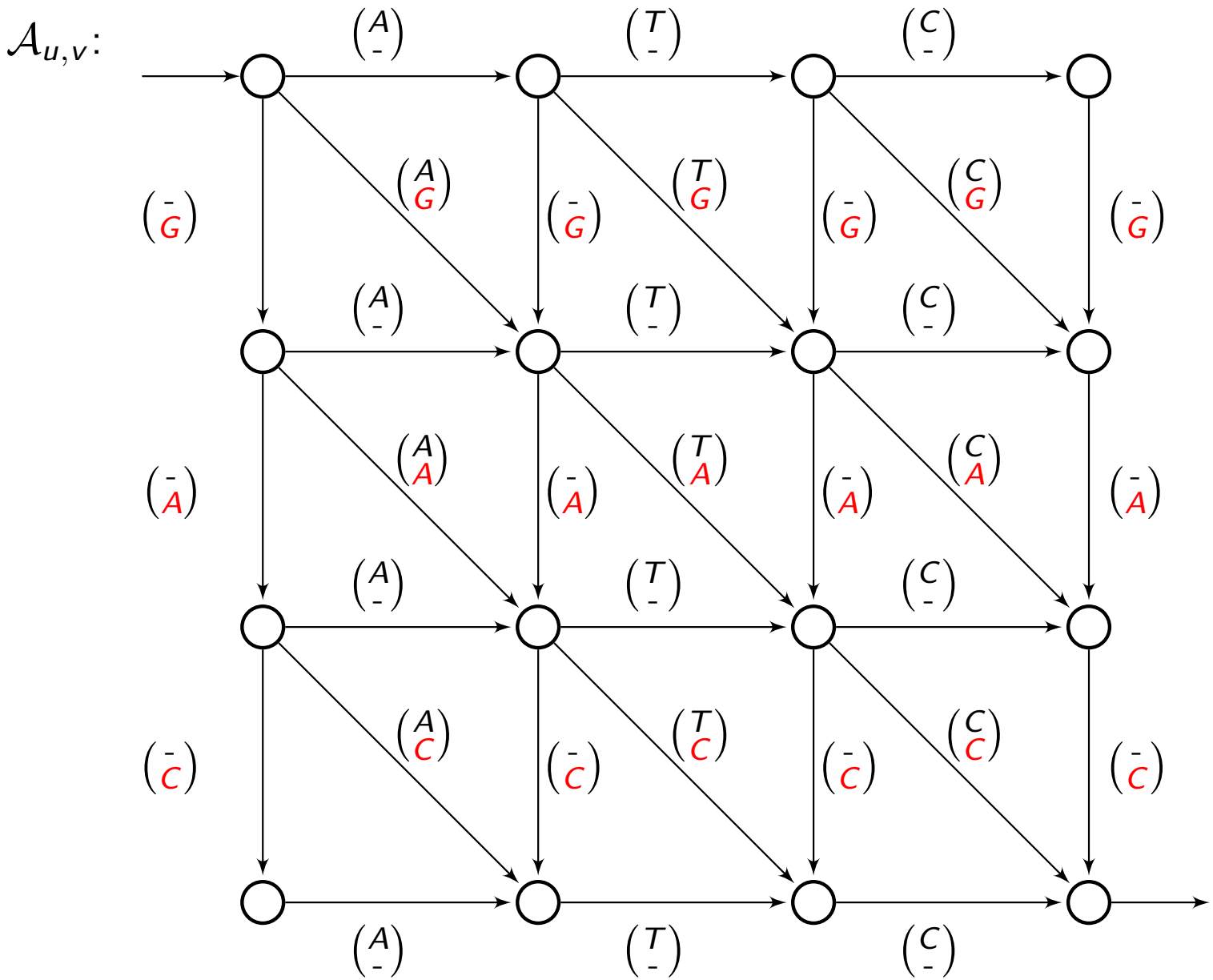


Example 1: Automaton for $u = ATC$ and $v = GAC$

$\mathcal{A}_{u,v}$:

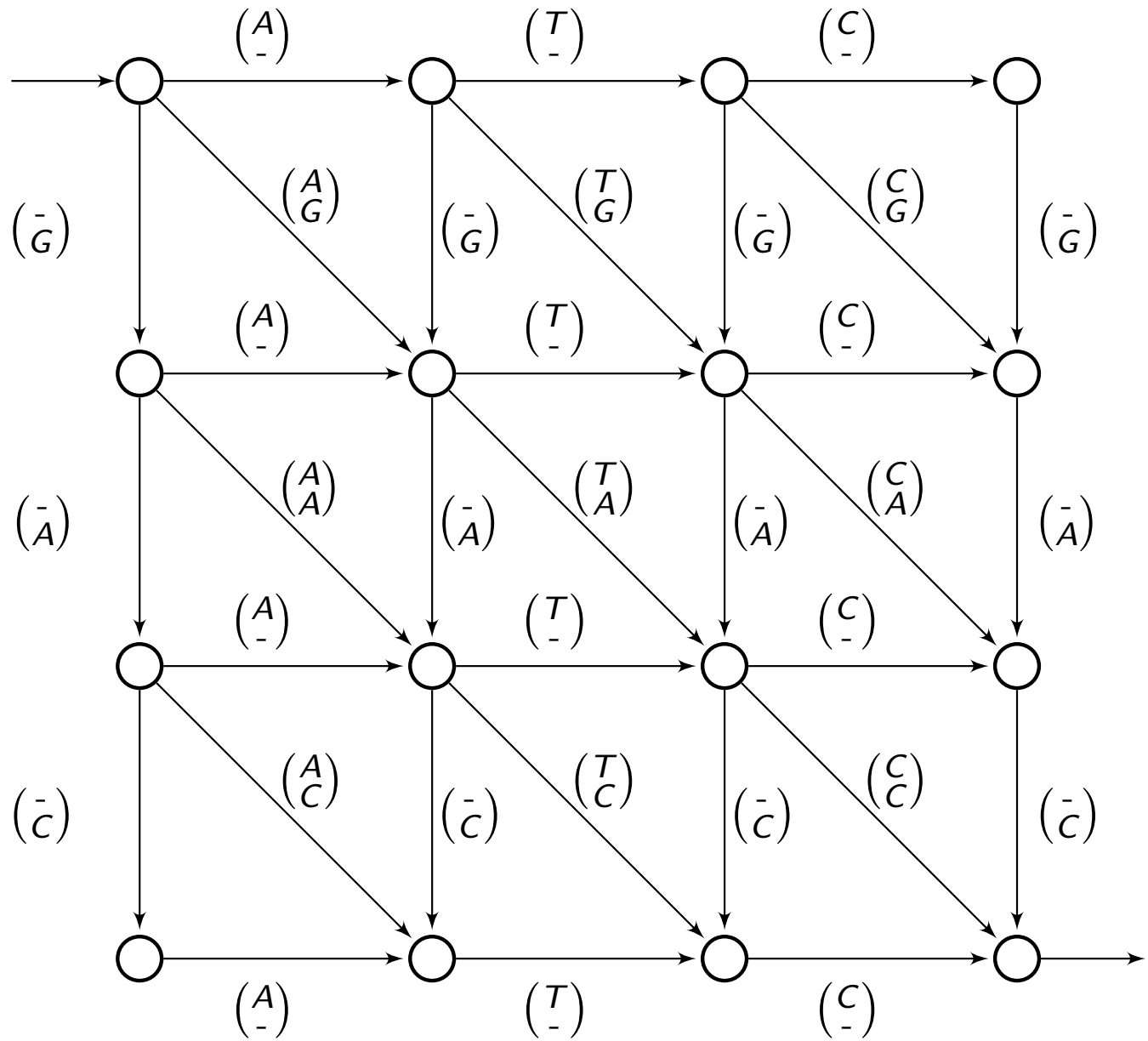


Example 1: Automaton for $u = ATC$ and $v = GAC$

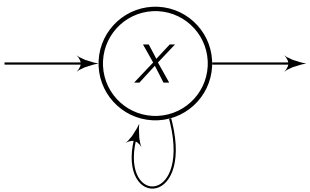


Example 1: Automaton for $u = ATC$ and $v = GAC$

$\mathcal{A} \cap \mathcal{A}_{u,v}$:



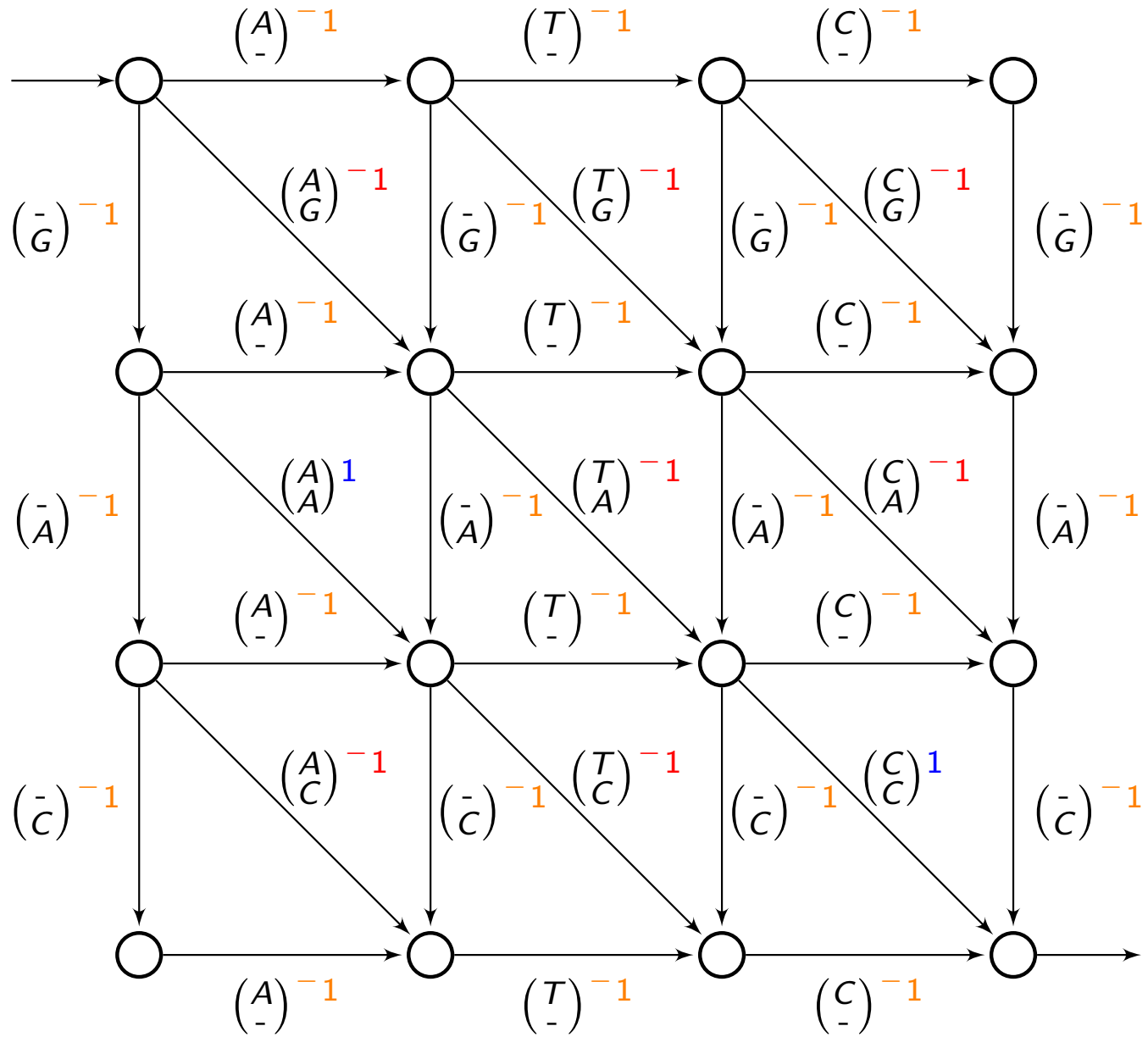
$\mathcal{A}$:



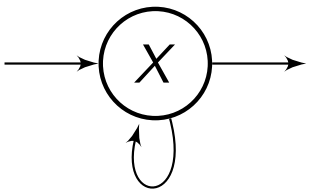
$\binom{u}{u}$ / 1
$\binom{u}{v}$ / -1
$\binom{u}{-}$ / -1
$\binom{-}{v}$ / -1

Example 1: Automaton for $u = ATC$ and $v = GAC$

$\mathcal{A} \cap \mathcal{A}_{u,v}$:



$\mathcal{A}$:



$\binom{u}{u} / \ 1$
$\binom{u}{v} / -1$
$\binom{u}{-} / -1$
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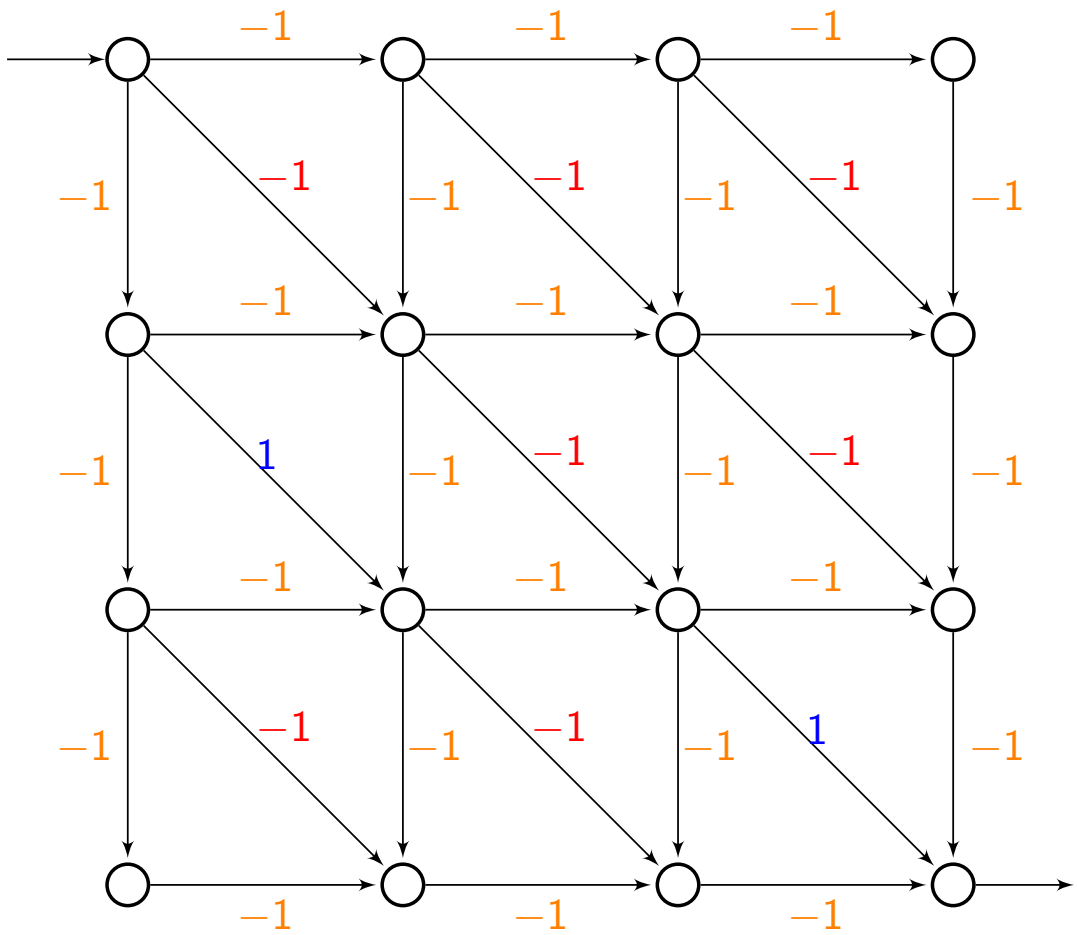
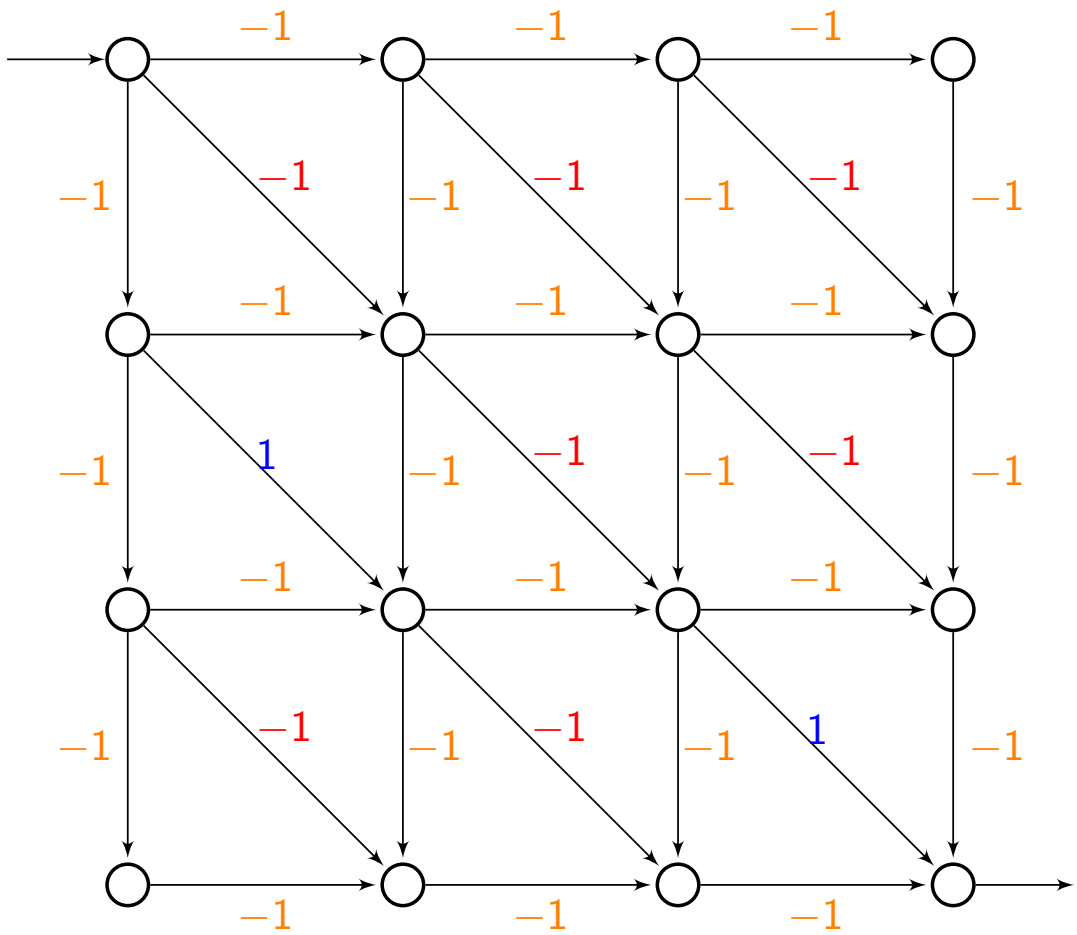
Example 1: Alignment for $u = ATC$ and $v = GAC$



Example 1: Alignment for $u = ATC$ and $v = GAC$

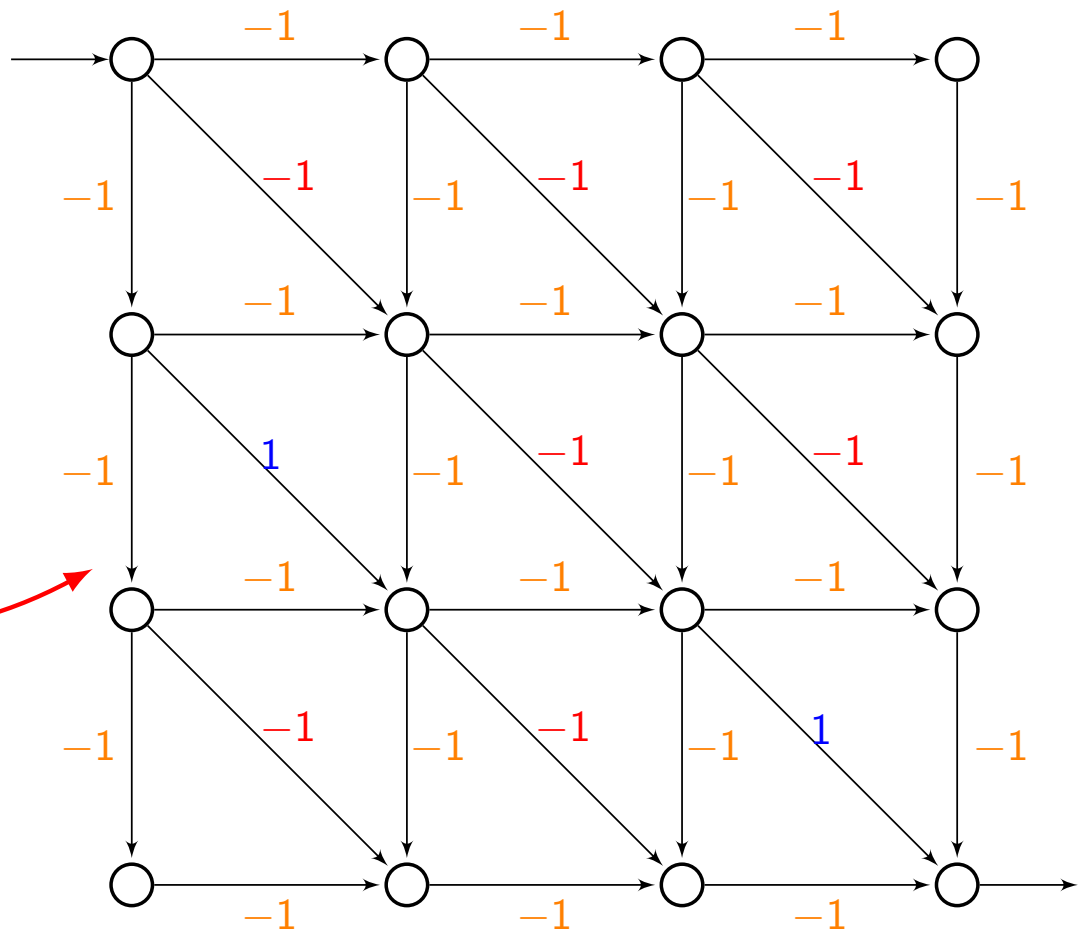
Apply Dijsktra!



Example 1: Alignment for $u = ATC$ and $v = GAC$

Apply Dijsktra!

acyclic

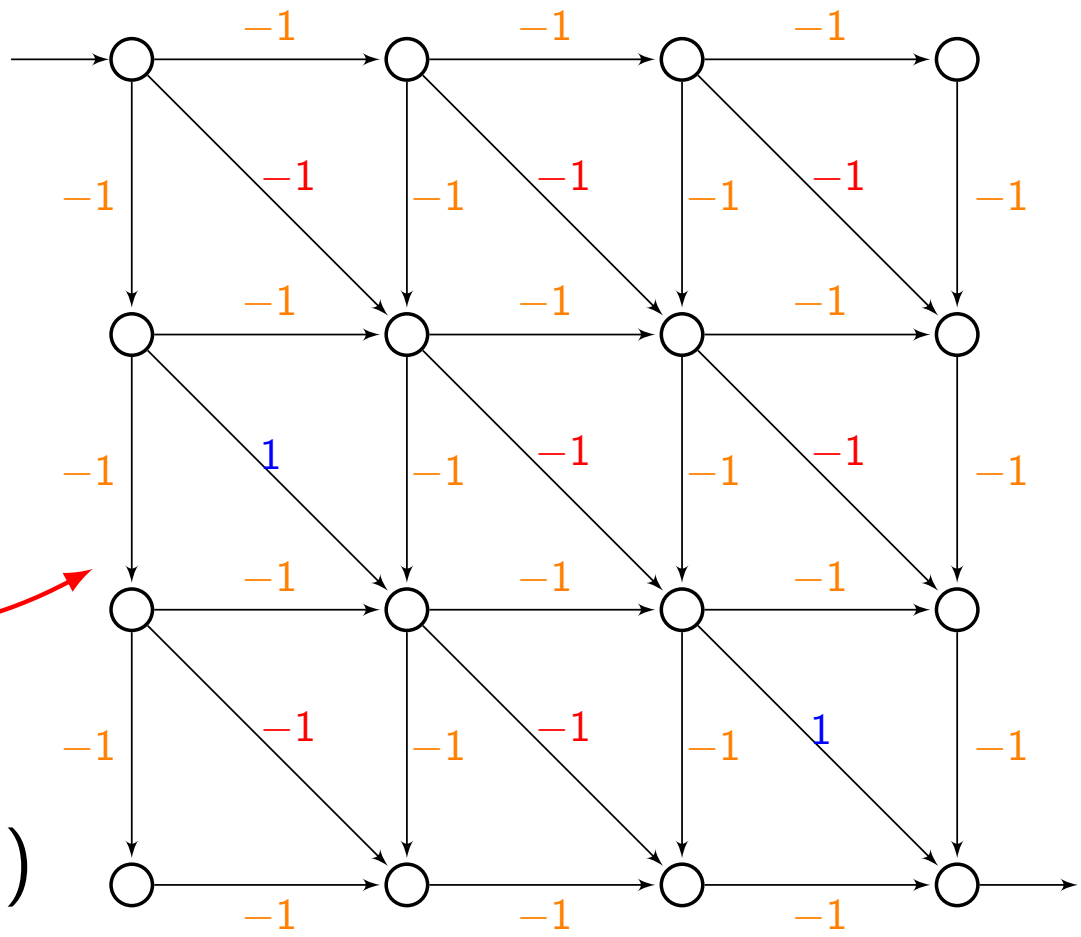


Example 1: Alignment for $u = ATC$ and $v = GAC$

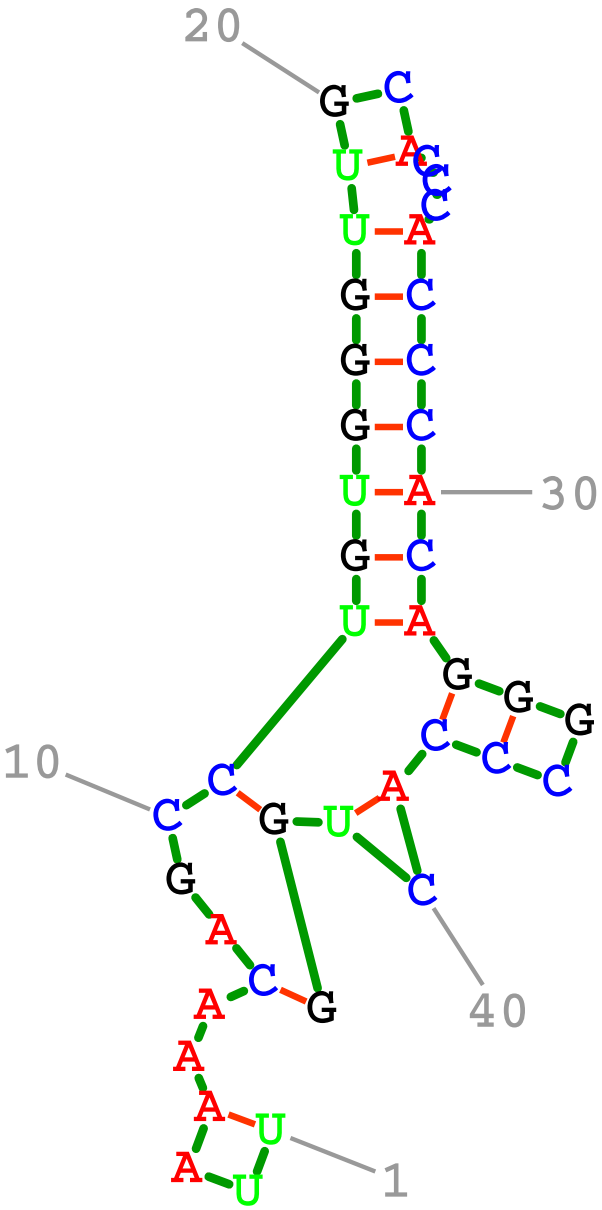
Apply Dijsktra!

acyclic

Complexity $O(|u||v|)$



Context-Free: Example Nussinov



Works as well!

different algorithm

Conclusion

DP works with automata!

Conclusion

DP works with automata!

Thank you for your attention!

Backup

Example 2: Folding of AATT

Folding 1

.	(	.	)
A	A	T	T

Score: 1 pair

bracket $= 1$
dot $= 0$

Example 2: Folding of AATT

Folding 1

. (.)
A A T T

Score: 1 pair

Folding 2

(( ))
A A T T

Score: 2 pairs

bracket $= 1$
dot $= 0$

Example 2: Folding Automaton

Grammar:

$$X \to \binom{\cdot}{N} X \mid \binom{(}{N} X \binom{)}{N^c} X \mid \epsilon$$

dot bracket end

$N \in \{A, T, C, G\}$

N^c complement of N

Example 2: Folding Automaton

Grammar:

Weight: 0 1 0

$$X \to \left(\dot{}_{N}\right)X \mid \left(^{(}_{N}\right)X\left(^{)}_{N^c}\right)X \mid \epsilon$$

dot bracket end

$N \in \{A, T, C, G\}$

N^c complement of N

Example 2: Folding Automaton

Grammar:

Weight: 0 1 0

$$X \to \binom{\cdot}{N} X \mid \binom{(}{N} X \binom{)}{N^c} X \mid \epsilon$$

dot bracket end

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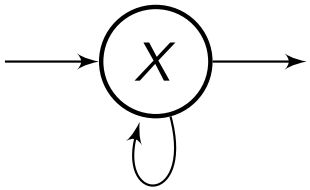
N^c complement of N

Automaton $\mathcal{A}$:



$\binom{\cdot}{N}$

$\binom{(}{N} \downarrow N^c$

$\binom{)}{N} \uparrow N$

Example 2: Folding Automaton

Grammar:

Weight: 0 1 0

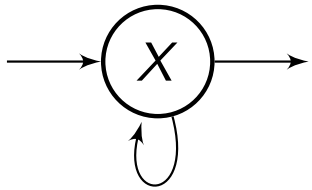
$$X \to \left(\dot{}_{N}\right) X \mid \left(\begin{smallmatrix}(\\ N\end{smallmatrix}\right) X \left(\begin{smallmatrix})\\ N^c\end{smallmatrix}\right) X \mid \epsilon$$

dot bracket end

$N \in \{A, T, C, G\}$

N^c complement of N

Automaton $\mathcal{A}$:



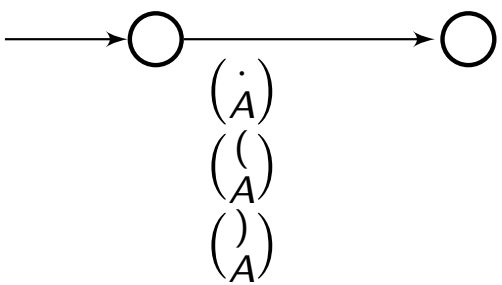
$\left(\dot{}_{N}\right)/0$

$\left(\begin{smallmatrix}(\\ N\end{smallmatrix}\right) \downarrow N^c/0$

$\left(\begin{smallmatrix})\\ N\end{smallmatrix}\right) \uparrow N/1$

Example 2: Automaton for $u = AATT$

$\mathcal{A}_u$:



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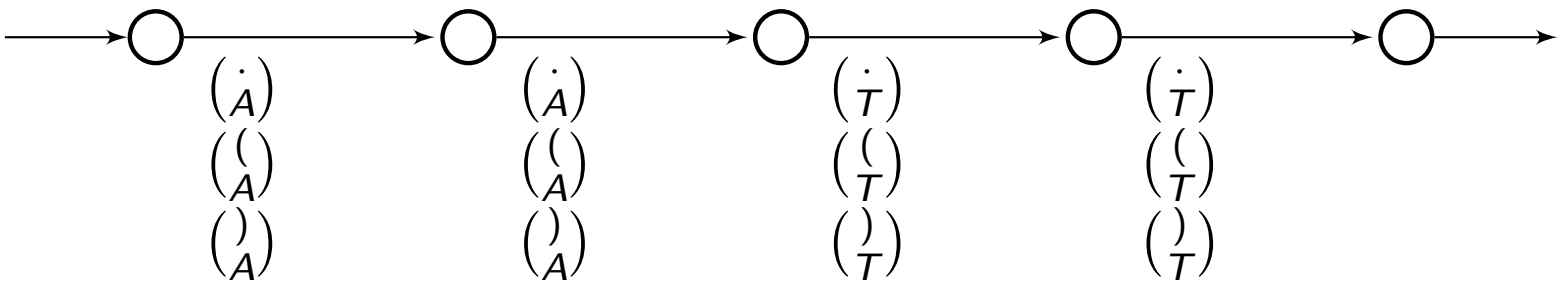
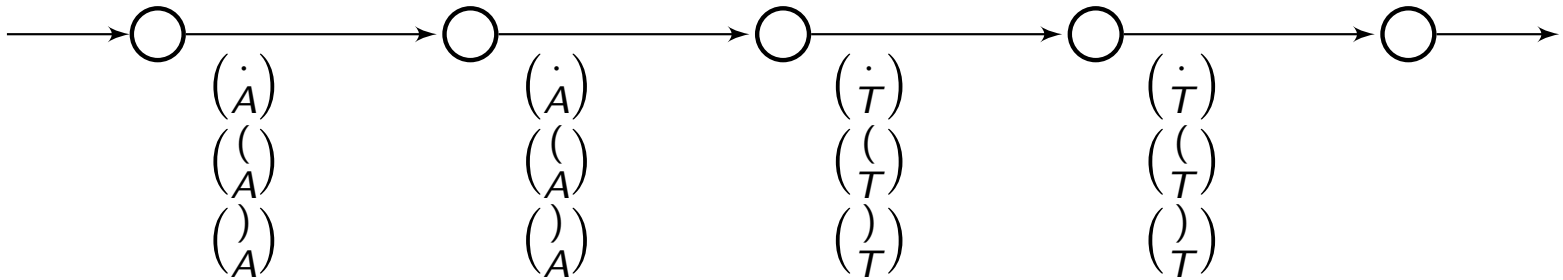


Example 2: Automaton for $u = AATT$

$\mathcal{A} \cap \mathcal{A}_u$:



Example 2: Automaton for $u = AATT$

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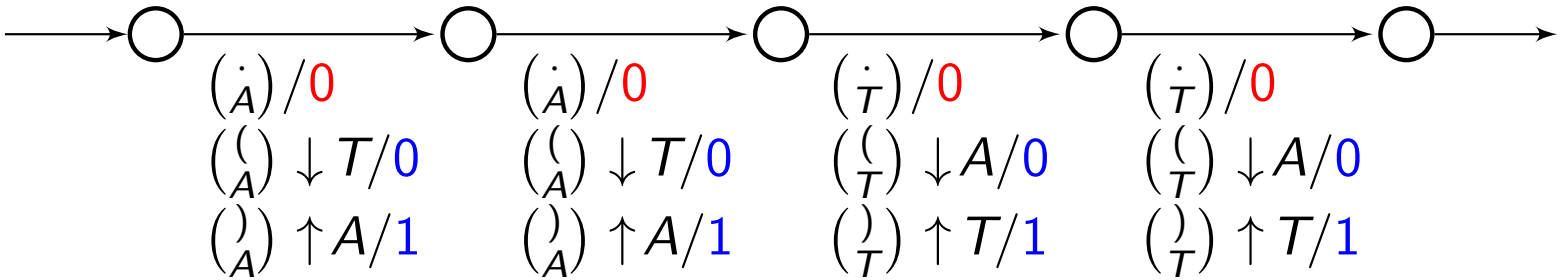
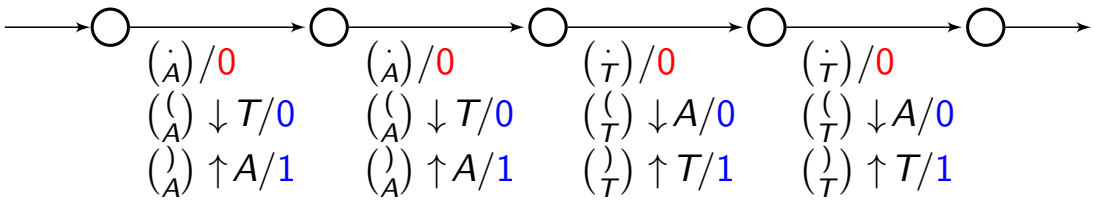


Example 2: Folding of $u = AATT$ (Overview)

$\mathcal{A} \cap \mathcal{A}_u$:

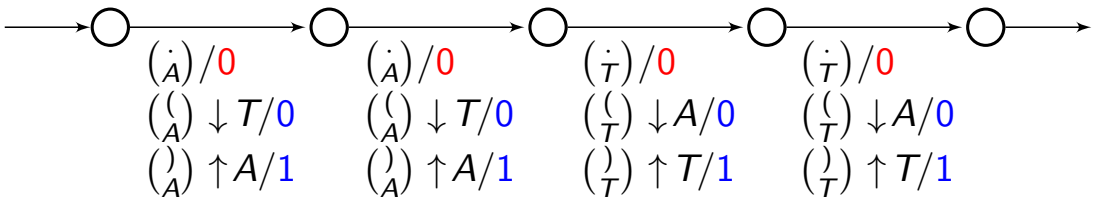


Find highest weighted (annotated) input word:

1. Convert to CF grammar ($O(n^2)$ non-terminals)

Example 2: Folding of $u = AATT$ (Overview)

$\mathcal{A} \cap \mathcal{A}_u$:

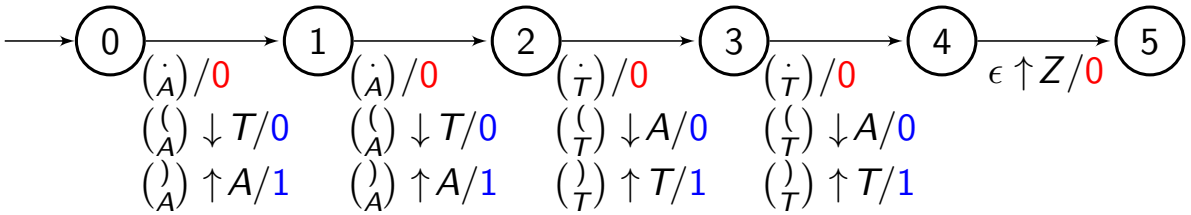


Find highest weighted (annotated) input word:

1. Convert to CF grammar ($O(n^2)$ non-terminals)
2. Mark for every non-terminal the maximal score (O(n) per non-terminal)

Complexity: $O(n^3)$

Example 2: Folding of $u = AATT$ (Details)



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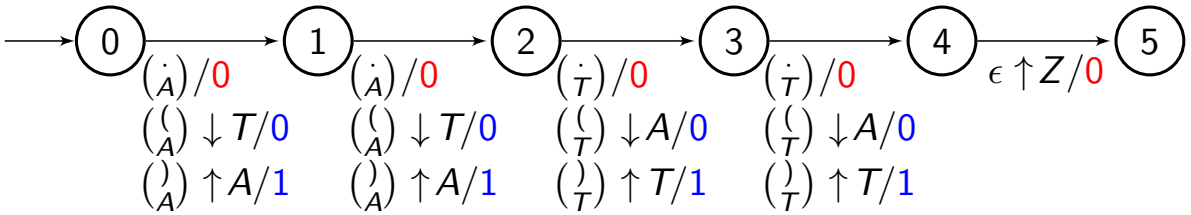


Grammar:

$$S \to Z_{05}$$

$$Z_{05} \to {}^{0}\binom{\cdot}{A} Z_{15} \mid {}^{0}\binom{(}{A} T_{12} Z_{25} \mid {}^{0}\binom{(}{A} T_{13} Z_{35} \mid {}^{0}\binom{(}{A} T_{14} Z_{45}$$

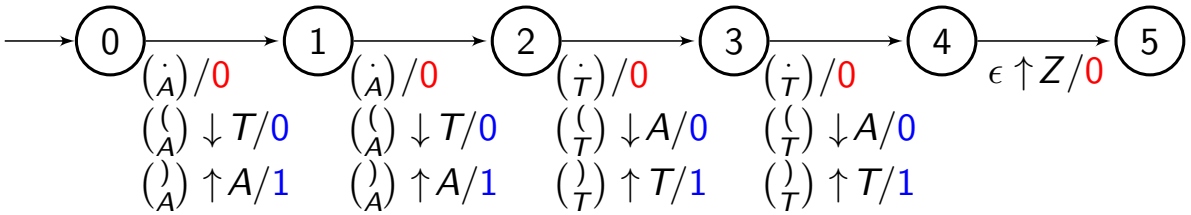
$$Z_{15} \to {}^{0}\binom{\cdot}{A} Z_{25} \mid {}^{0}\binom{(}{A} T_{23} Z_{35} \mid {}^{0}\binom{(}{A} T_{24} Z_{45}$$

$$Z_{25} \to {}^{0}\binom{\cdot}{T} Z_{35} \mid {}^{0}\binom{(}{T} A_{34} Z_{45}$$

$$Z_{35} \to {}^{0}\binom{\cdot}{T} Z_{45}$$

$$Z_{45} \to {}^{0}\epsilon$$

Example 2: Folding of $u = AATT$ (Details)



Grammar:

$$S \to Z_{05}$$

$$Z_{05} \to {}^{0}(\dot{A})\, Z_{15} \mid {}^{0}(A)\, T_{12} Z_{25} \mid {}^{0}(A)\, T_{13} Z_{35} \mid {}^{0}(A)\, T_{14} Z_{45}$$

$$Z_{15} \to {}^{0}(\dot{A})\, Z_{25} \mid {}^{0}(A)\, T_{23} Z_{35} \mid {}^{0}(A)\, T_{24} Z_{45}$$

$$Z_{25} \to {}^{0}(\dot{T})\, Z_{35} \mid {}^{0}(T)\, A_{34} Z_{45}$$

$$Z_{35} \to {}^{0}(\dot{T})\, Z_{45}$$

$$Z_{45} \to {}^{0}\epsilon$$

$$T_{12} \to \text{no path available}$$

$$T_{13} \to {}^{0}(\dot{A})\, T_{23}$$

$$T_{14} \to {}^{0}(\dot{A})\, T_{24} \mid {}^{0}(A)\, T_{23} T_{34}$$

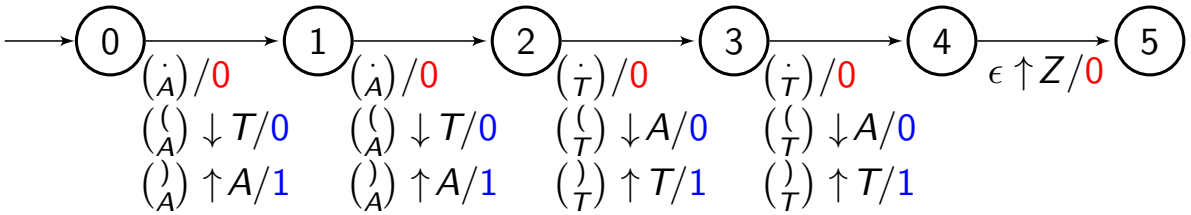
$$T_{23} \to {}^{1})_{T}$$

$$T_{24} \to {}^{0}(\dot{T})\, T_{34}$$

$$T_{34} \to {}^{1})_{T}$$

$$A_{34} \to \text{no path available}$$

Example 2: Folding of $u = AATT$ (Details)



Grammar:

$$S \to Z_{05}$$

$$Z_{05} \to {}^{0}(\dot{A})\, Z_{15} \mid {}^{0}(A)\, \cancel{T_{12}\, Z_{25}} \mid {}^{0}(A)\, T_{13}\, Z_{35} \mid {}^{0}(A)\, T_{14}\, Z_{45}$$

$$Z_{15} \to {}^{0}(\dot{A})\, Z_{25} \mid {}^{0}(A)\, T_{23}\, Z_{35} \mid {}^{0}(A)\, T_{24}\, Z_{45}$$

$$Z_{25} \to {}^{0}(\dot{T})\, Z_{35} \mid {}^{0}(\cancel{T})\, \cancel{A_{34}\, Z_{45}}$$

$$Z_{35} \to {}^{0}(\dot{T})\, Z_{45}$$

$$Z_{45} \to {}^{0}\epsilon$$

$$\cancel{T_{12}} \to \text{no path available}$$

$$T_{13} \to {}^{0}(\dot{A})\, T_{23}$$

$$T_{14} \to {}^{0}(\dot{A})\, T_{24} \mid {}^{0}(A)\, T_{23}\, T_{34}$$

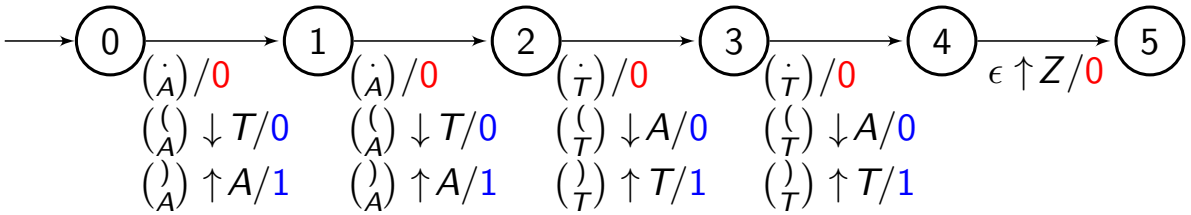
$$T_{23} \to {}^{1}(T)$$

$$T_{24} \to {}^{0}(\dot{T})\, T_{34}$$

$$T_{34} \to {}^{1}(T)$$

$$\cancel{A_{34}} \to \text{no path available}$$

Example 2: Folding of $u = AATT$ (Details)



Grammar:

$^{2}S \to Z_{05}$

$^{2}Z_{05} \to {}^{0}\binom{\cdot}{A} Z_{15} \mid {}^{0}\binom{(}{A} \cancel{T_{12} Z_{25}} \mid {}^{0}\binom{(}{A} T_{13} Z_{35} \mid {}^{0}\binom{(}{A} T_{14} Z_{45}$

$^{1}Z_{15} \to {}^{0}\binom{\cdot}{A} Z_{25} \mid {}^{0}\binom{(}{A} T_{23} Z_{35} \mid {}^{0}\binom{(}{A} T_{24} Z_{45}$

$^{0}Z_{25} \to {}^{0}\binom{\cdot}{T} Z_{35} \mid {}^{0}\binom{(}{T} \cancel{A_{34} Z_{45}}$

$^{0}Z_{35} \to {}^{0}\binom{\cdot}{T} Z_{45}$

$^{0}Z_{45} \to {}^{0}\epsilon$

$\cancel{T_{12}} \to$ no path available

$^{1}T_{13} \to {}^{0}\binom{\cdot}{A} T_{23}$

$^{2}T_{14} \to {}^{0}\binom{\cdot}{A} T_{24} \mid {}^{0}\binom{(}{A} T_{23} T_{34}$

$^{1}T_{23} \to {}^{1}\binom{)}{T}$

$^{1}T_{24} \to {}^{0}\binom{\cdot}{T} T_{34}$

$^{1}T_{34} \to {}^{1}\binom{)}{T}$

$\cancel{A_{34}} \to$ no path available