Approximate Graph Products

A new concept for graph approximation by Brakensiek and Davies

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TBI Winterseminar
Bled, Slovenia

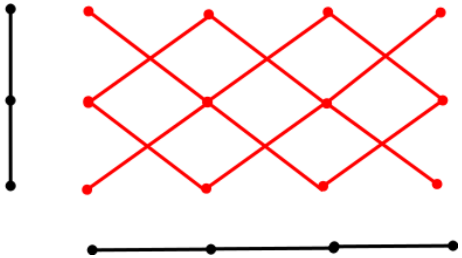
February 13, 2023

Motivation

Motivation for the investigation of approximate graph products
– graph drawing
– computational engineering and
– theoretical biology

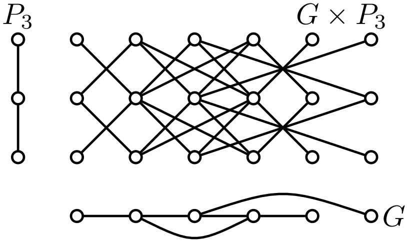
In theoretical biology graph products arise in two contexts
1. Sequence spaces. They are a convenient framework to discuss the evolution of genetic sequences. They give rise to Hamming graphs, which are Cartesian products of complete graphs (Eigen, Dress, Rumschitzki).
2. Topological theory of the relationships between genotypes and phenotypes. If recombination and sexual inheritance are disregarded, this framework reduces to strong products of graphs.

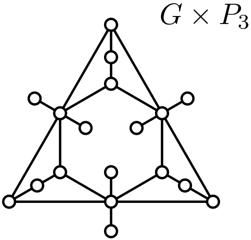
The direct product



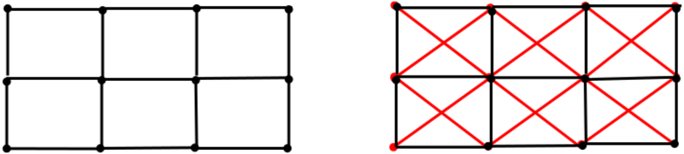
The direct product of a P_3 by a P_2

Another direct product





The Cartesian and the strong product



The Cartesian and the strong product of a P_3 by a P_2

Distance between graphs

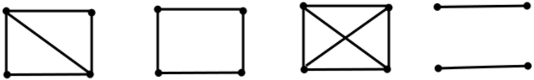
The distance $d(G, H)$ between two graphs G and H is at most k if they have representations G',H' on the same vertex sets such that

$$|E(G') \triangle E(H')| \leq k.$$

A graph G is a k-approximate graph product if there is a product H such that

$$d(G, H) \leq k.$$

Which product does the graph on the left approximate? With which distance?



Numerous papers have been written on the subject, notably by Peter Stadler, who introduced the concept with Bärbel, Hellmuth Marc and Tomas Kupka.

All work was mainly on the strong and the Cartesian product

But, last year Brakensiek and Davis, in their arXiv paper from July 2022, entitled

Robust Factorization and Colorings of Tensor Graphs

extended the concept to the direct product.

ϵ-nearness

We considered disturbed products H, where k edges could be either deleted from or added to a product P.

Given a direct product P, Brakensiek and Davis only allow graphs H that have the same vertex set as P, and

$$E(H) = E(P) \setminus E', \text{ where } E' \subseteq E(P)$$

and where no vertex v in H has more than an ϵ-fraction of its incident edges in E'.

Such graphs H are called ϵ-near to P.

Reconstruction goal

Given P and H as before, the goal is to find a direct product $\tilde{H}$ such that

$$|E(H)\Delta E(\tilde{H}| \leq O(\epsilon|E(H)|.$$

Theorem

Assume $\epsilon = \Omega(|V(H)|/|E(H)|)$. Let H be ϵ-near $K_3 \times G$. Then, there is a polynomial time algorithm that constructs $\tilde{H} = K_3 \times \tilde{G}$ with $V(H) = V(\tilde{H})$ achieving the reconstruction goal.

In other words,
if H is ϵ-close to an unknown $K_3 \times G$, then one can find a $\tilde{G}$ such that the edge sets of $\tilde{H} = K_3 \times \tilde{G}$ satisfies

$$|E(H)\Delta E(\tilde{H}| \leq O(\epsilon|E(H)|.$$

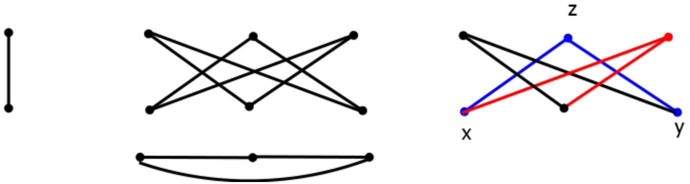
Question

The authors claim that they can also treat the case when K_3 was replaced by K_n or an odd cycle.

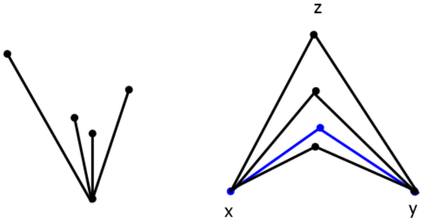
Can one do more?

Triangles give rise to hexagons

$K_3 \times K_2$ is a hexagon, consisting of three paths of length 2.

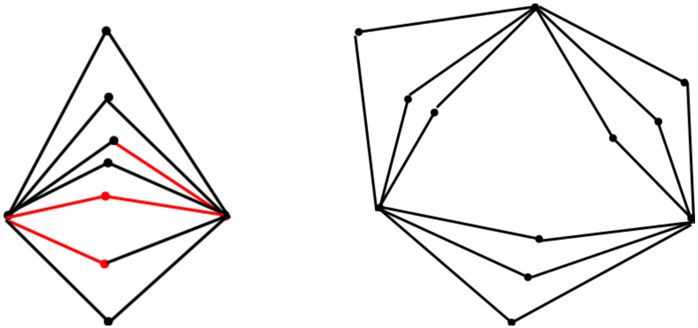


If we consider the blue path and multiply by a star instead of an edge we obtain

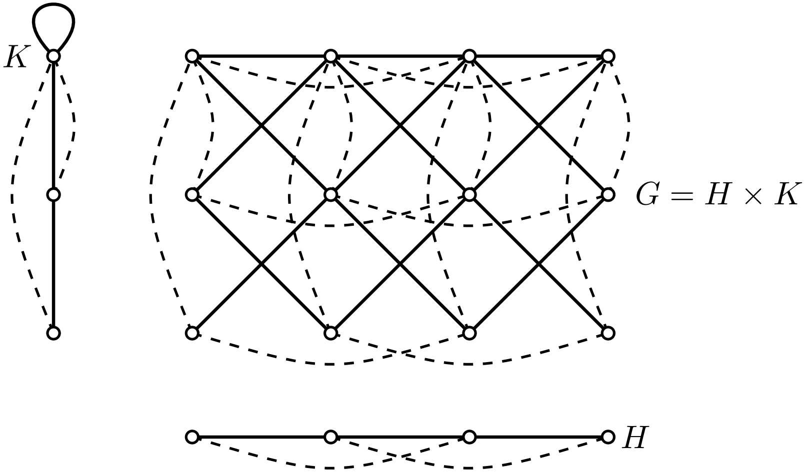


Key observation

The authors then recover the parts that arise from the 2-paths in the triangles of the first factor, and then the hexagons that arise from it.

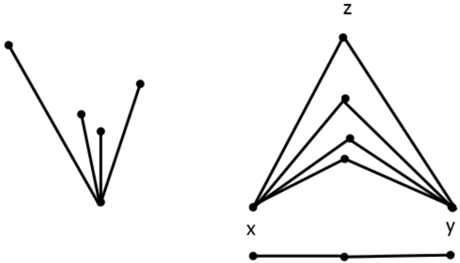


This reminds of the construction of the Cartesian skeleton:



With the methods of the Cartesian skeleton we, see Hammack, Imrich, Klavžar, Handbook of product graphs, CRC, 2011, would have said the x,y are "horizontal" in

$$N_P(x) \cap N_P \subset N_P(x) \cap N_P(z) \text{ or } N_P(x) \subset N_P(z) \subset N_P(y).$$



But, the inequalities may be disturbed, because the neighborhoods in H are not the same as in P, due to the possible ϵ-changes between them.

Conclusion

Although similar, the construction of Brakensiek and Davies is not the same, and has lots of caveats.

But once Brakensiek and Davies have the Cartesian skeleton, they factor it as usual and recover the direct product.

It would be challenging to generalize their ideas.

THANK YOU FOR YOUR ATTENTION