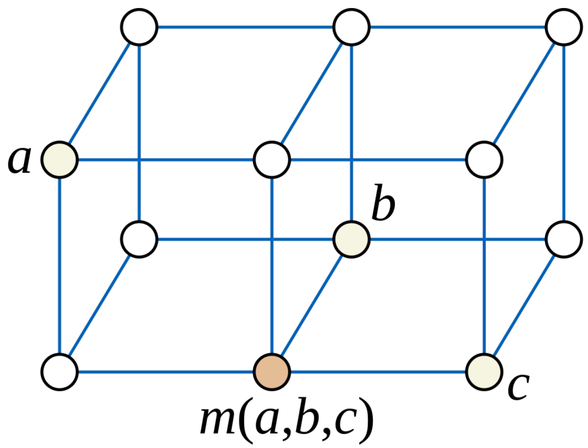
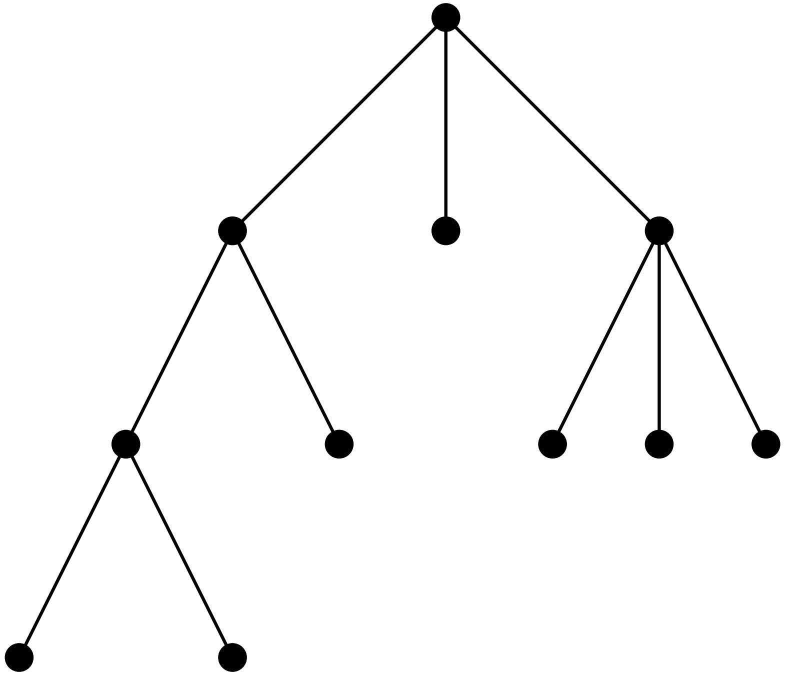


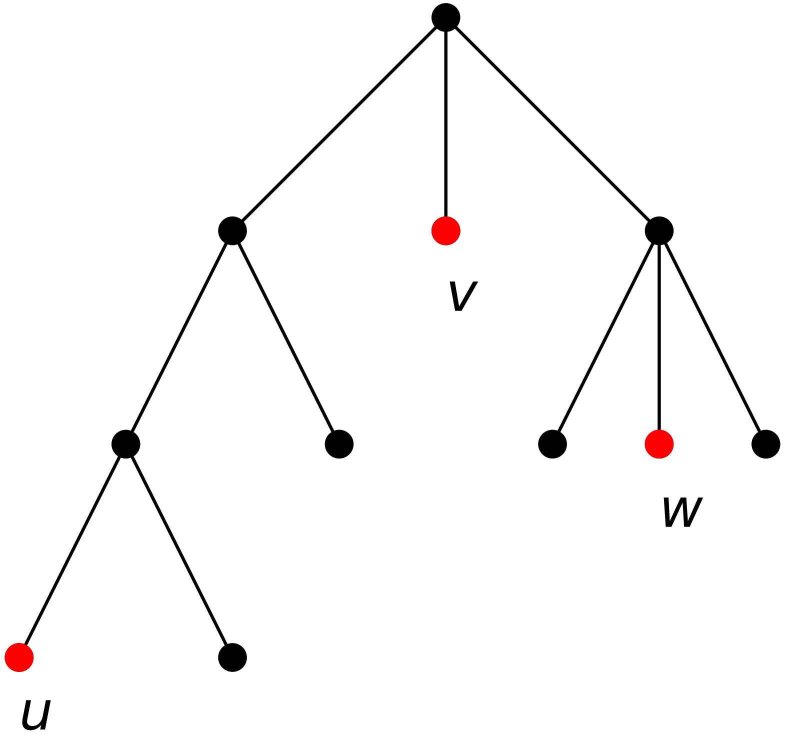
k-Median Graphs

Marc Hellmuth, Sandhya Thekkumpadan Puthiyaveedu

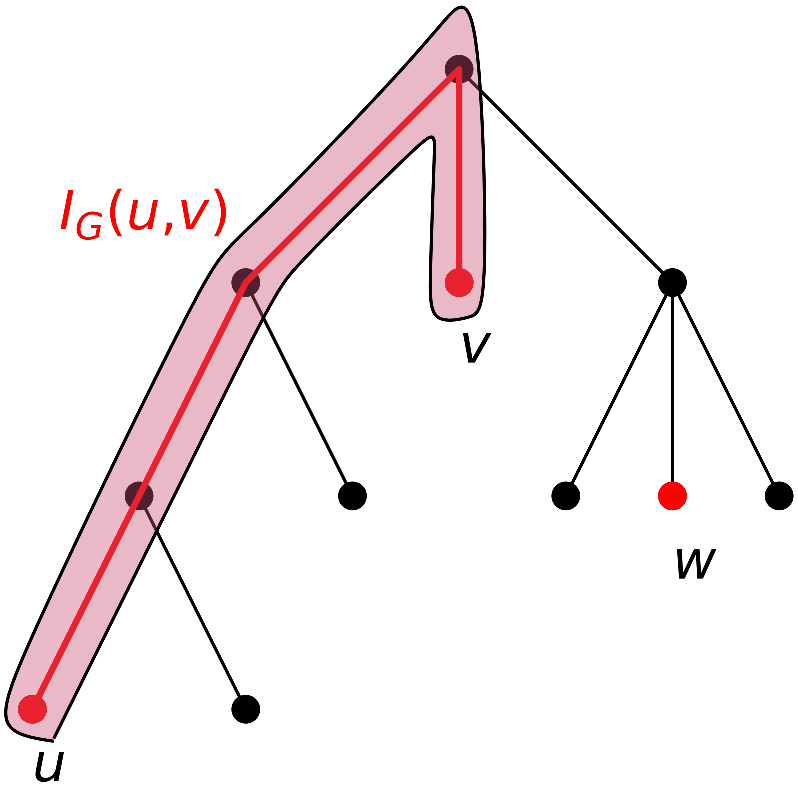
Department of Mathematics
Faculty of Science
Stockholm University

TBI Winterseminar 2023




v
w
u

$I_G(x,y)$: vertices in every shortest $x - y$ path



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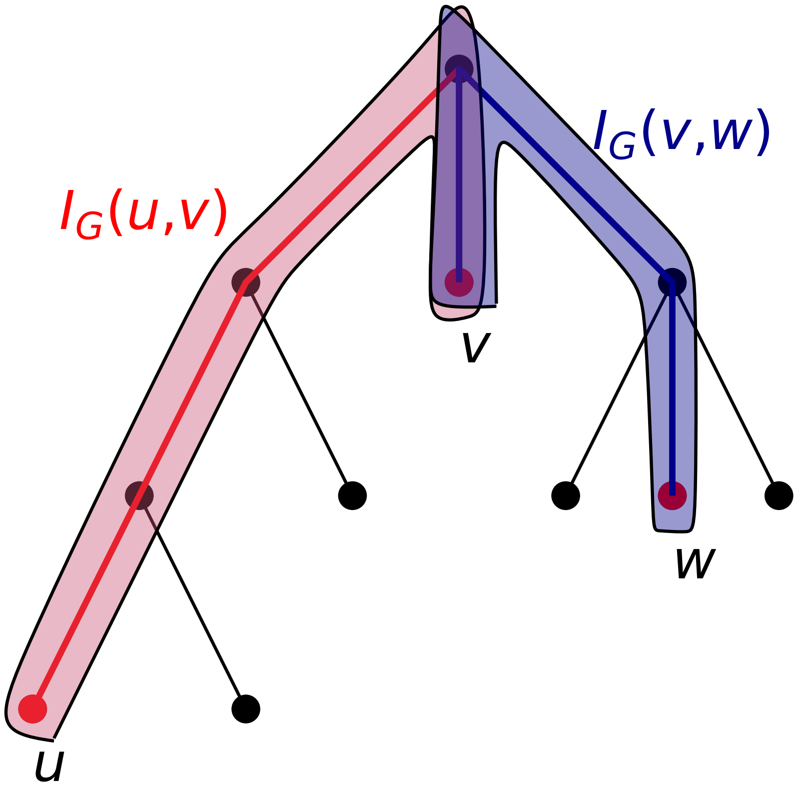

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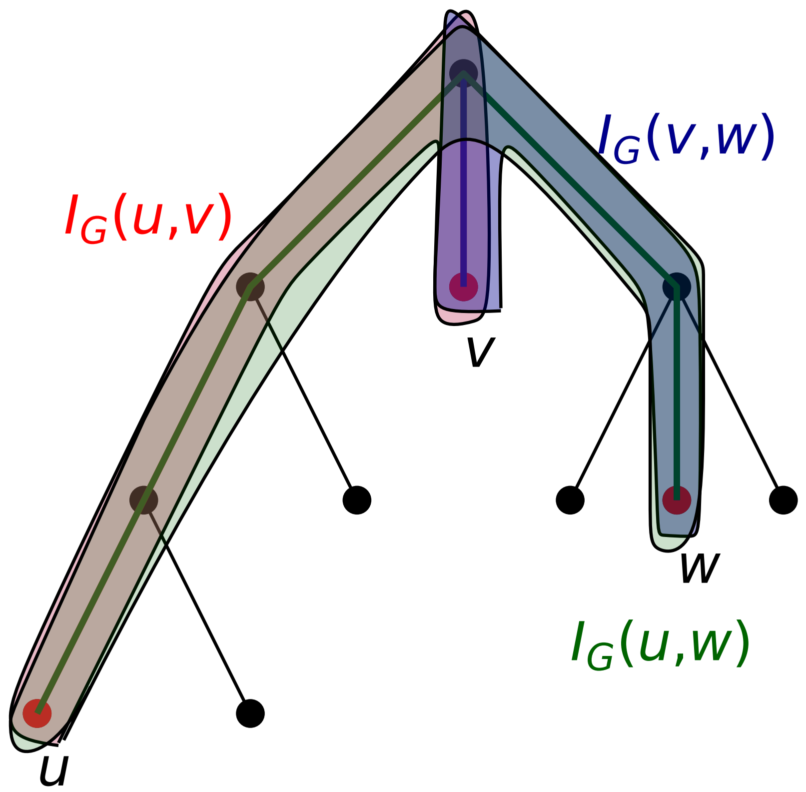

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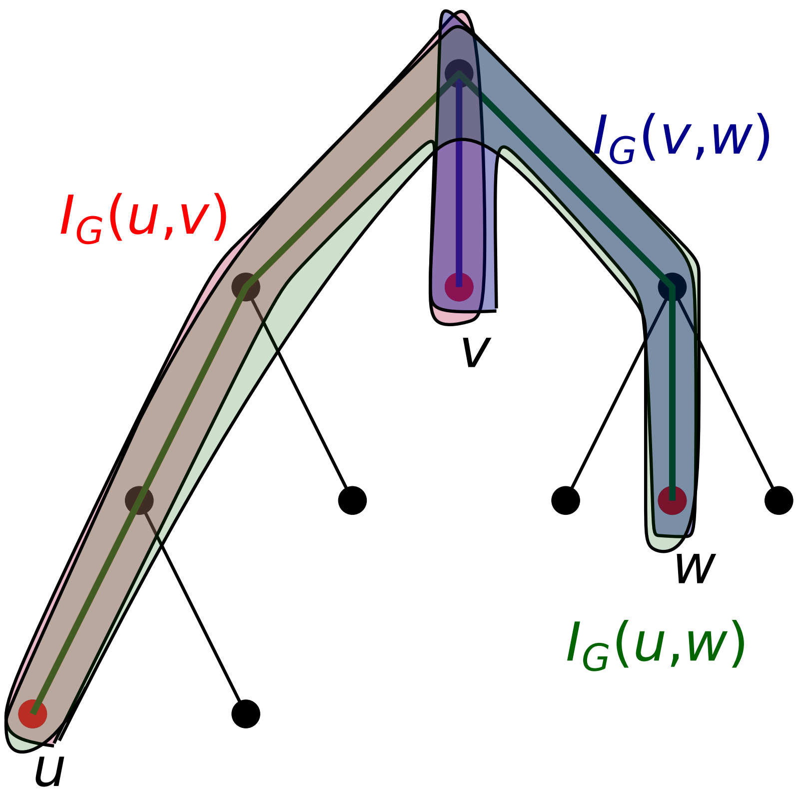

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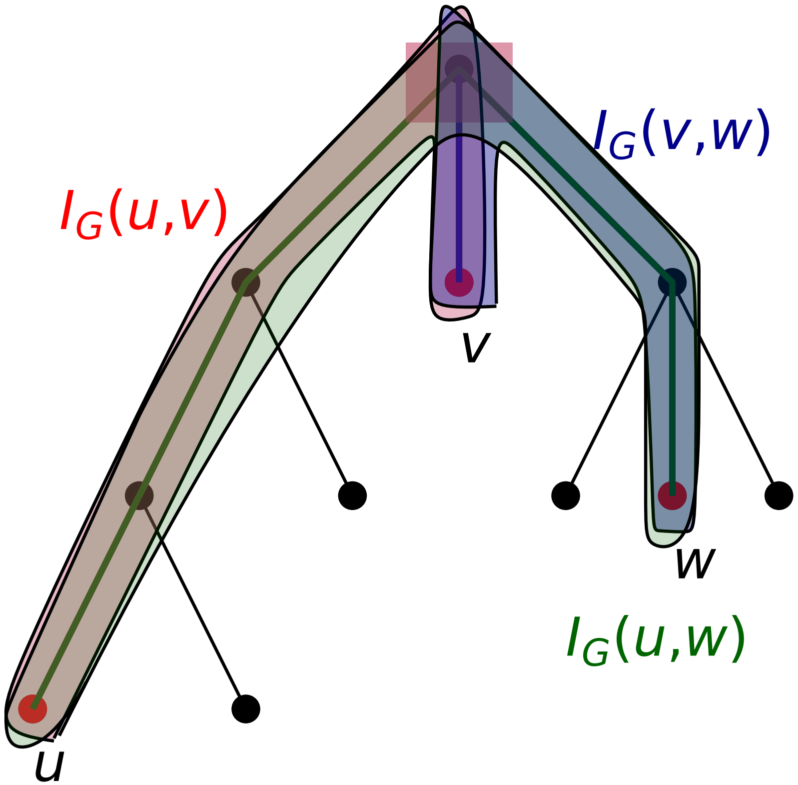

$|I_G(u,v) \cap I_G(v,w) \cap I_G(u,w)| = 1$

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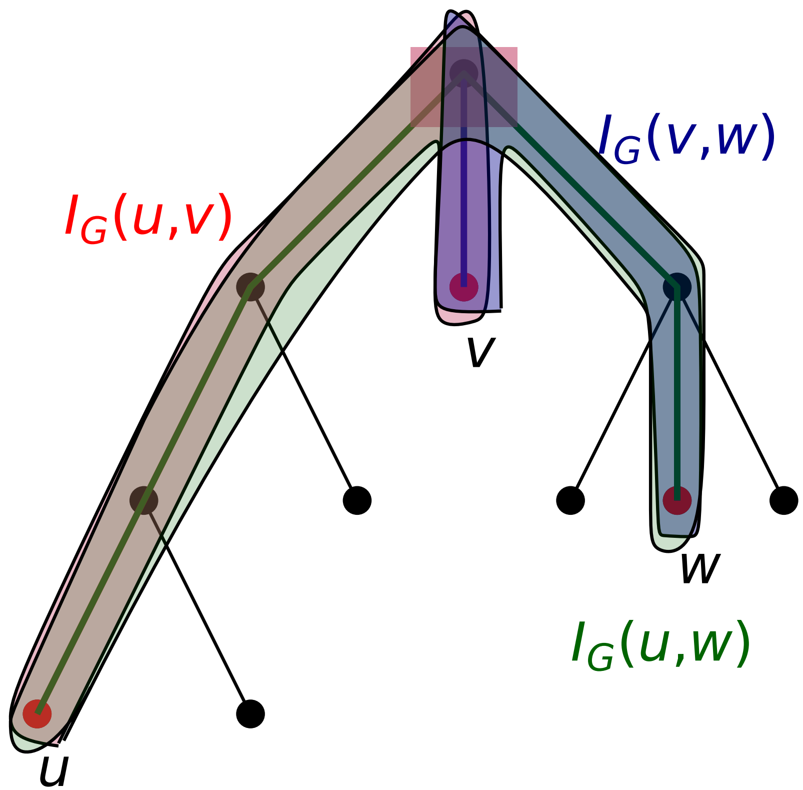

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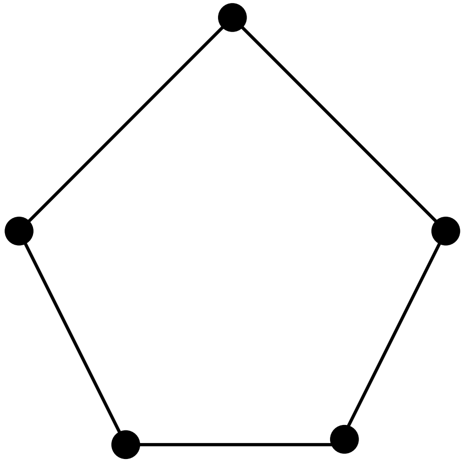

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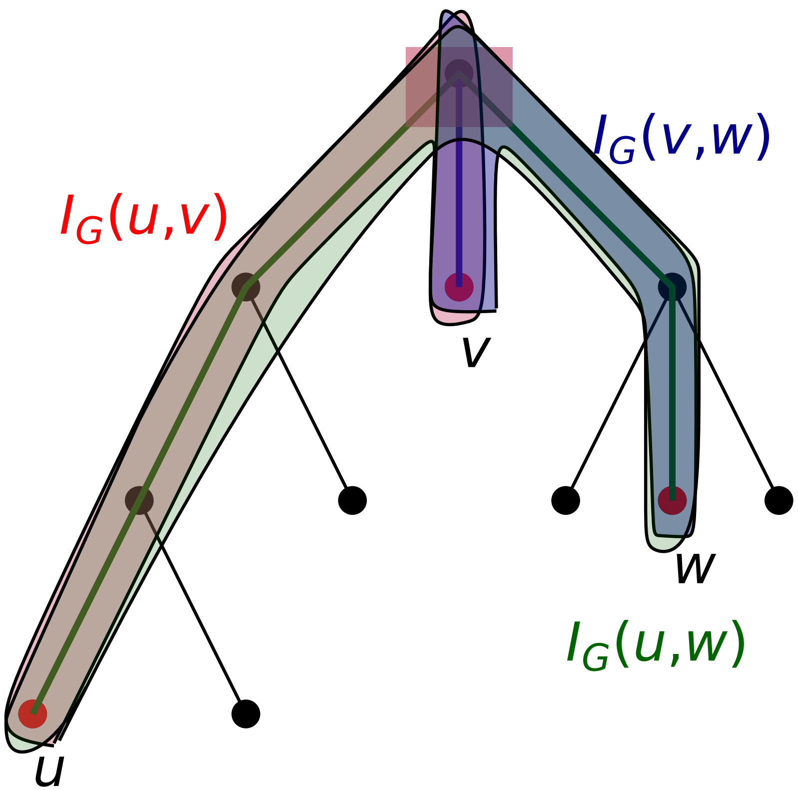

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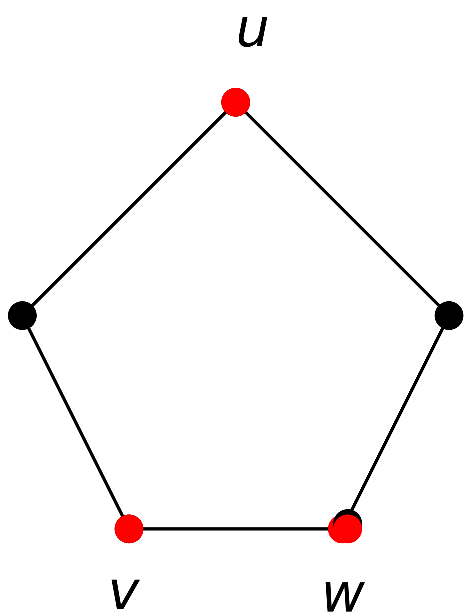

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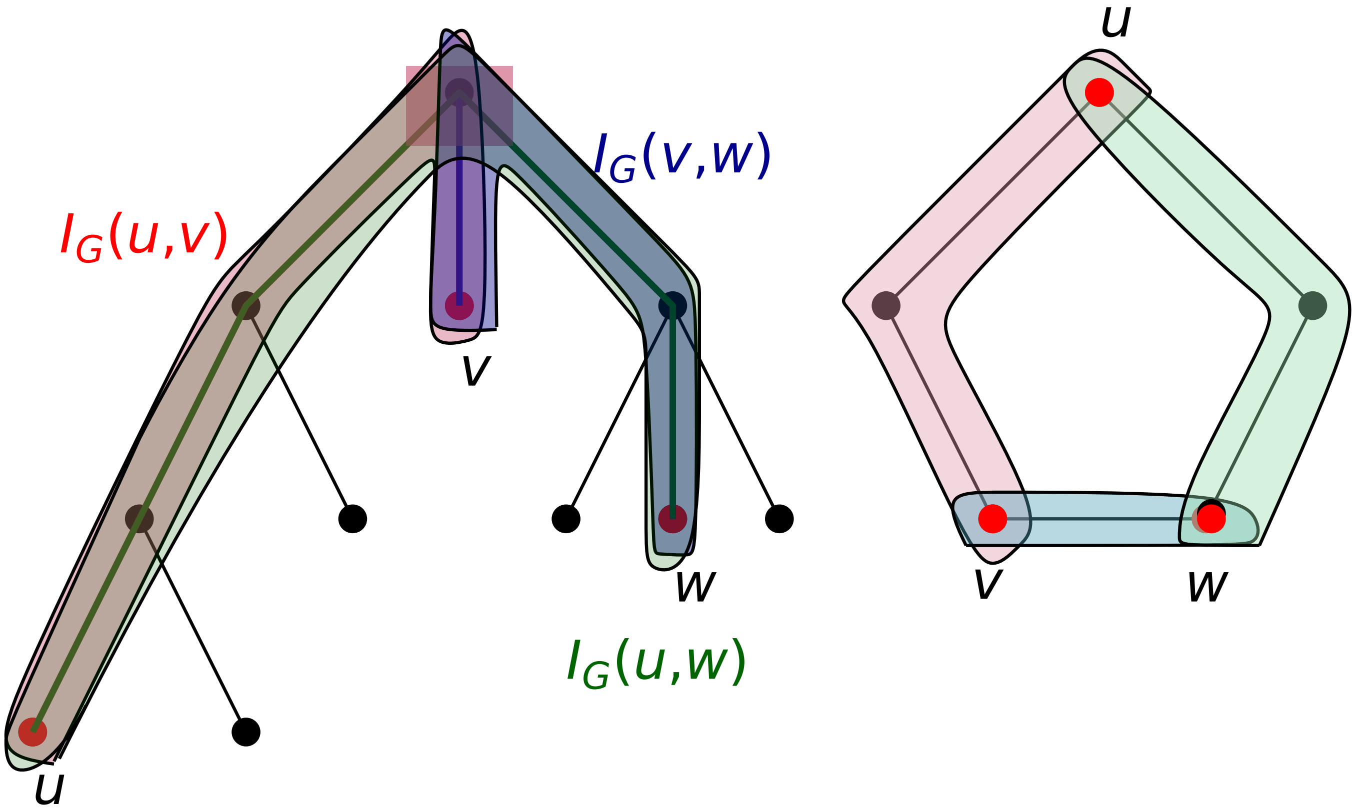

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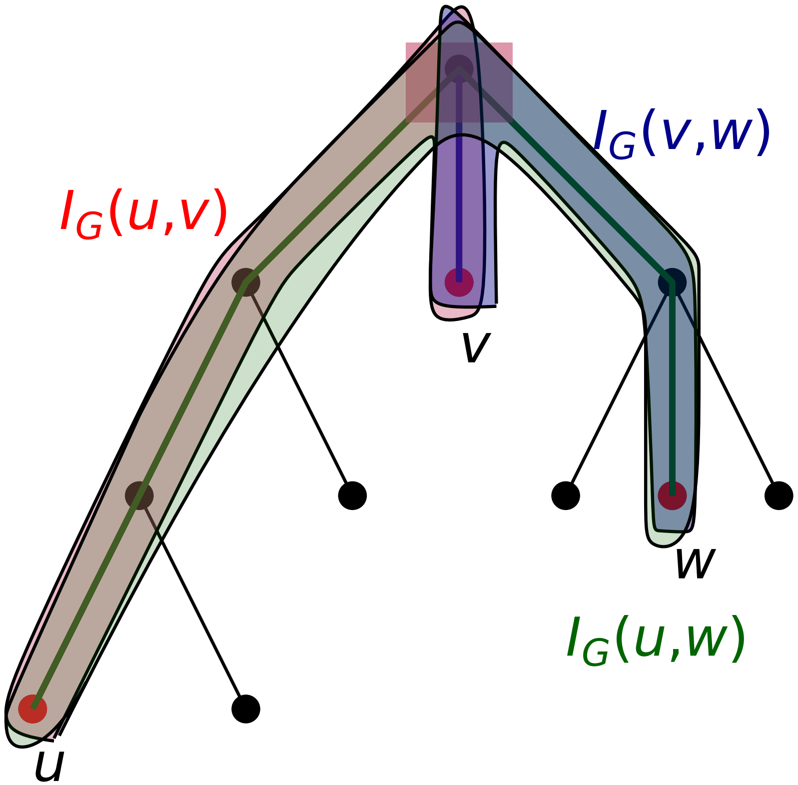

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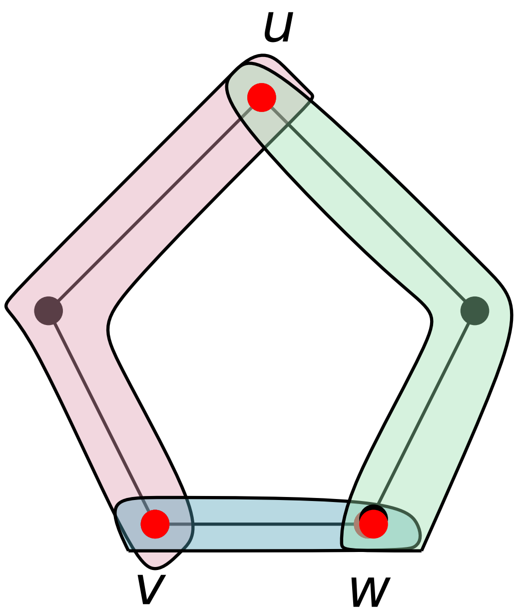
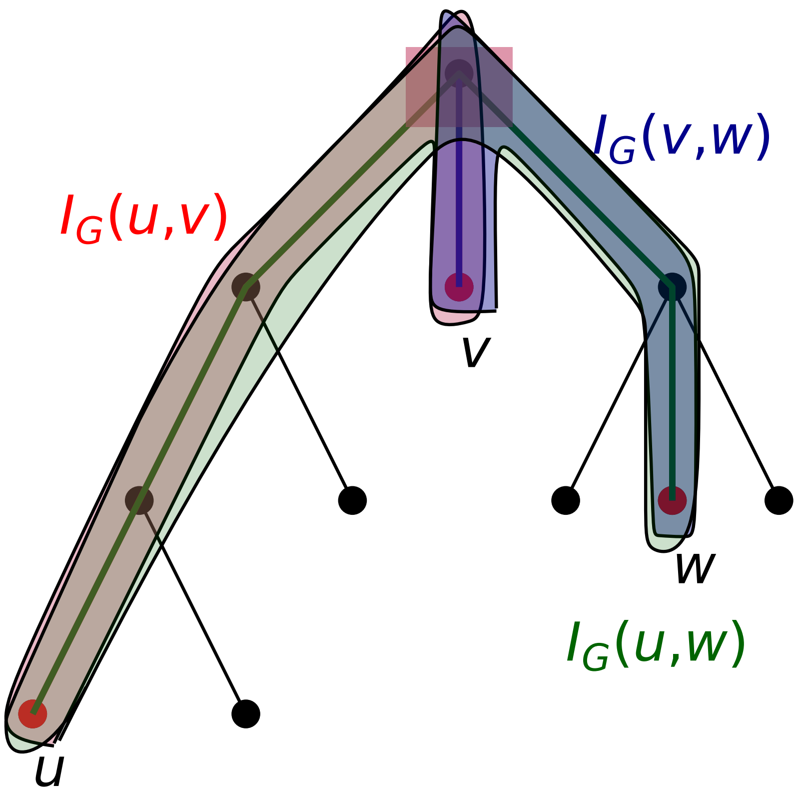


$|I_G(u,v) \cap I_G(v,w) \cap I_G(u,w)| = 0$

$I_G(x,y)$: vertices in every shortest $x - y$ path



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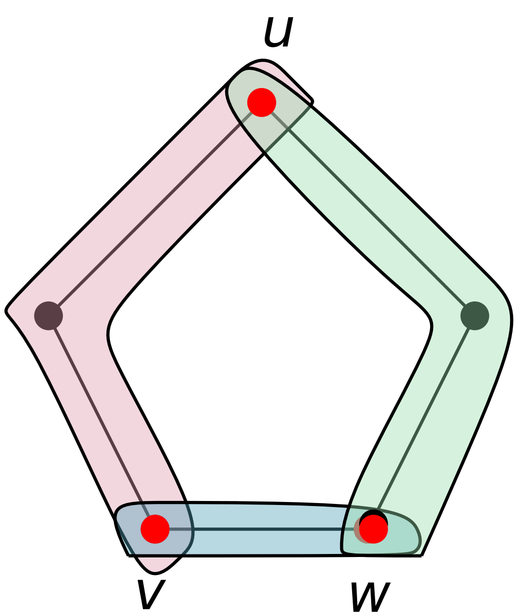
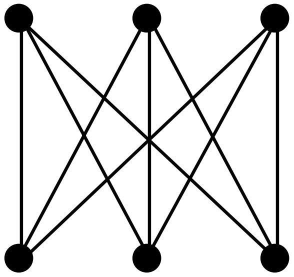


$|I_G(u,v) \cap I_G(v,w) \cap I_G(u,w)| = 0$



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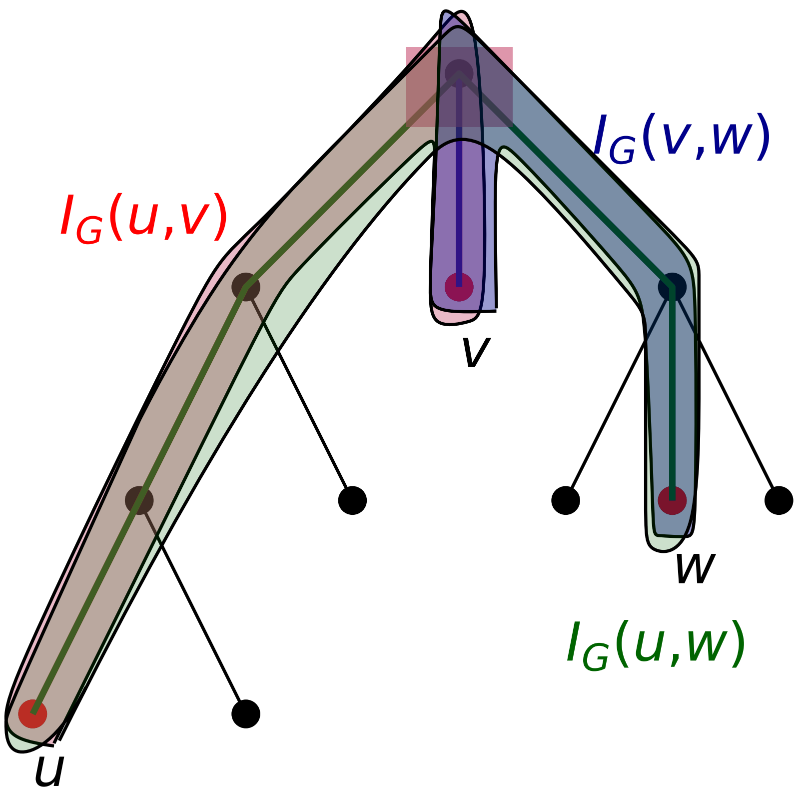

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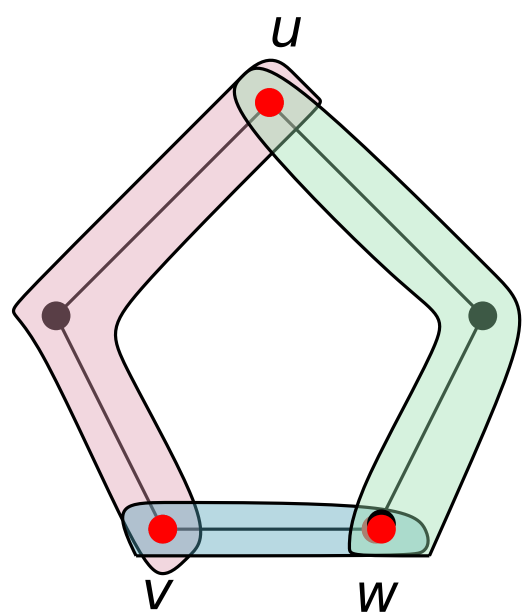
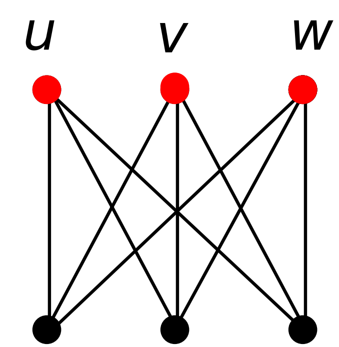


$|I_G(u,v) \cap I_G(v,w) \cap I_G(u,w)| = 0$



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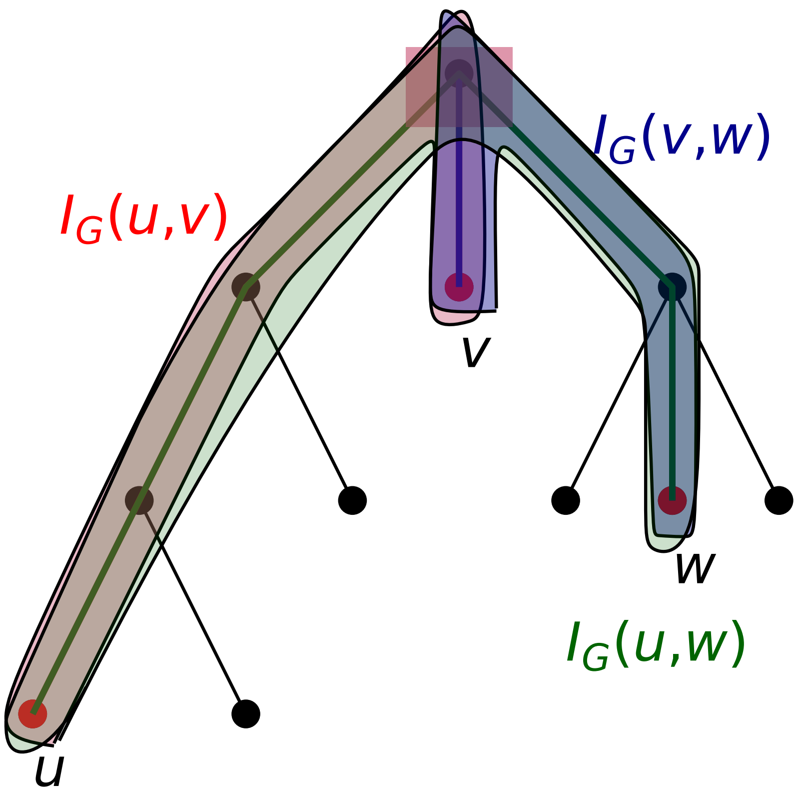

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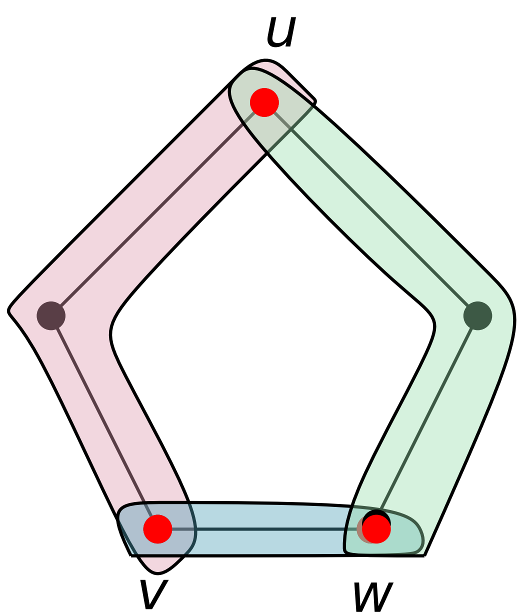
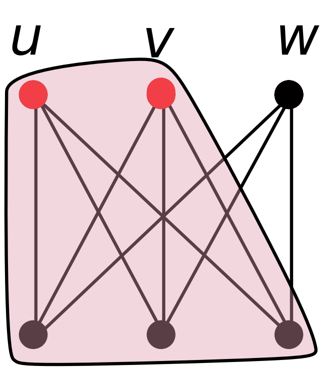


$|I_G(u,v) \cap I_G(v,w) \cap I_G(u,w)| = 0$



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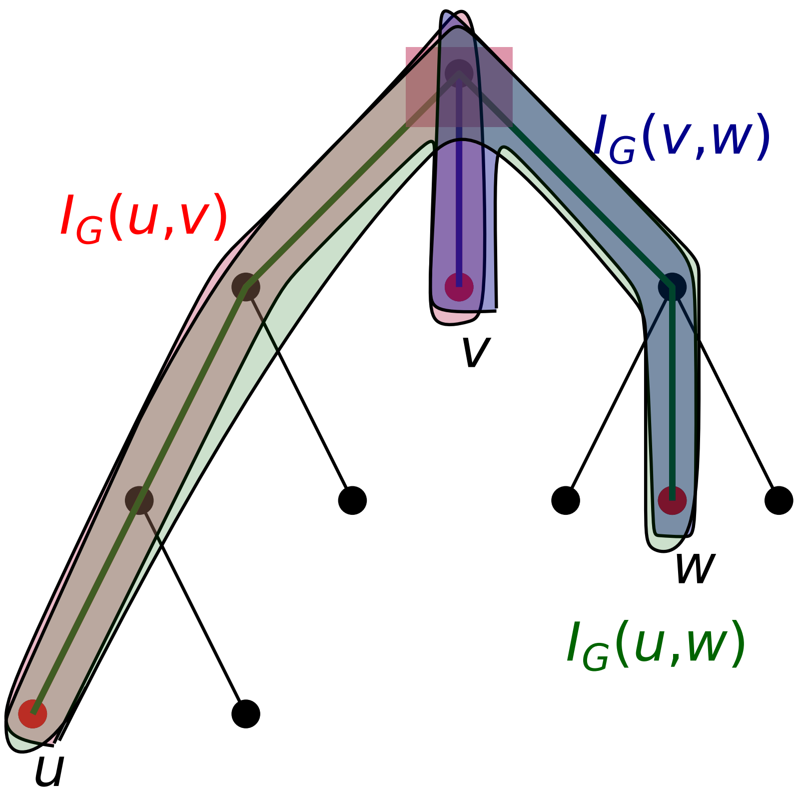

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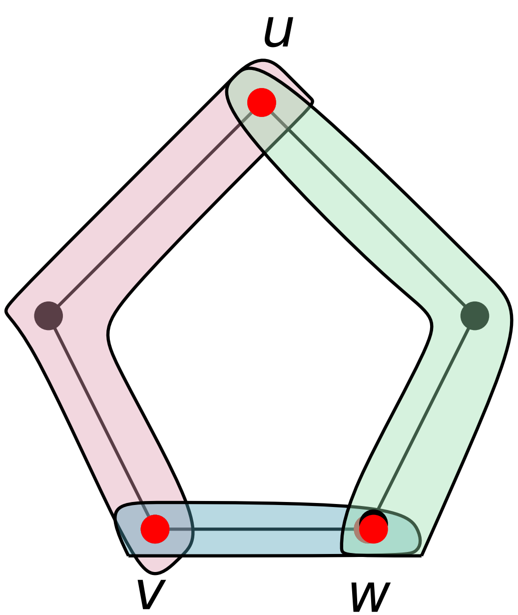

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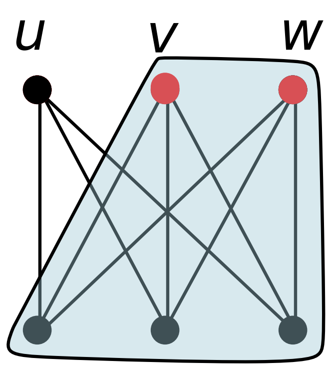

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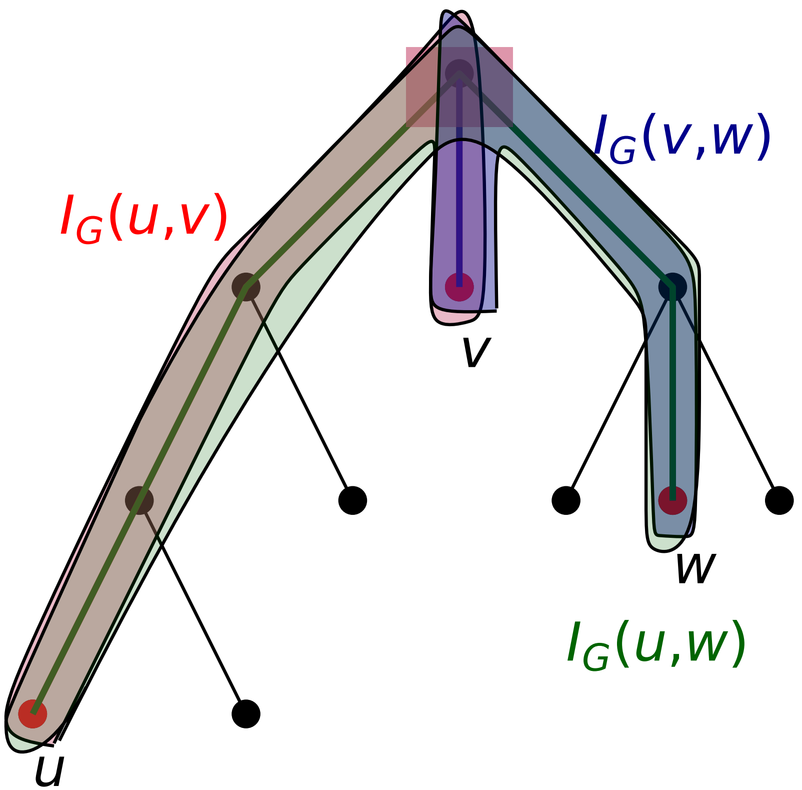

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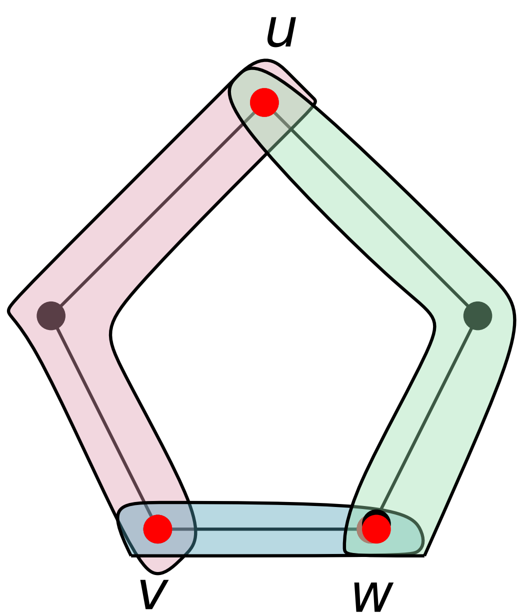

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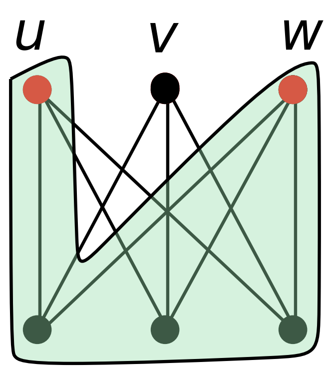

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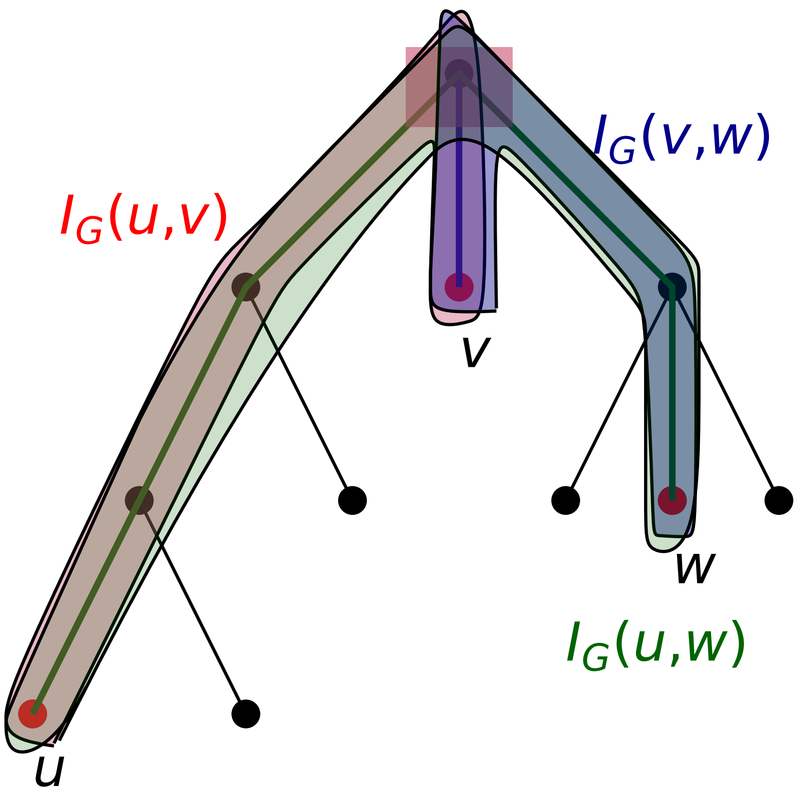

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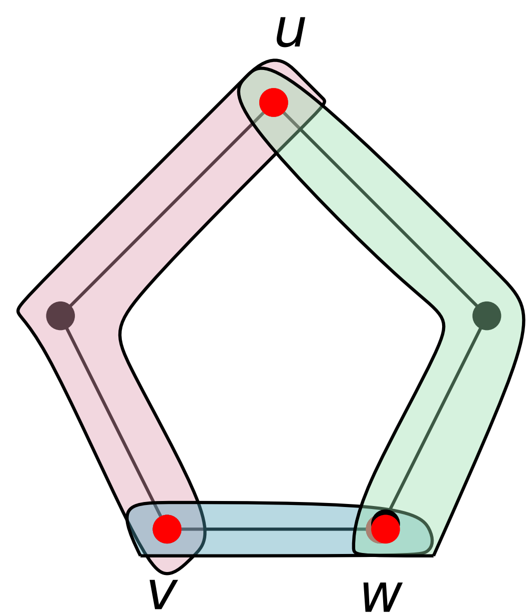
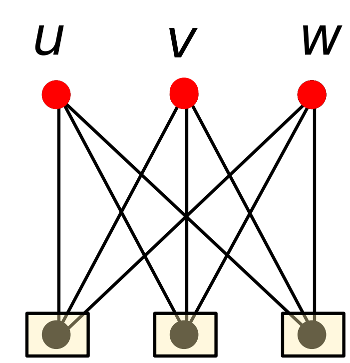
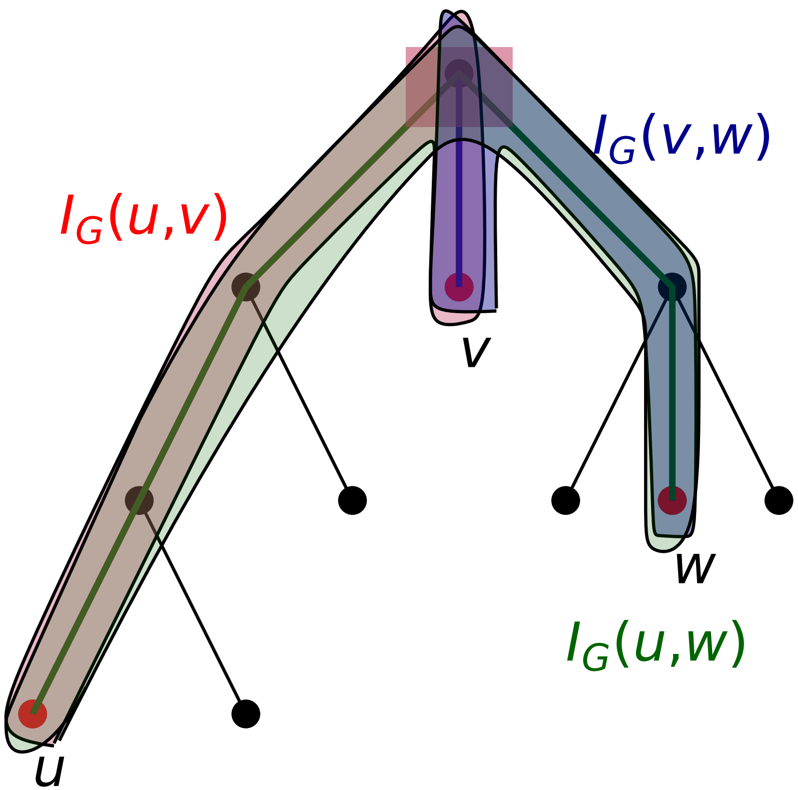


$|I_G(u,v) \cap I_G(v,w) \cap I_G(u,w)| = 0$



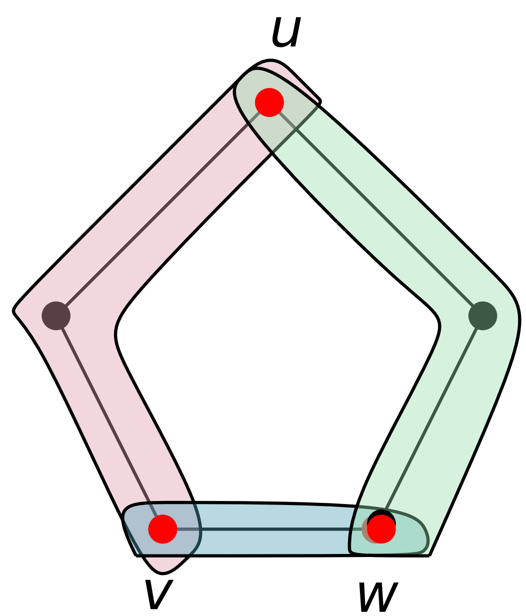
$|I_G(u,v) \cap I_G(v,w) \cap I_G(u,w)| = 3$

$I_G(x,y)$: vertices in every shortest $x - y$ path

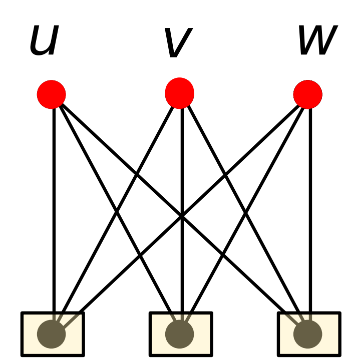


$|I_G(u,v) \cap I_G(v,w) \cap I_G(u,w)| = 1$

✓



$|I_G(u,v) \cap I_G(v,w) \cap I_G(u,w)| = 0$



$|I_G(u,v) \cap I_G(v,w) \cap I_G(u,w)| = 3$

Median Graphs

G : Connected graph

- $|I_G(u,v) \cap I_G(u,w) \cap I_G(v,w)| = 1, \forall (u,v,w) \in V \times V \times V.$

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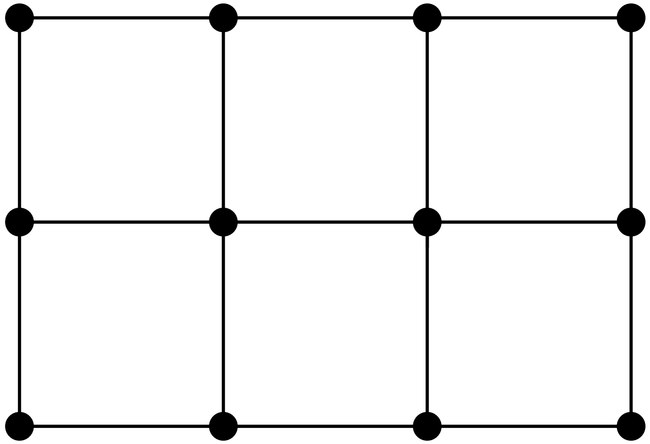

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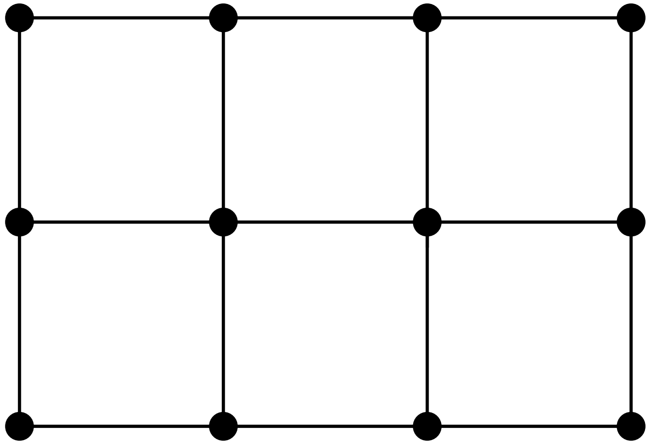

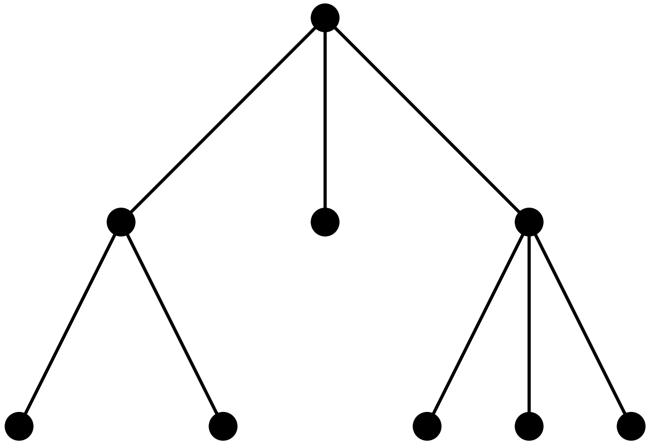

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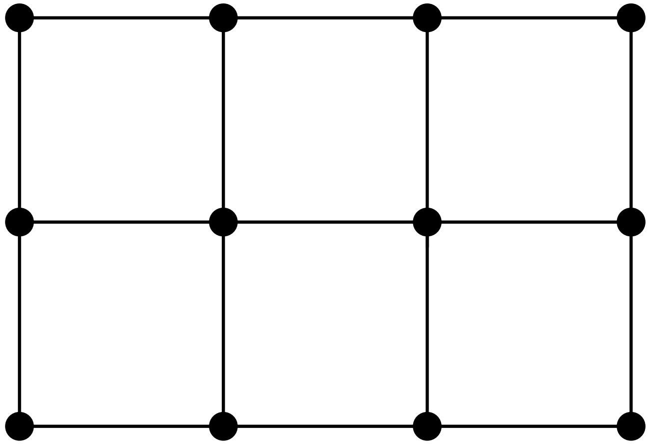
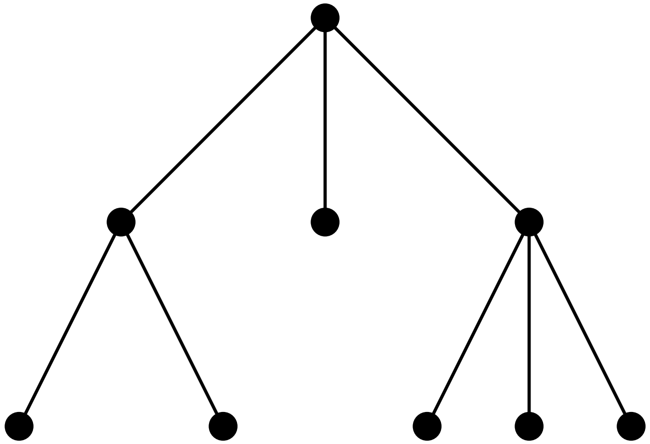
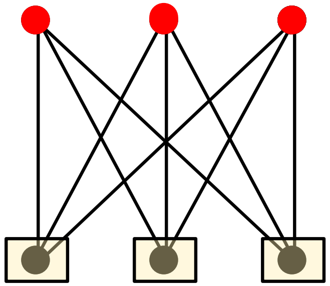

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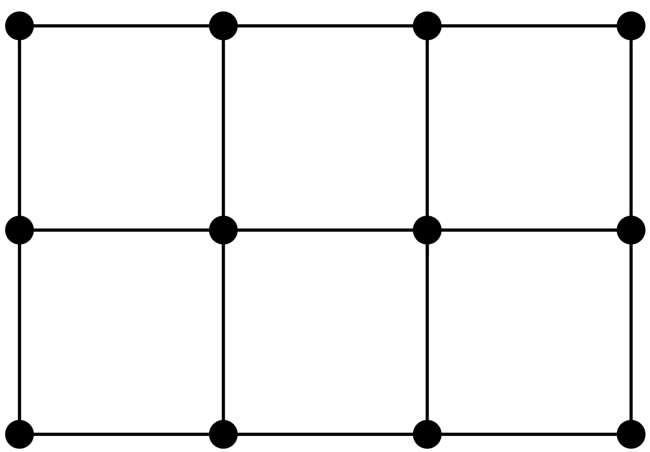

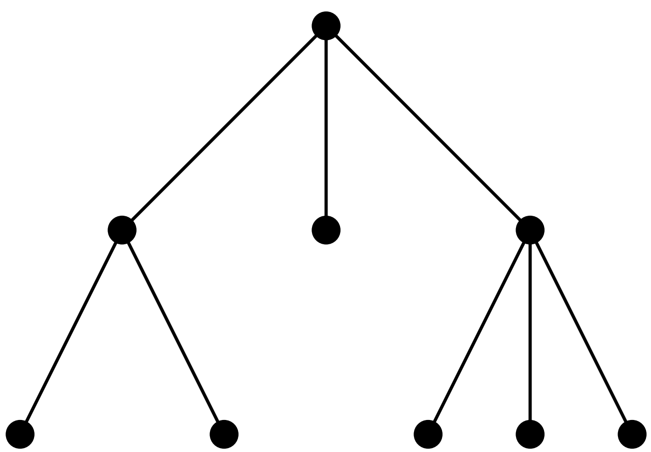

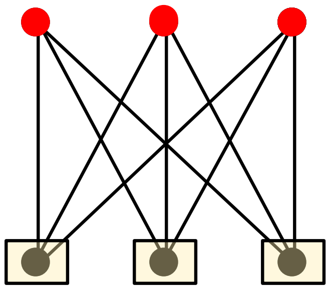

No

Properties

- $C_n : n \geq 5 \rightarrow$ Not median

Properties

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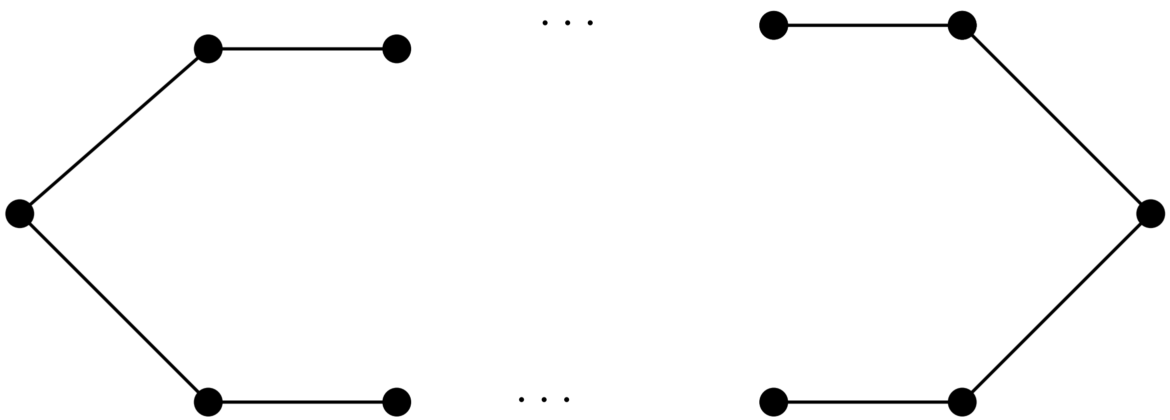
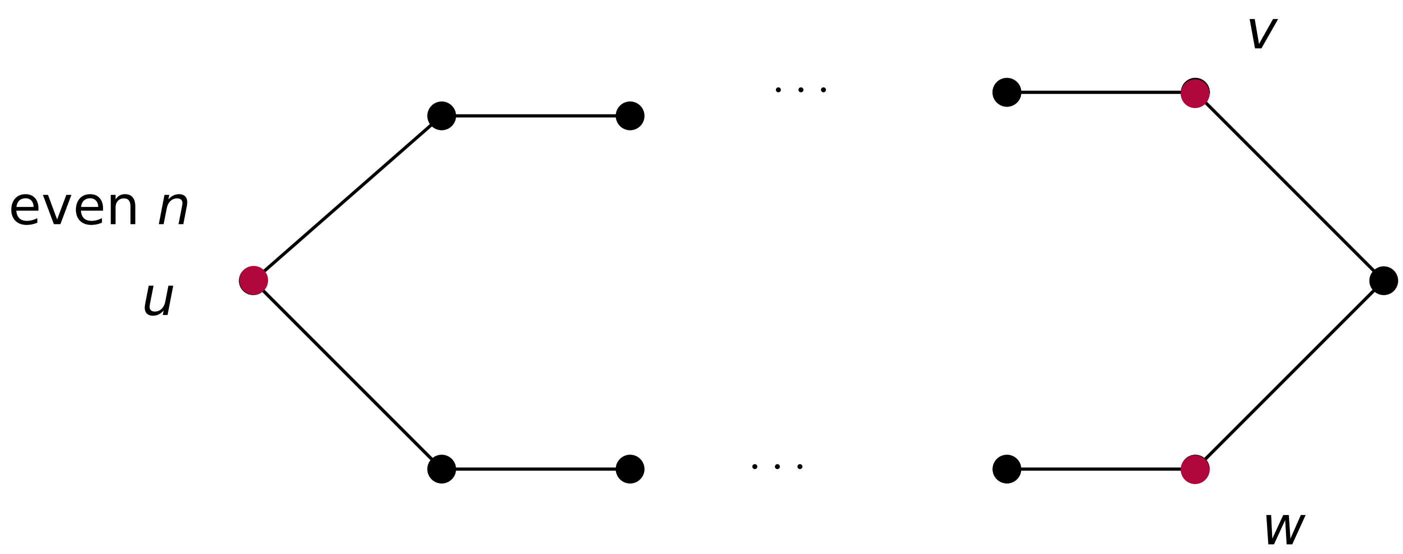
even n



Properties

- $C_n : n \geq 5 \rightarrow$ Not median



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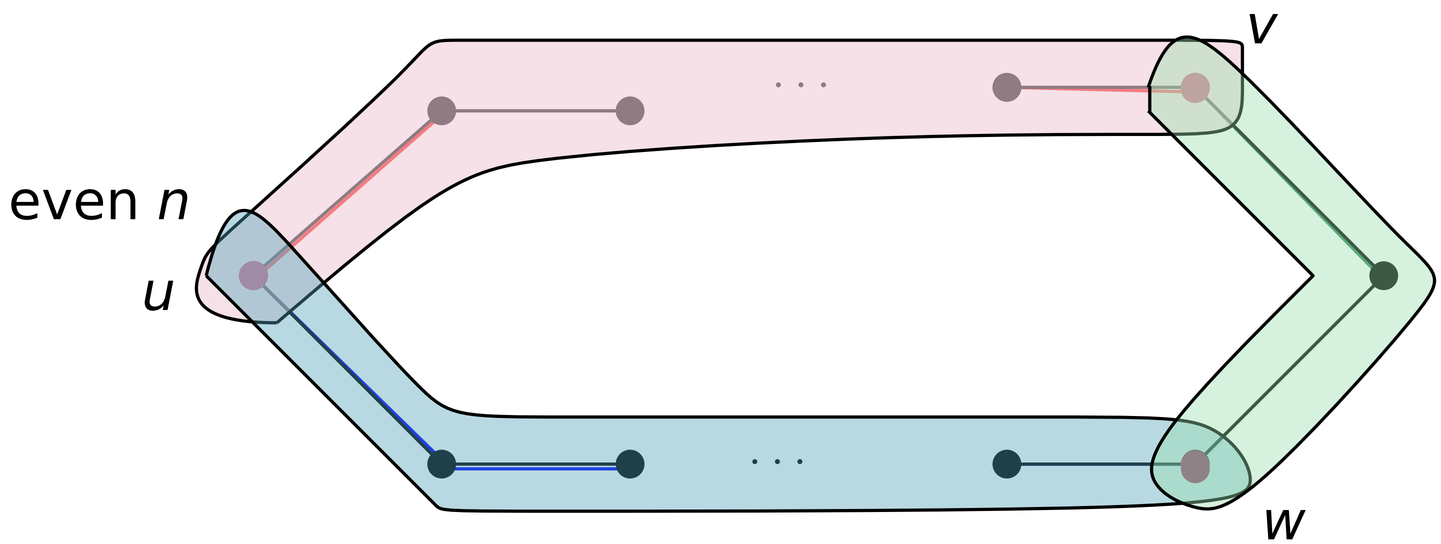

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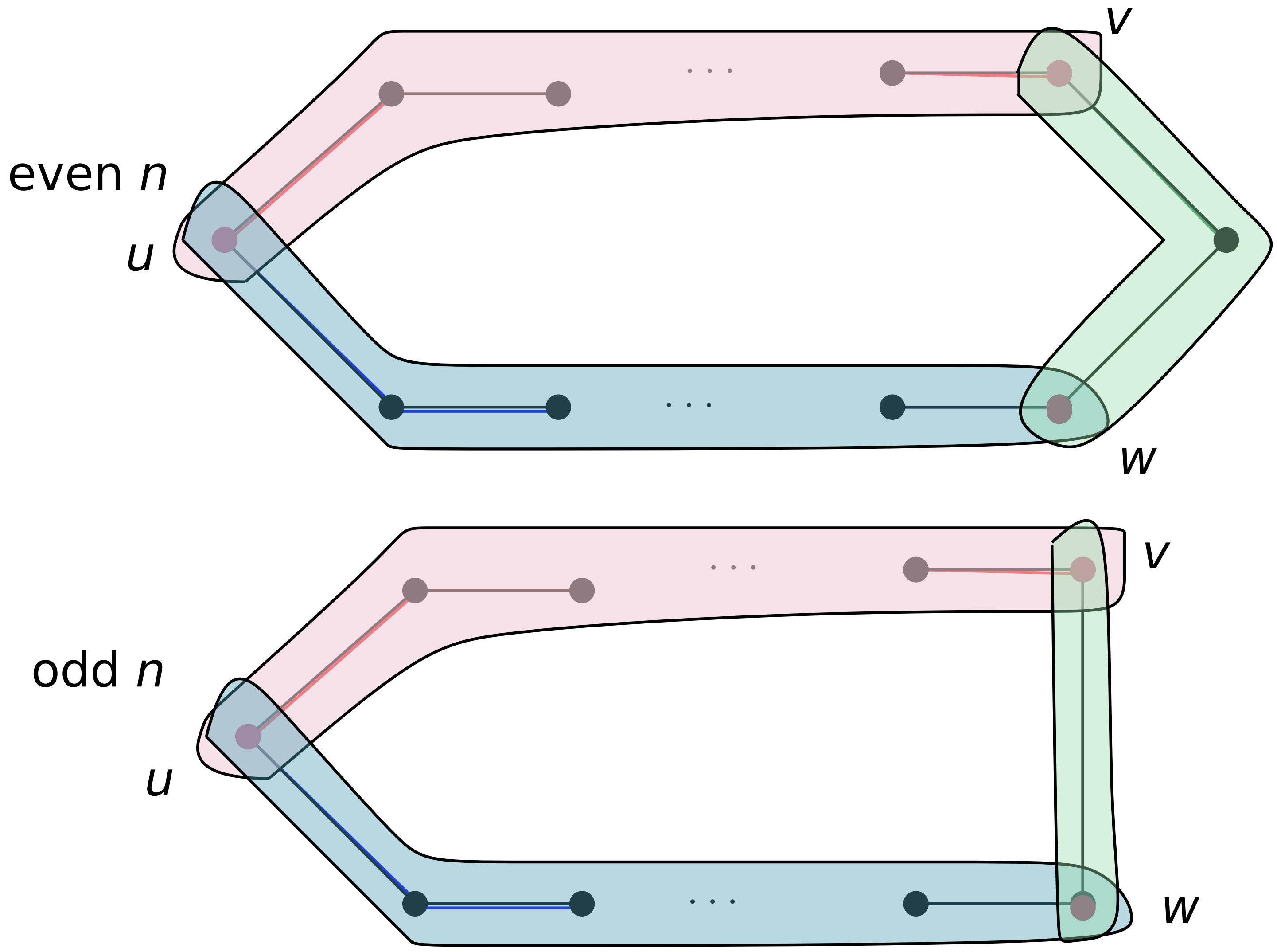
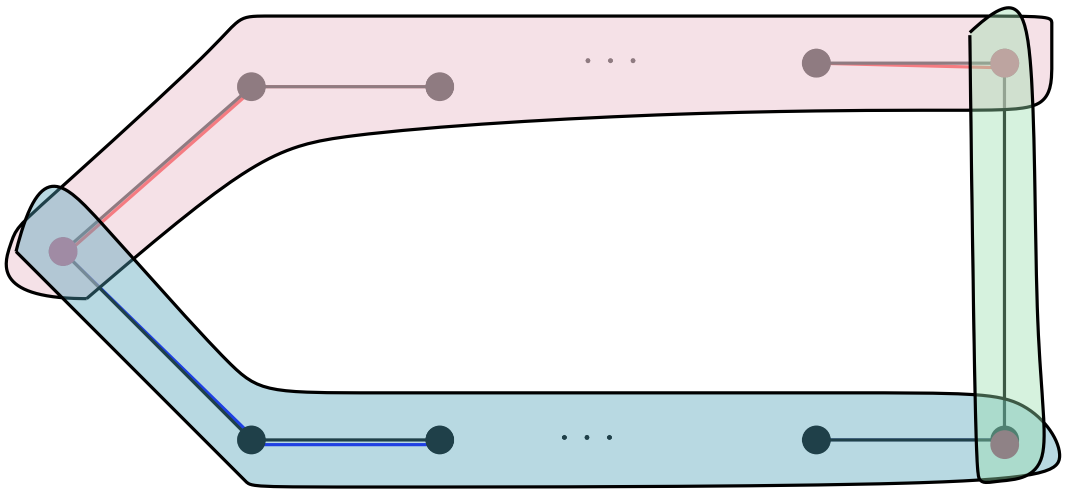


- Median $\Rightarrow$ Biparite

Select the shortest odd cycle



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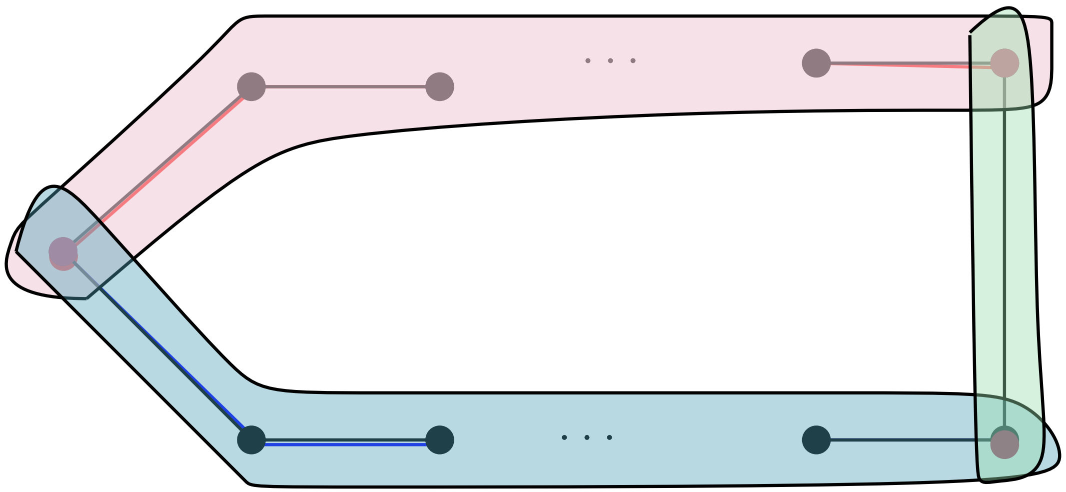

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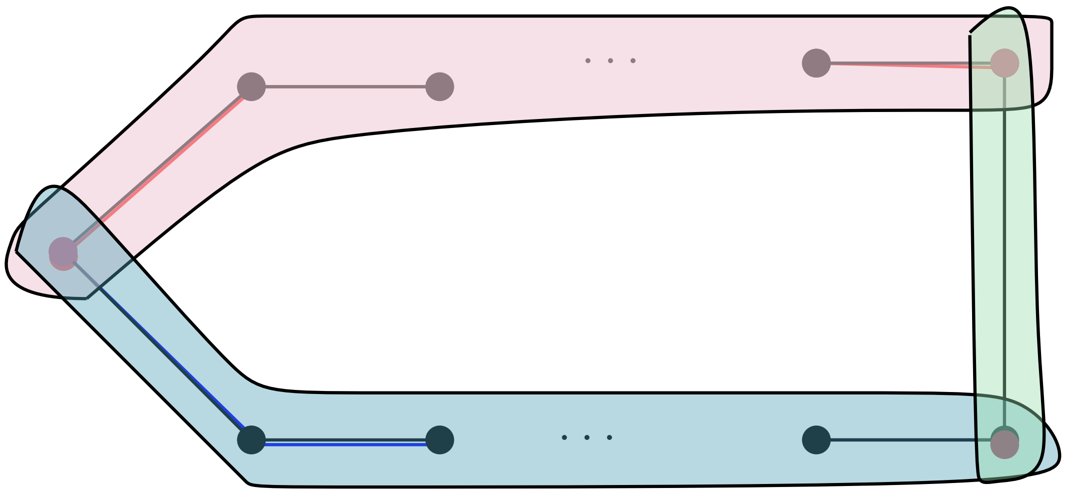
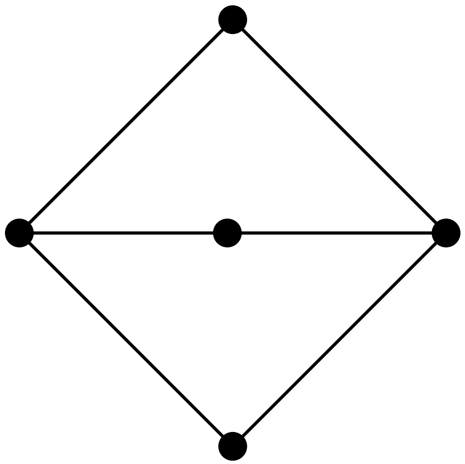


- Median $\Rightarrow K_{2,3}$-free



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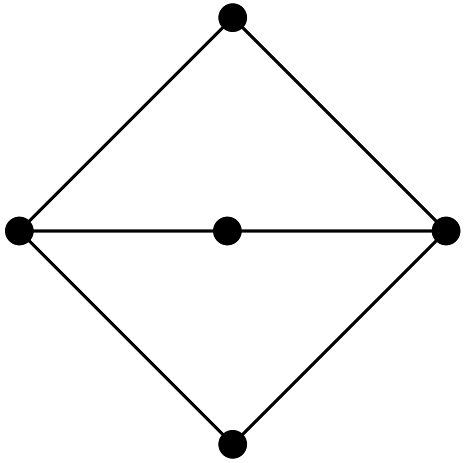
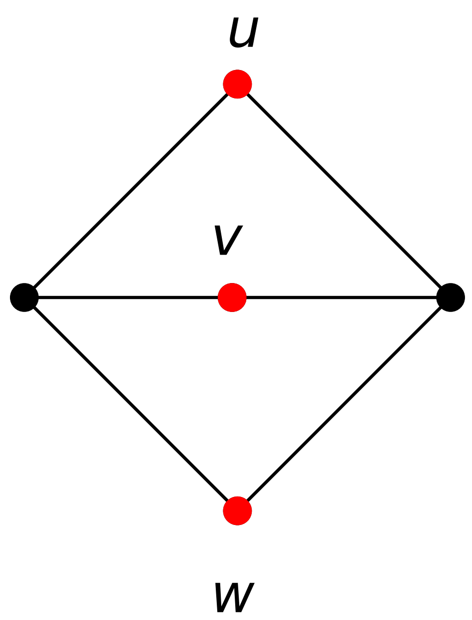


- Median $\Rightarrow$ $K_{2,3}$-free



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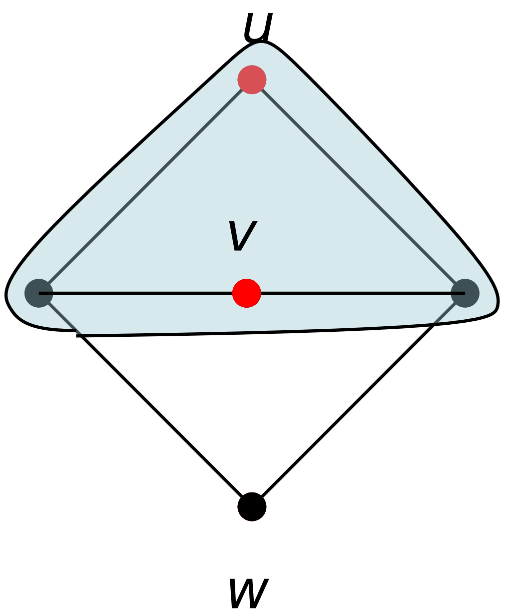

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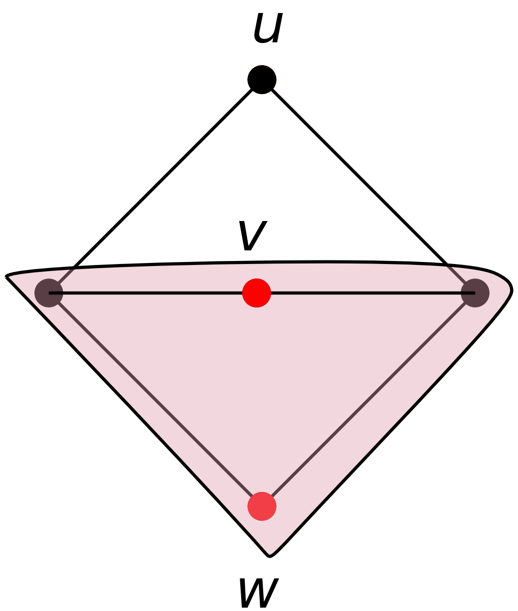

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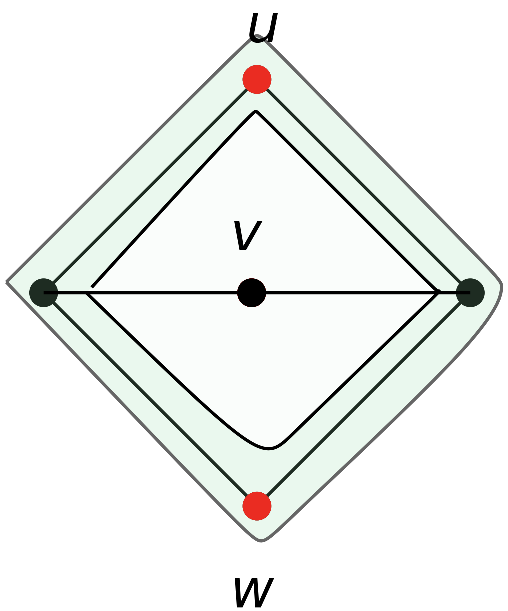

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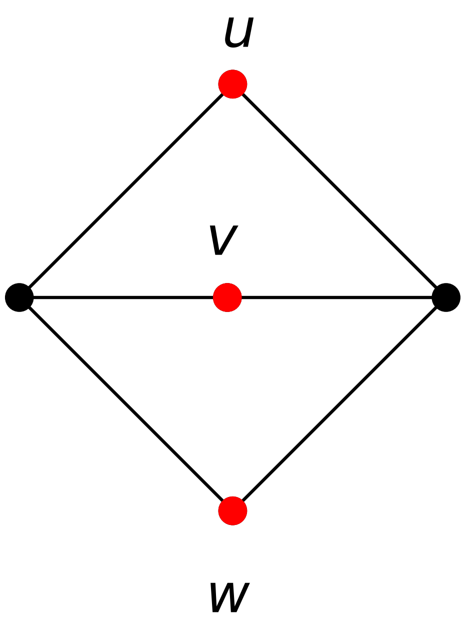

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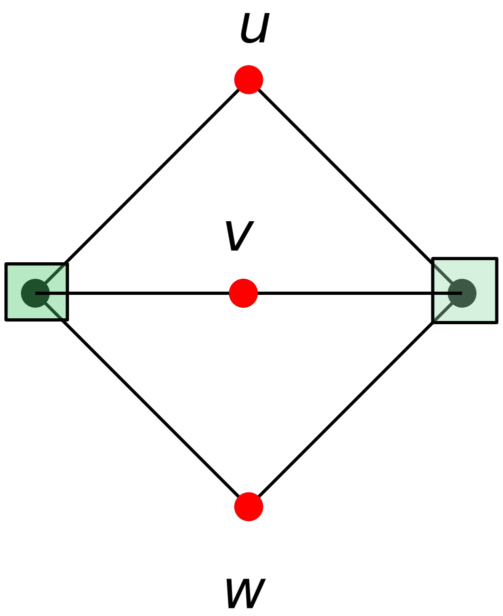
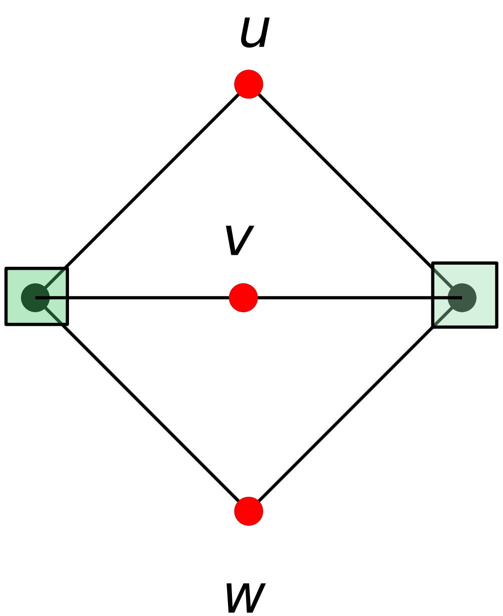


$$|I_G(u,v) \cap I_G(v,w) \cap I_G(u,w)| = 2$$

- Median $\Rightarrow K_{2,3}$-free



$$|I_G(u,v) \cap I_G(v,w) \cap I_G(u,w)| = 2$$

- Median $\Rightarrow$ every edge $e \in E$ that lies on some cycle must be contained in some C_4

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$\exists$ vertex x with (x,u,v) has a unique median, $\forall(u,v)$

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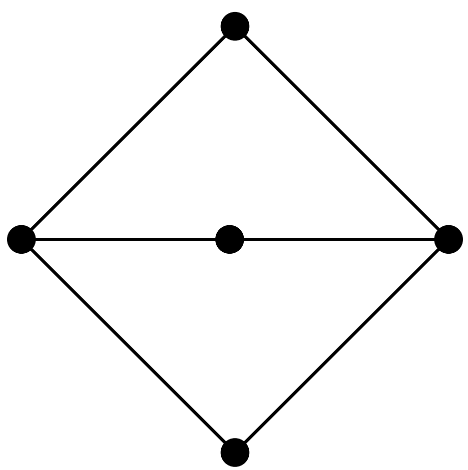

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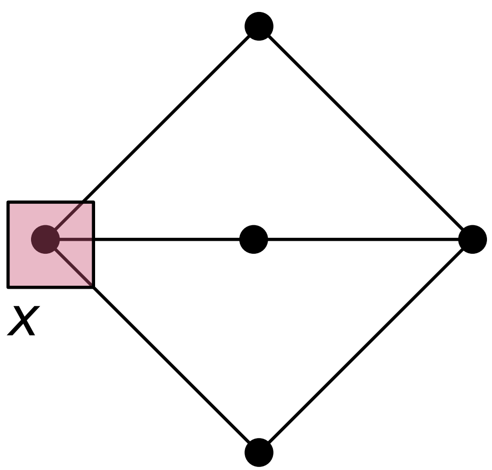

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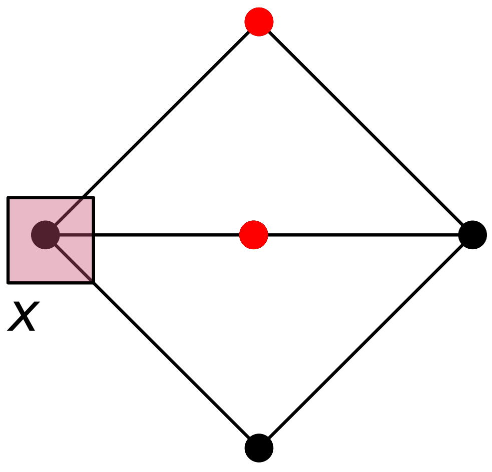

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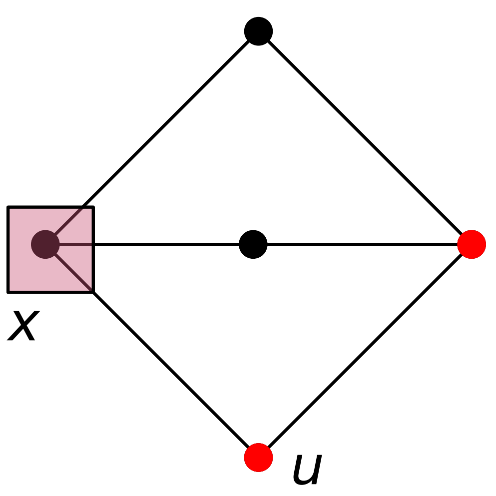

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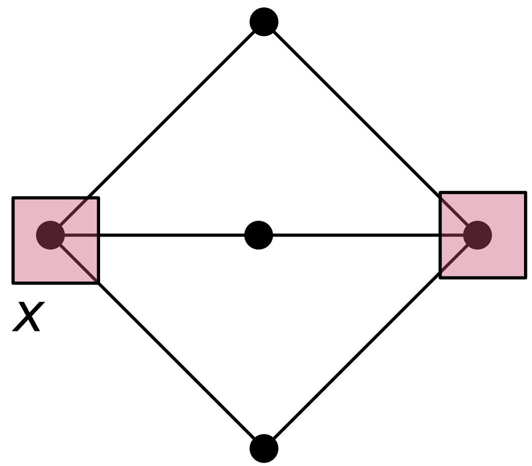

k-Median graphs

- A connected graph G
- $\exists\ k$ vertices $\rho_1, \ldots, \rho_k$, such that $|I_G(\rho_i, v) \cap I_G(v, w) \cap I_G(\rho_i, w)| = 1$, for all distinct $v, w \in V \setminus \{\rho_i\}$, $1 \leq i \leq k$

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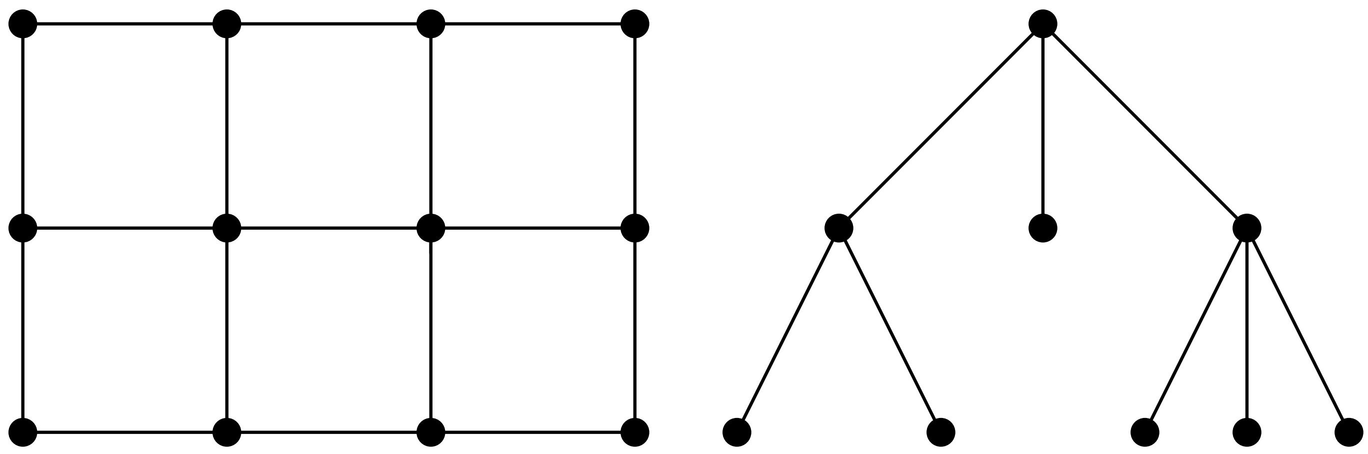

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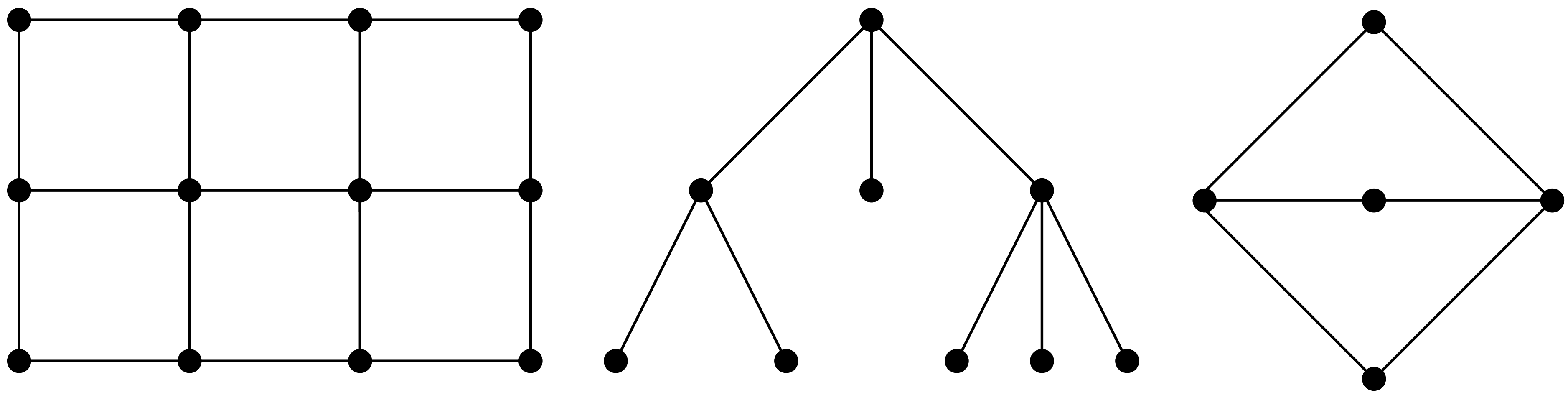

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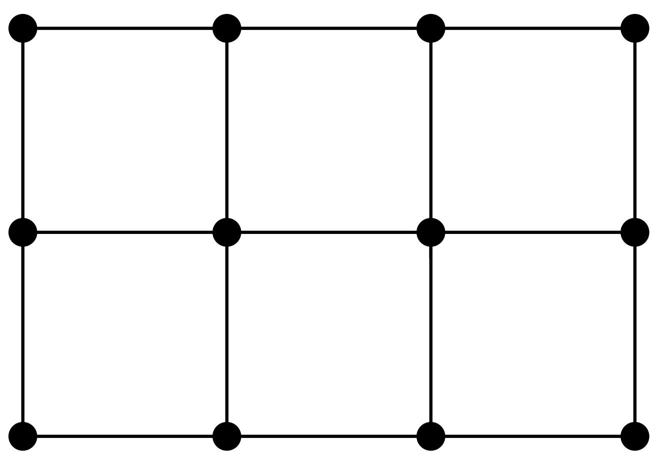

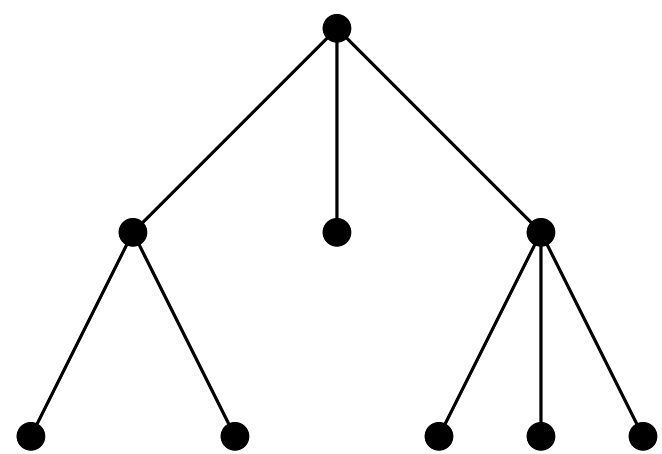

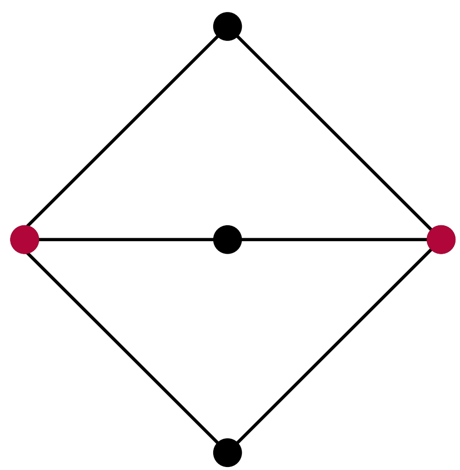

2 – Median

Not Median

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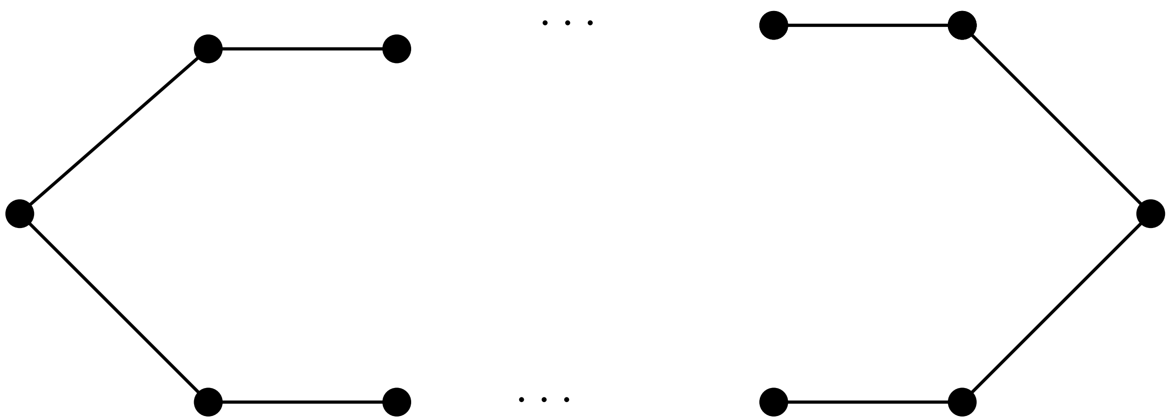

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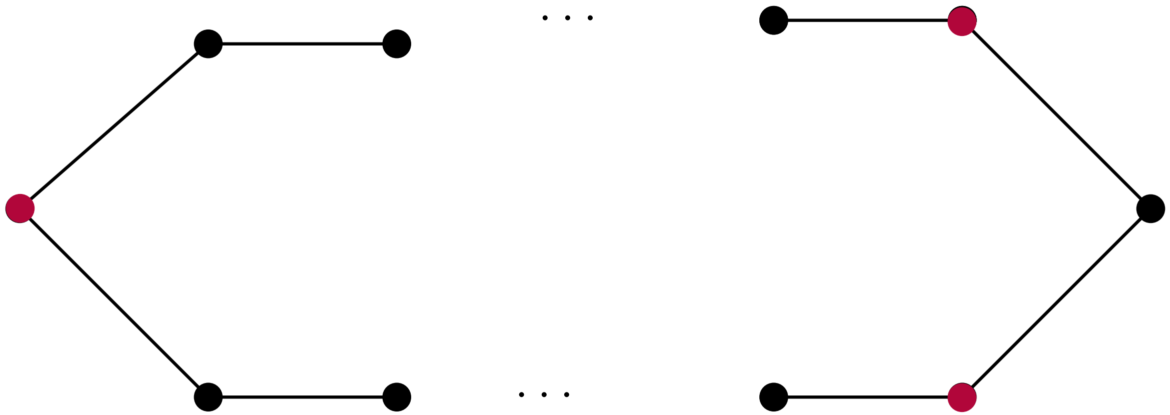

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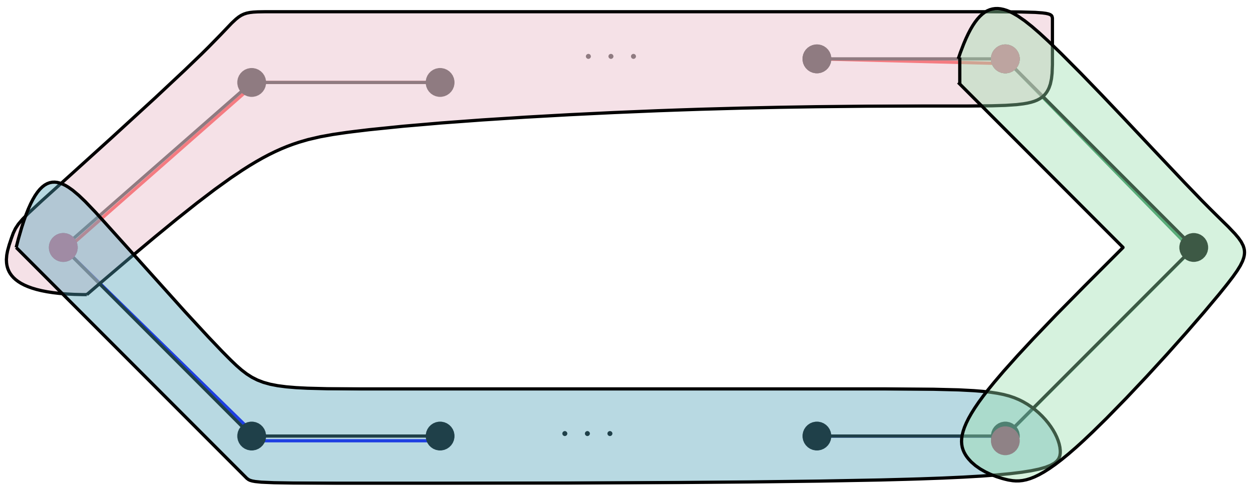

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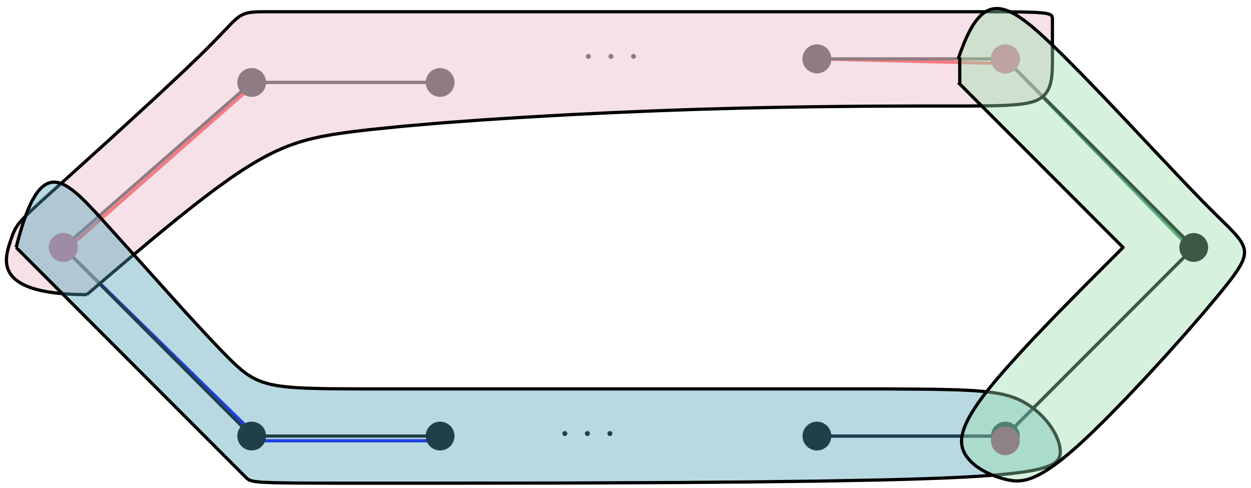

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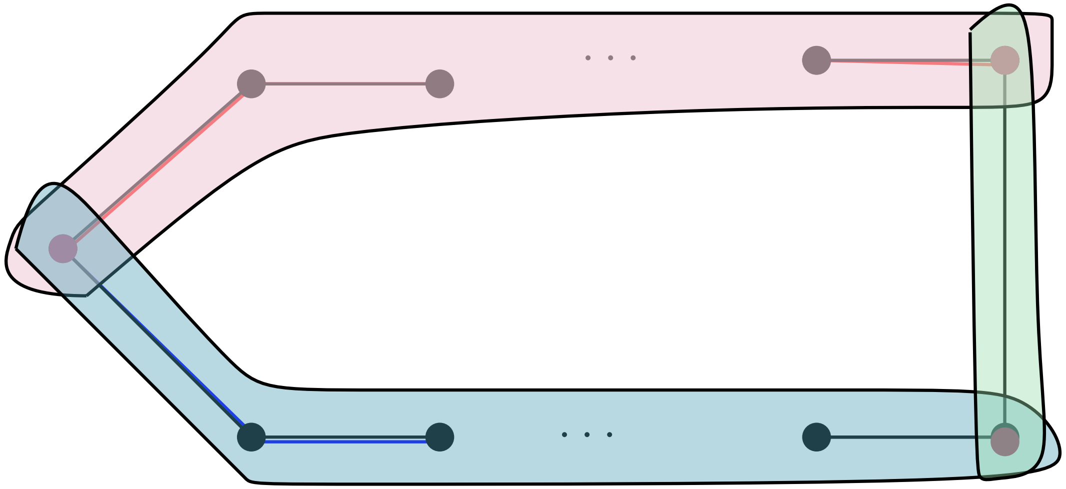

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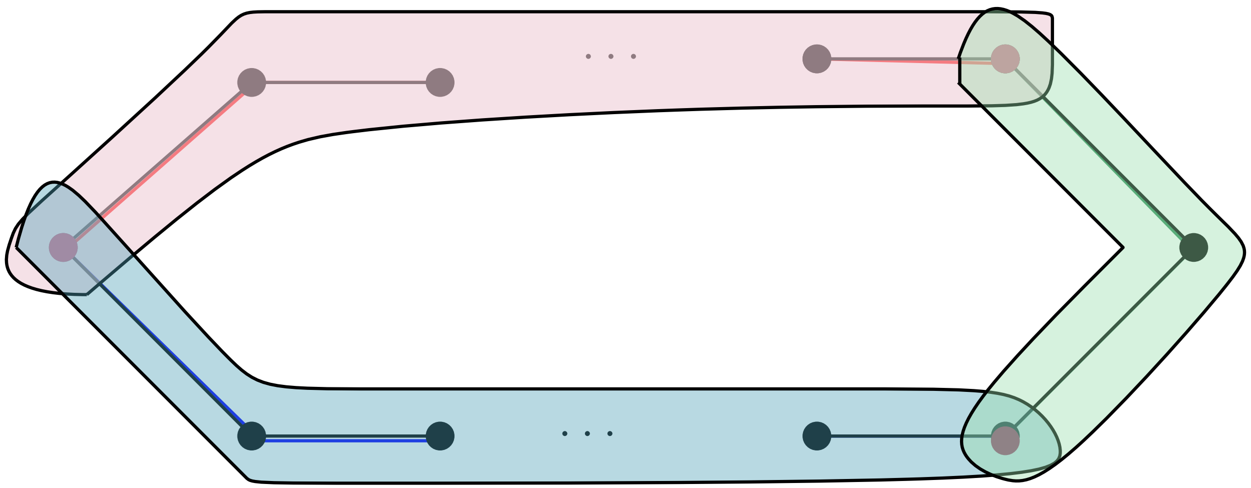

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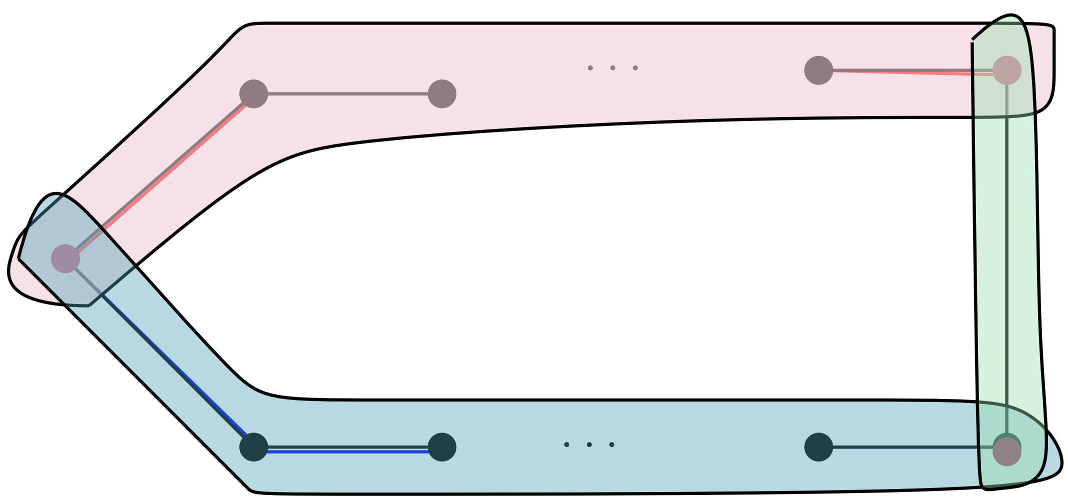

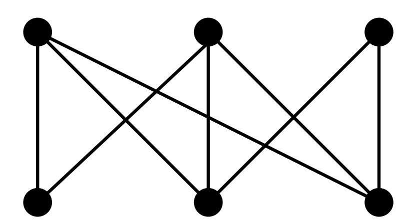

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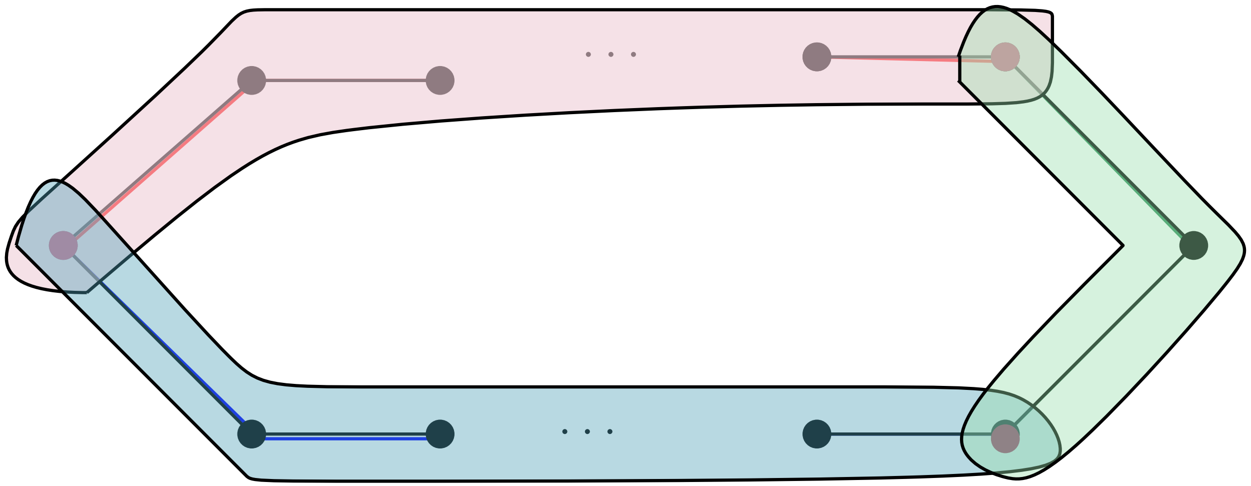

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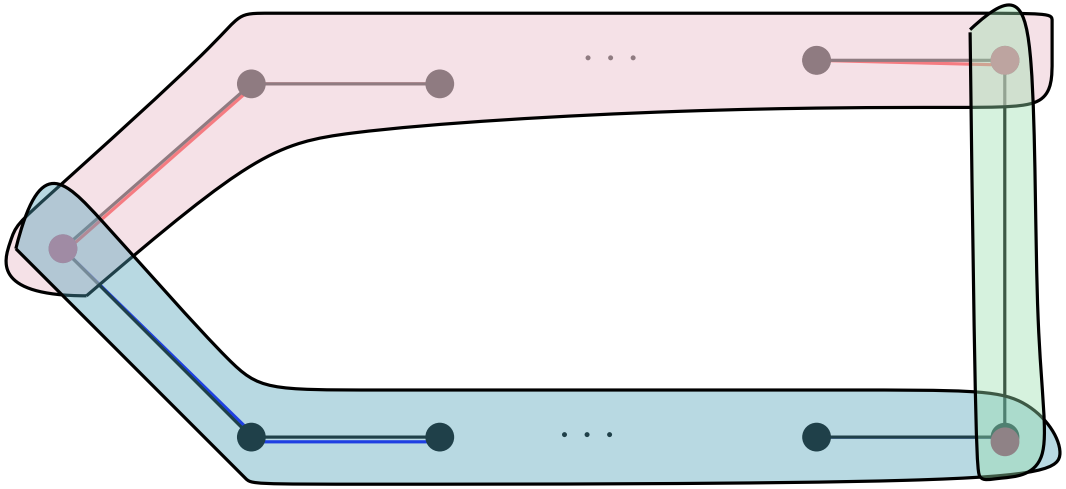

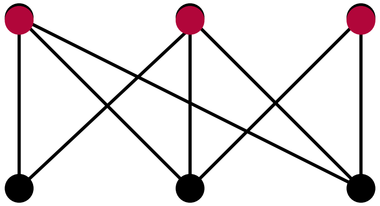

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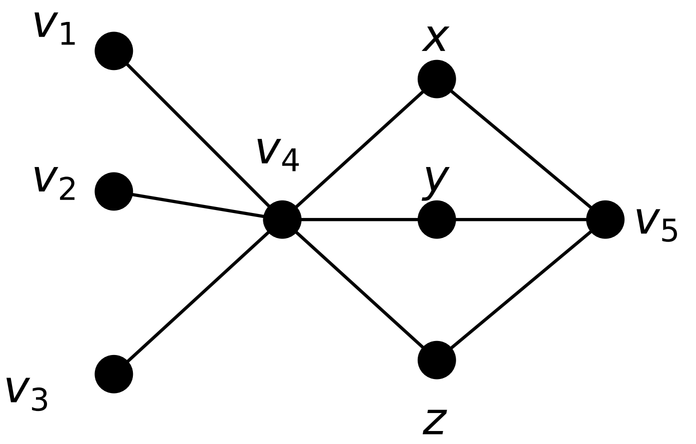

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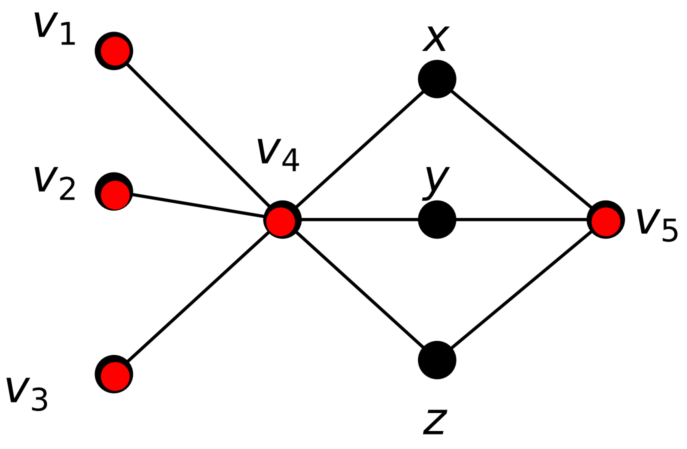

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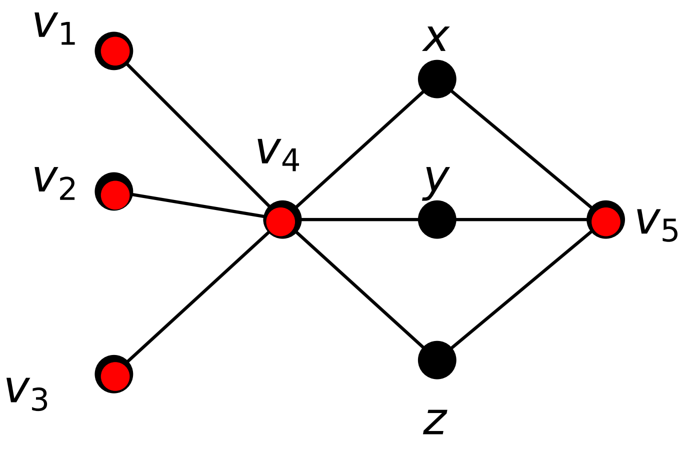

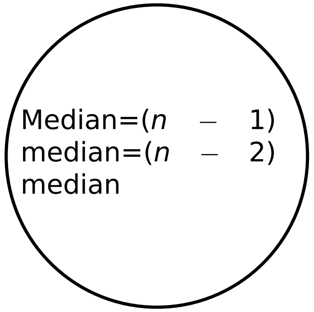

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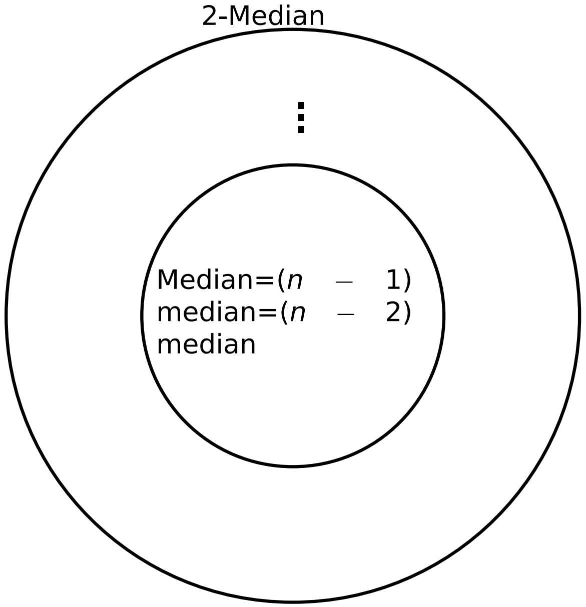

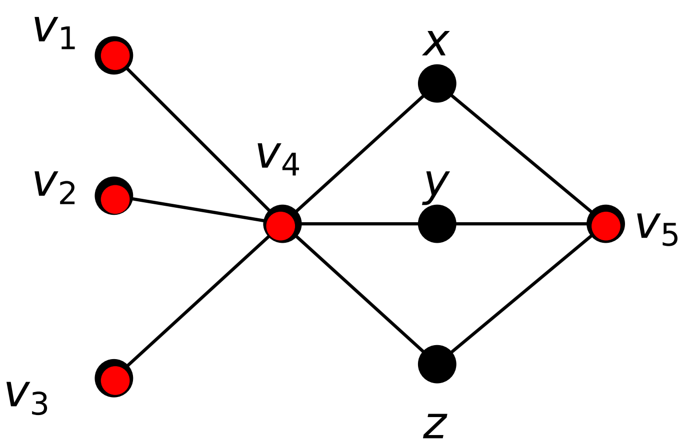

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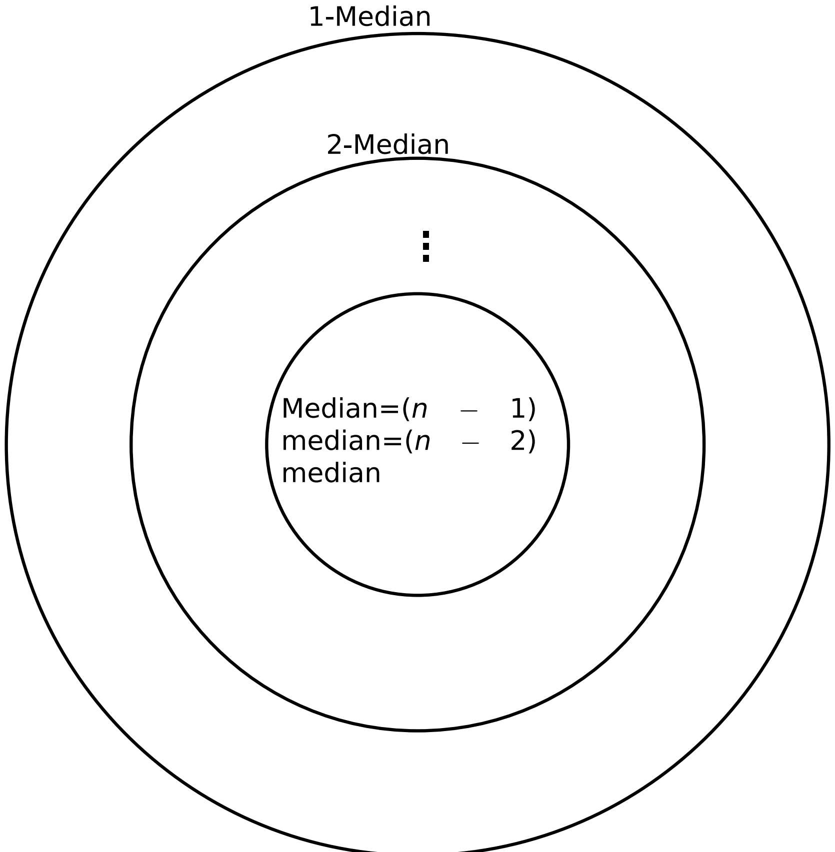

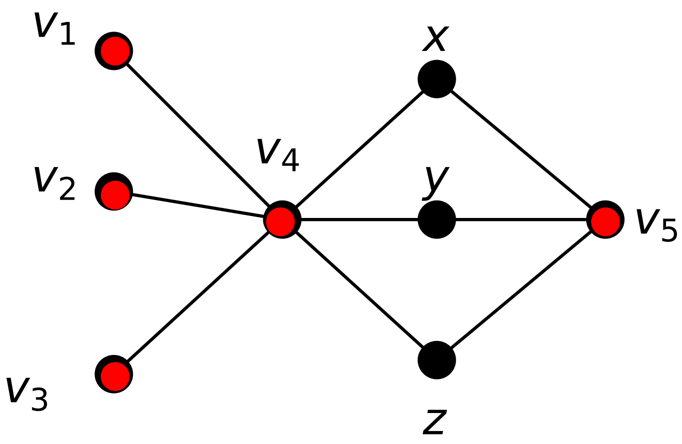

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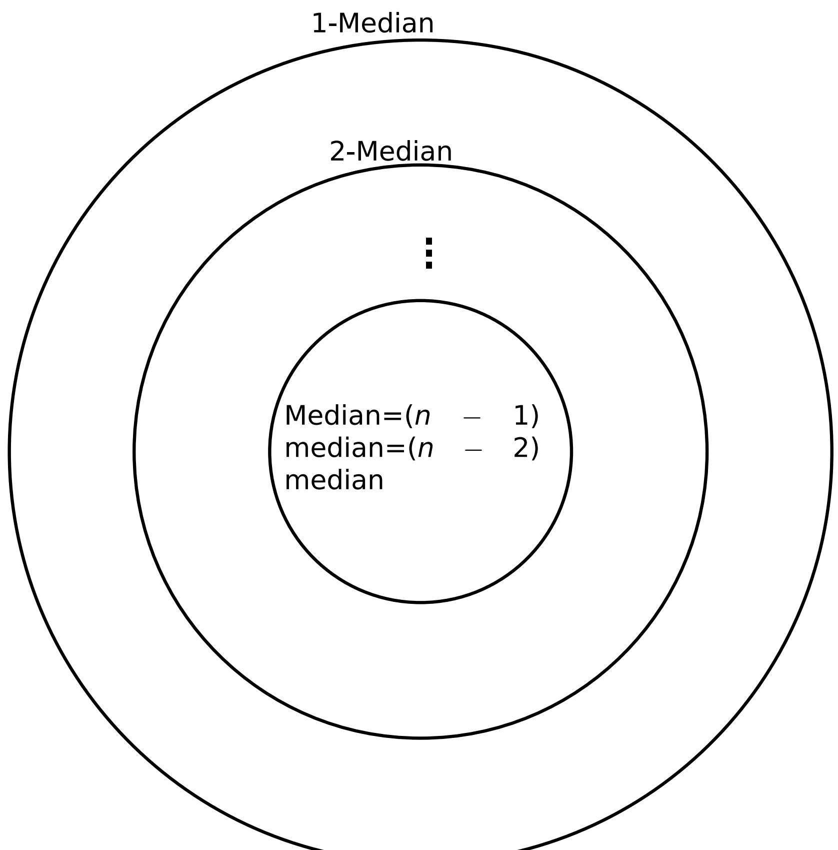
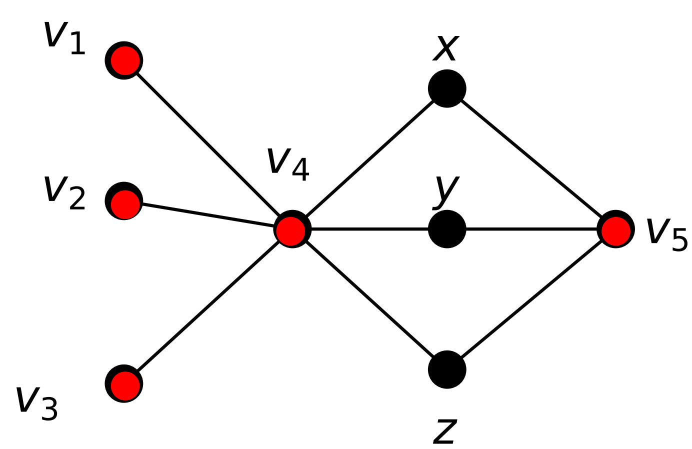
$K_{3,3}$





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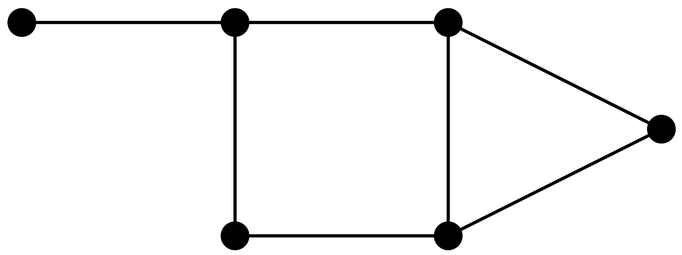
Thank You

Characterization: Convex Expansion

Convex Graphs: An induced subgraph $G[W]$ with all shortest paths in G lie entirely in $G[W]$.



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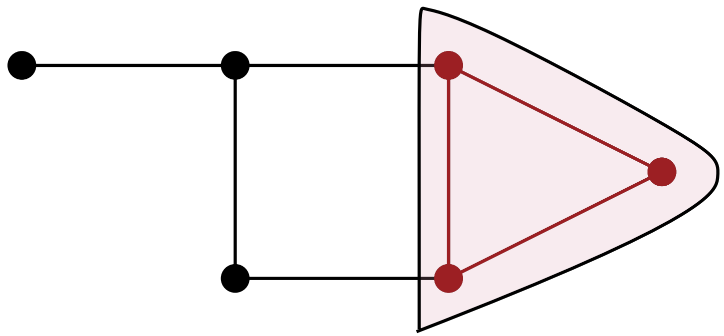

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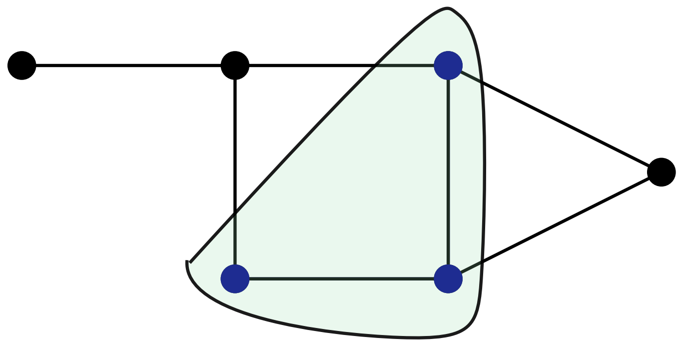

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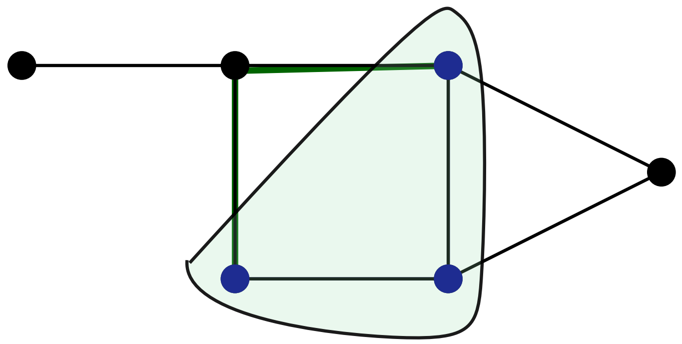

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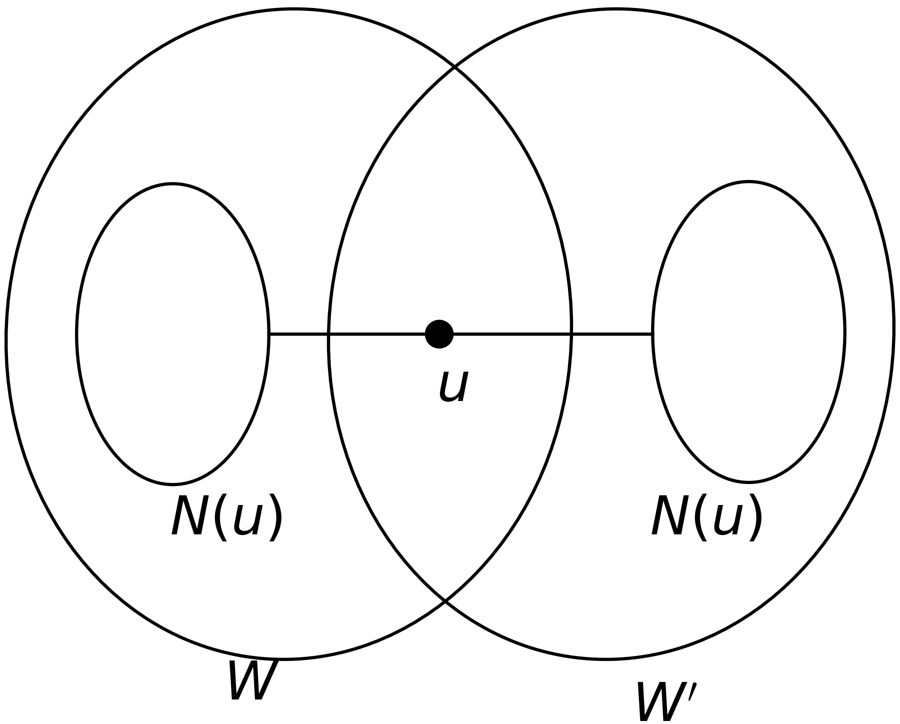

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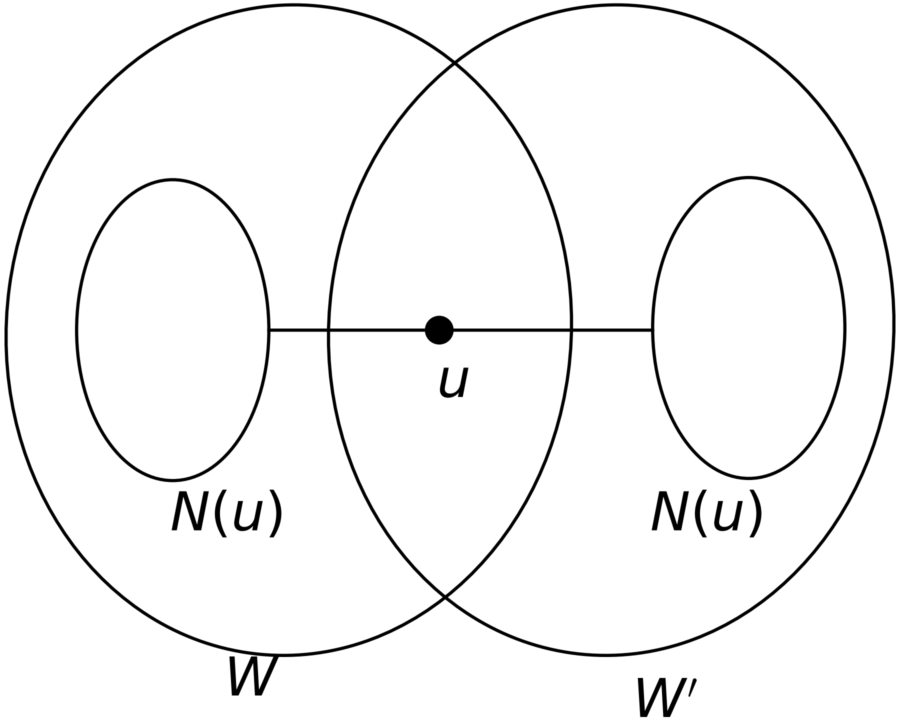

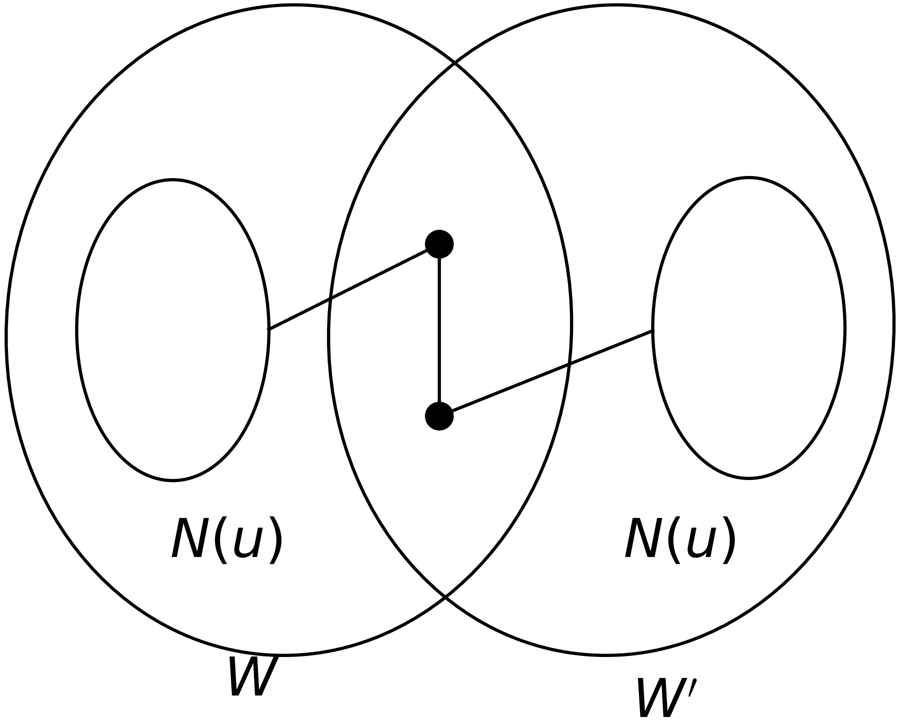

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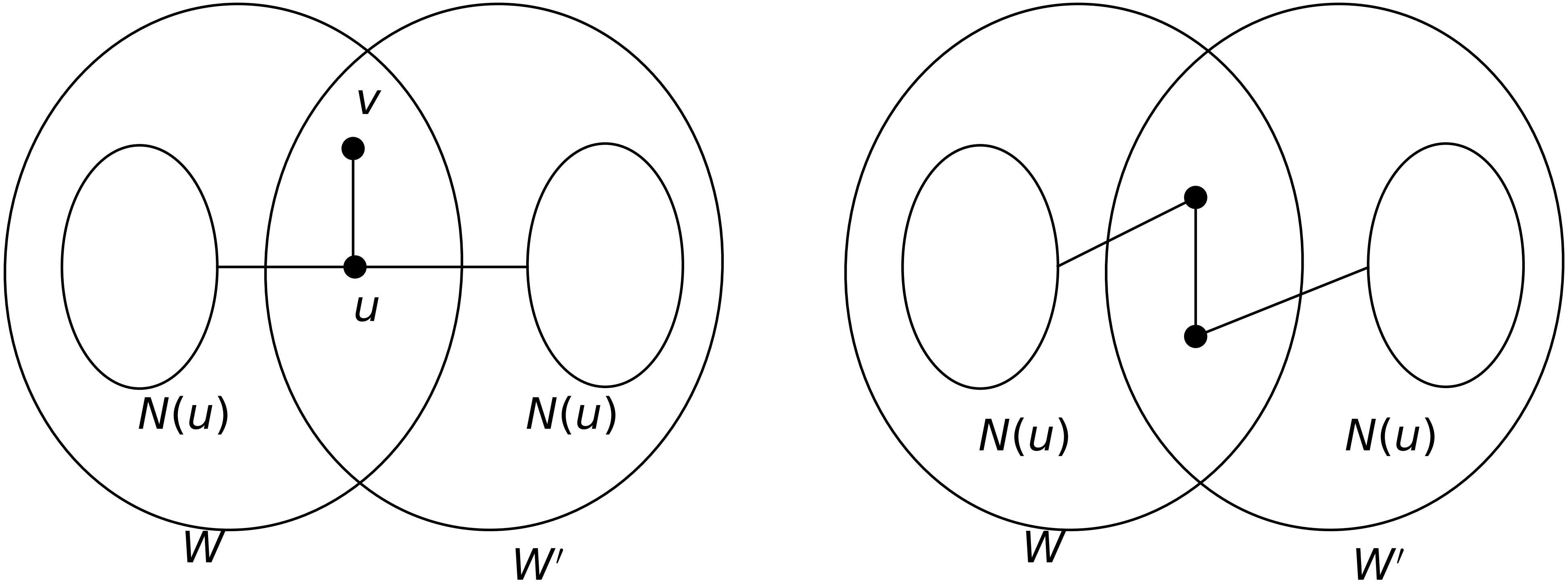

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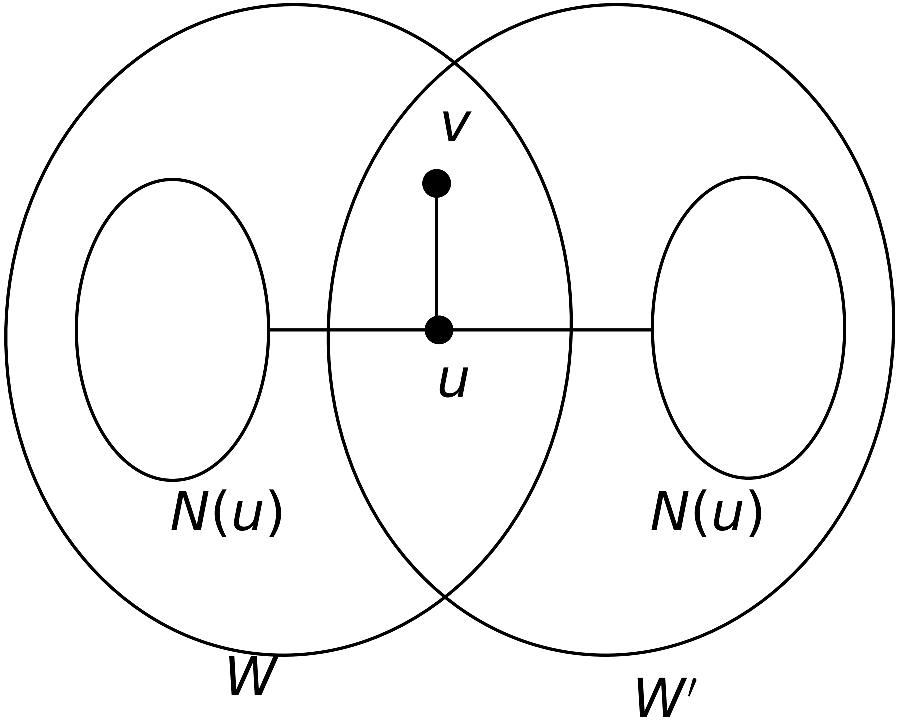

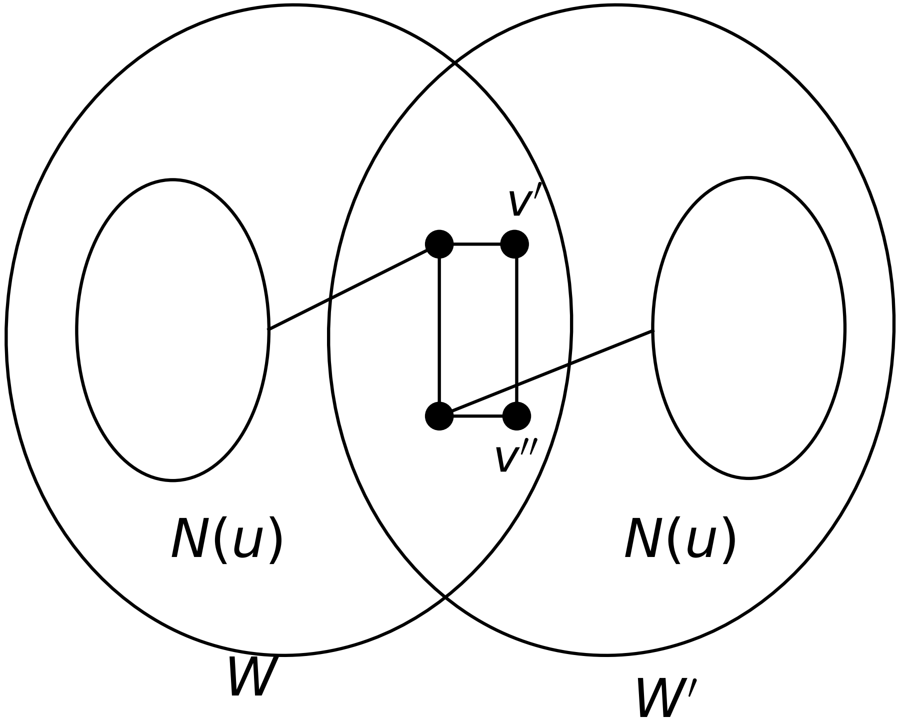

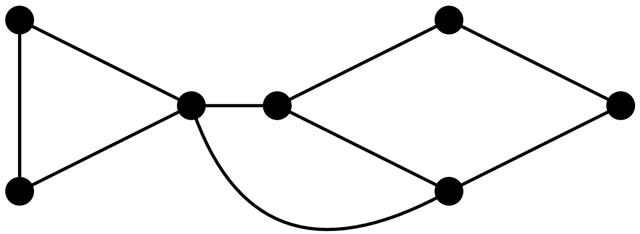

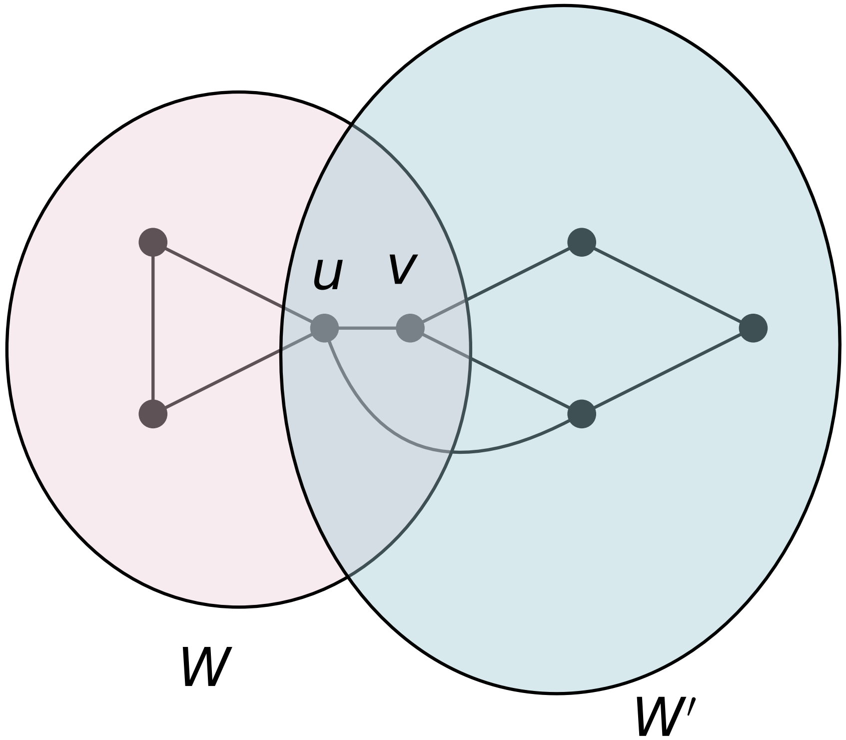
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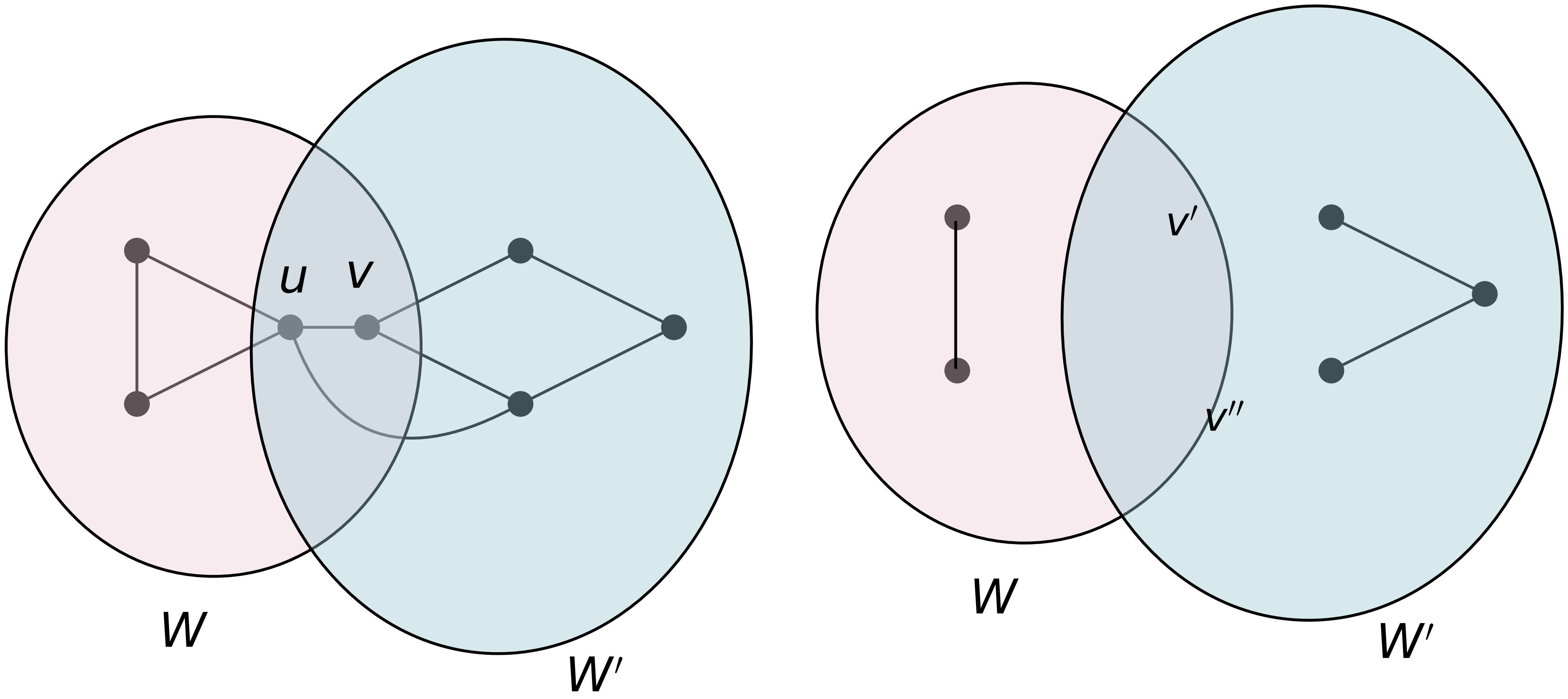
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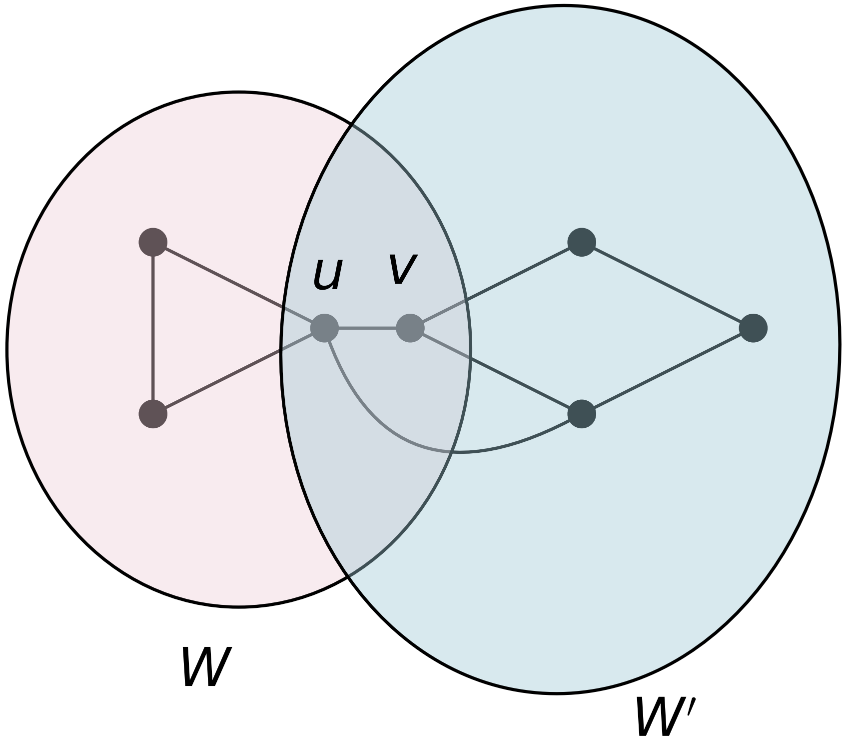
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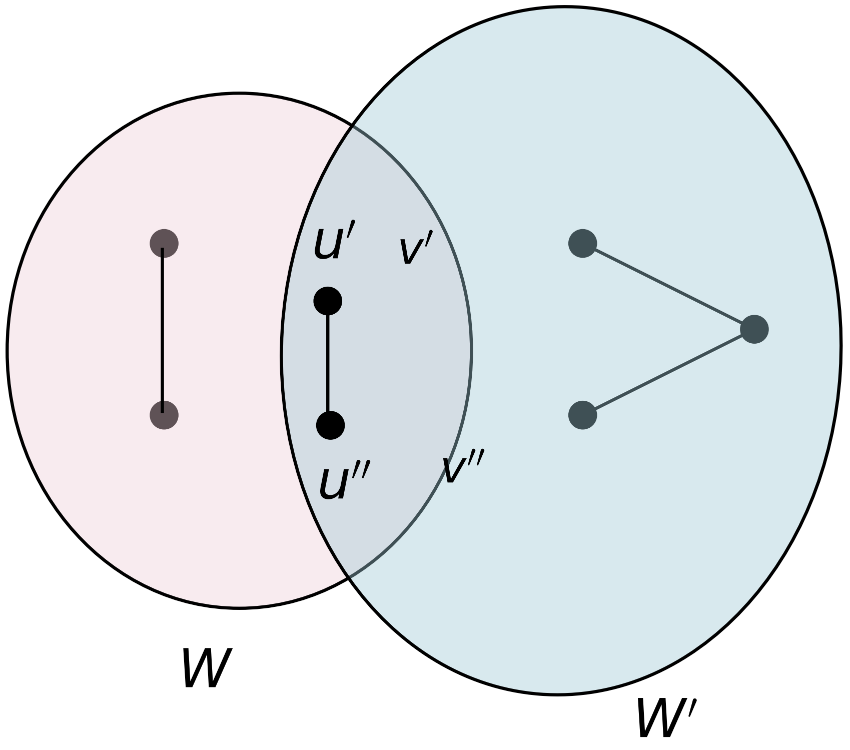
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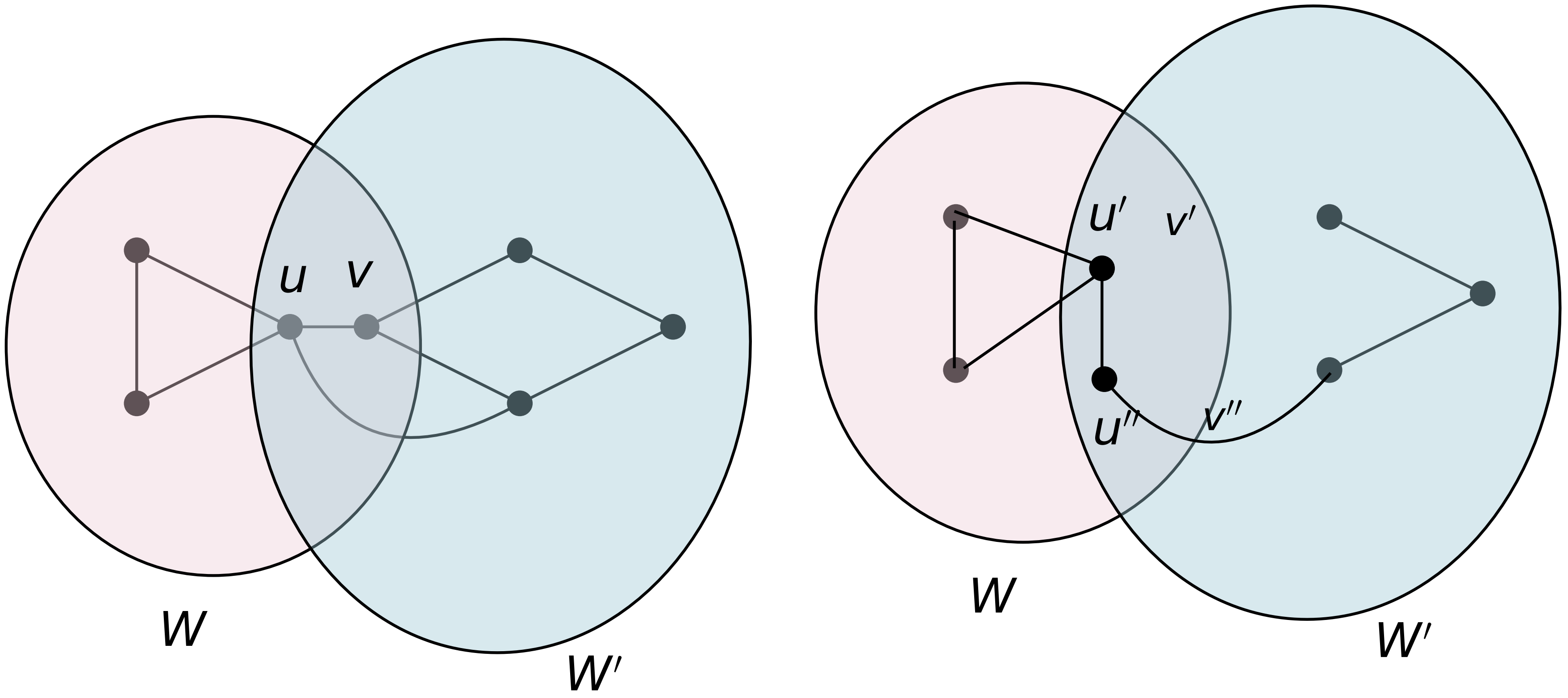
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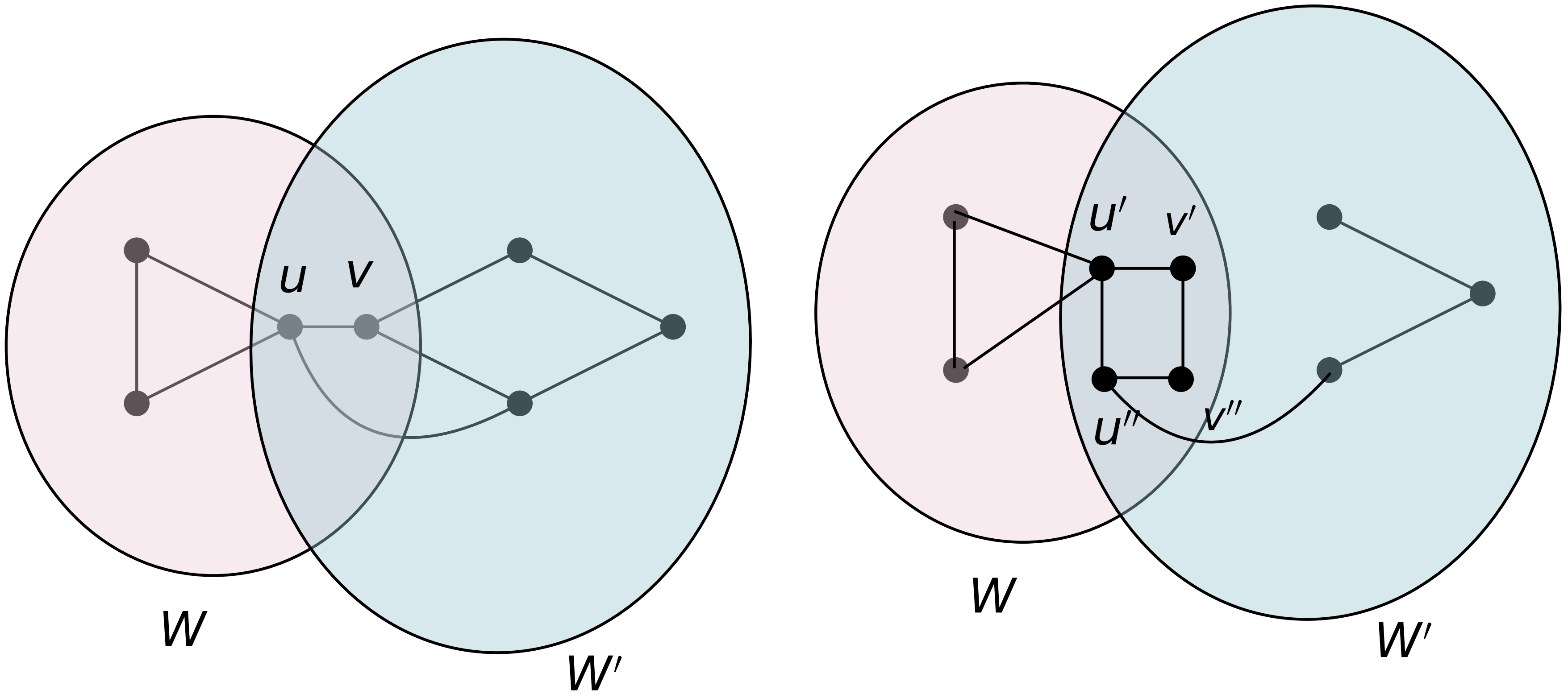
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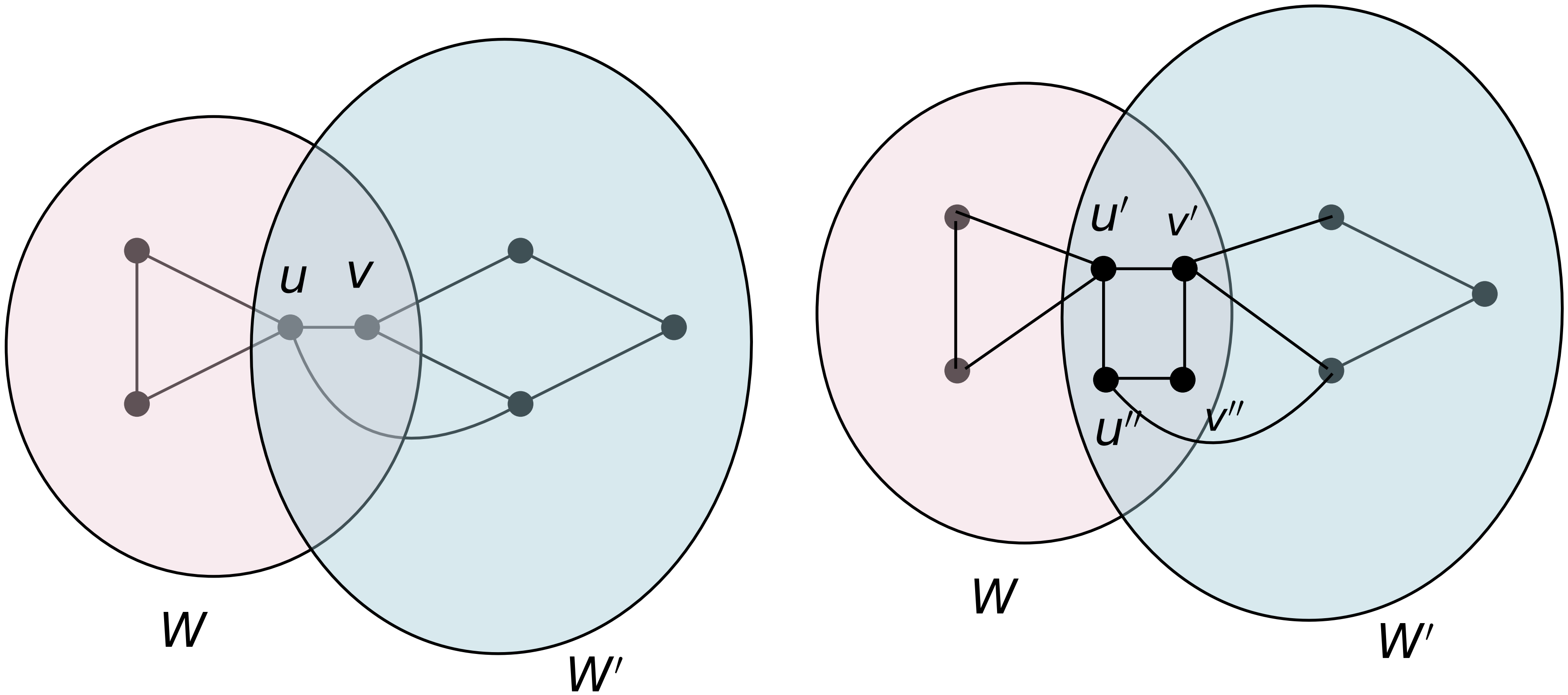
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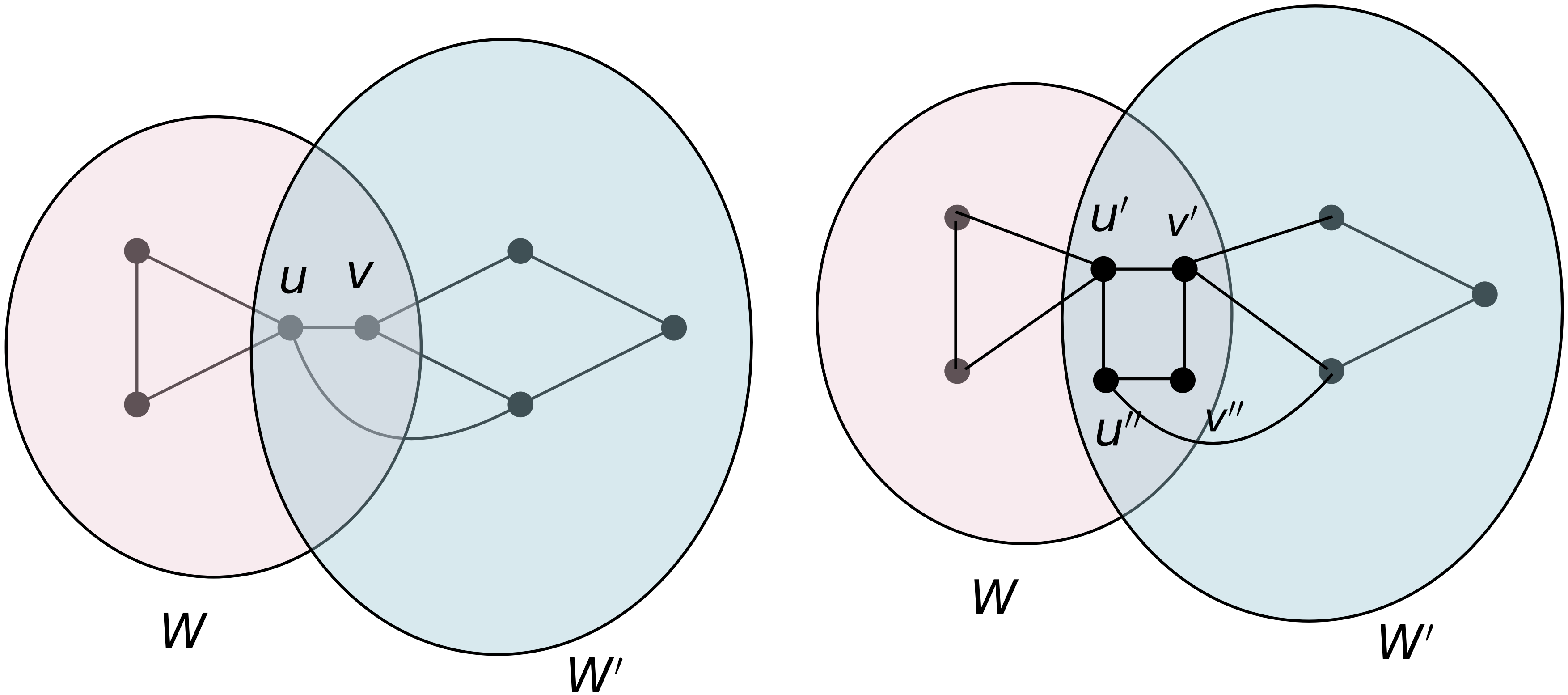

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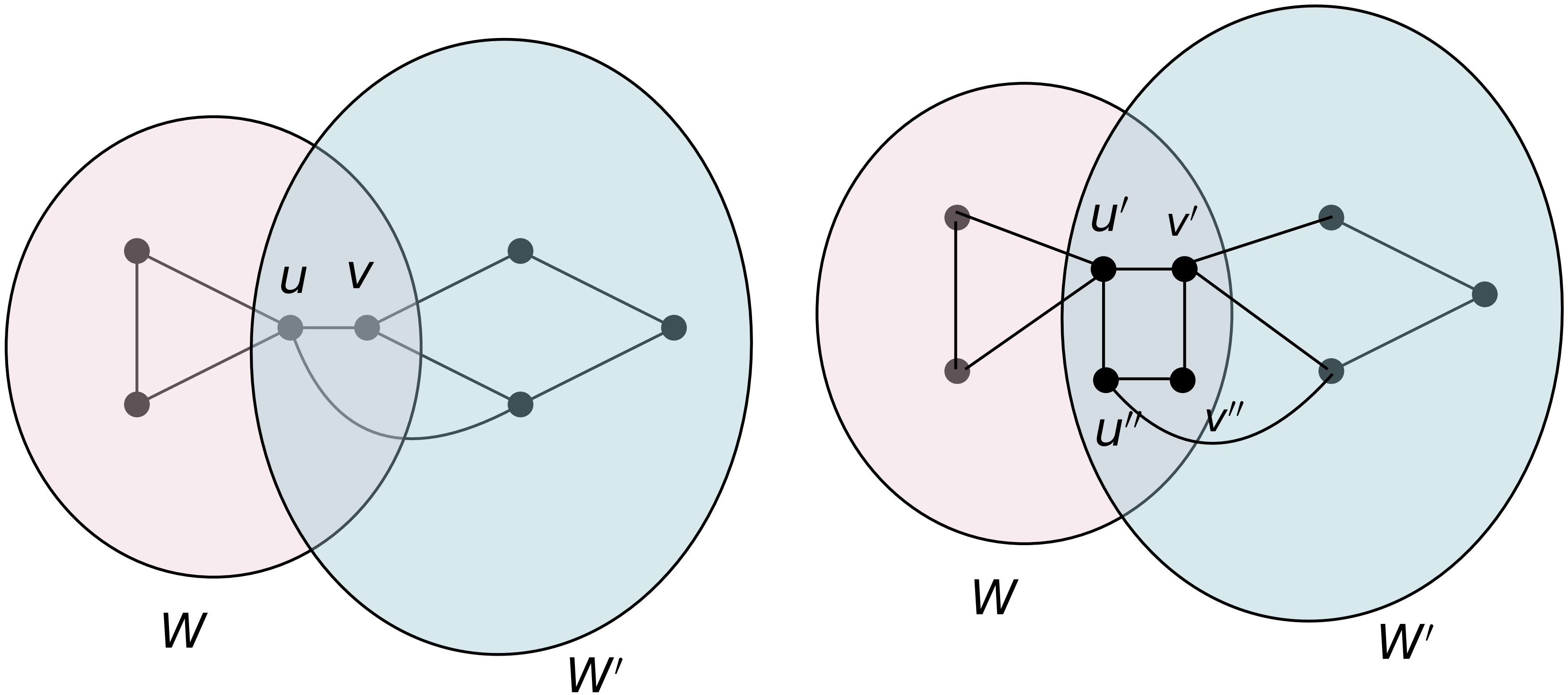

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G is median $\iff$ it can be obtained from K_1 by successive convex expansions.