Clustering Systems of Phylogenetic Networks

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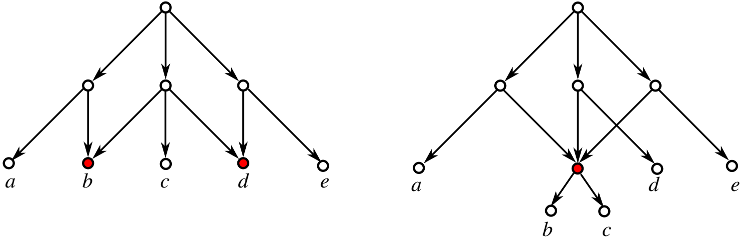
TBI Winterseminar, 2024

Basics

All networks $N = (V, E)$ considered here are

1. DAGs with a single root
2. phylogenetic (= no indegree 1 and outdegree 1 vertices)



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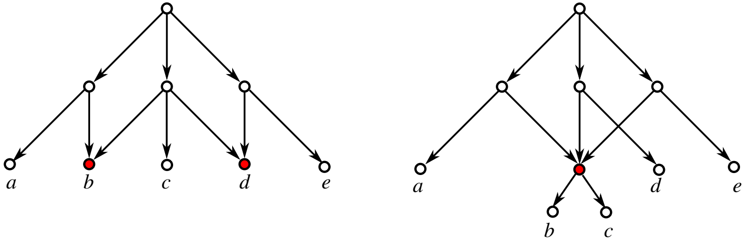

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(In)comparable vertices

$u \preceq_N v$ if v is an ancestor of u, i.e., there is a directed path from v to u.

u and v are $\preceq_N$-comparable if $u \preceq_N v$ or $v \preceq_N u$

Otherwise, u and v are $\preceq_N$-incomparable

Clustering systems

A **clustering system** on X is a set $\mathscr{C} \subseteq 2^X \setminus \{\emptyset\}$ such that $X \in \mathscr{C}$ and $\{x\} \in \mathscr{C}$ for all $x \in X$.

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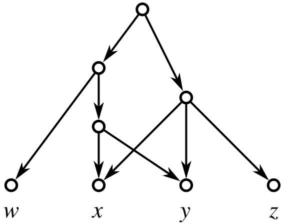

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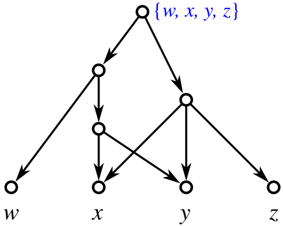

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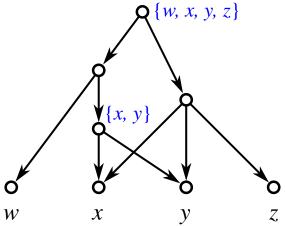

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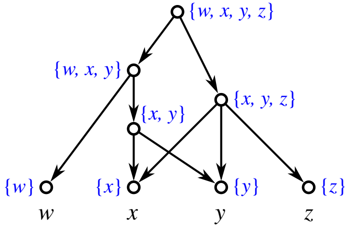

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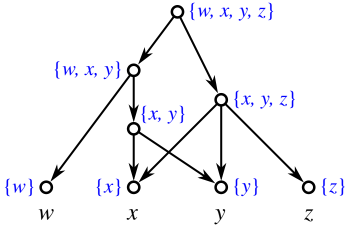

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How much information about N is contained in $\mathscr{C}_N$?

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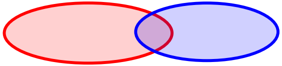
Folklore (Trees):

- $C, C' \in \mathscr{C}$ **overlap**, if $C \cap C' \notin \{C, C', \emptyset\}$.

  A clustering system is a **hierarchy** if it does not contain pairwise overlapping sets.



OVERLAP

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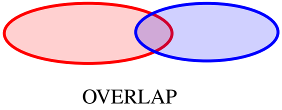
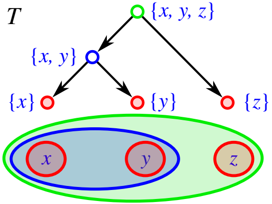
- **There is a 1-to-1 correspondence between rooted phylogenetic trees and hierarchies.**
  [keyword: Hasse-diagram]





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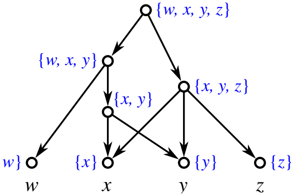

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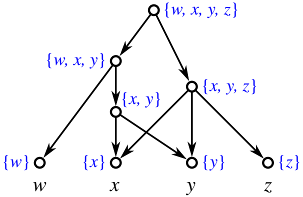

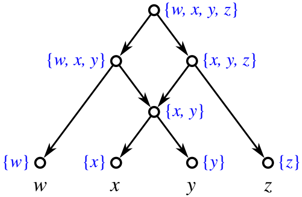

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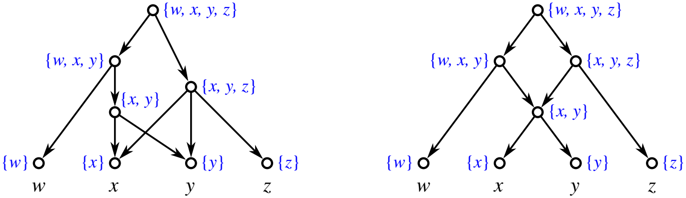
- *No* 1-to-1 correspondence between networks and clustering systems.



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  type = tree, level-k, binary, galled tree, tree-child, tree-based, regular, normal, . . .

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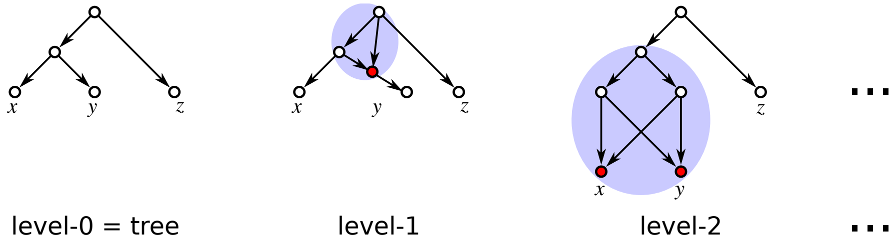
Let us try to answer some of the questions for level-1 networks.

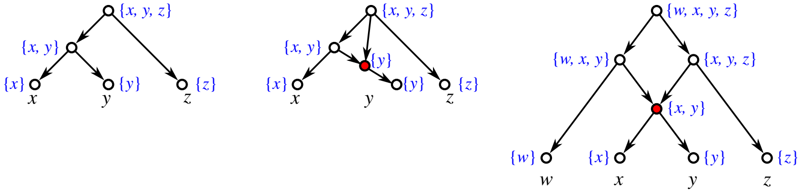
Level-k networks

A **block B** in a network is a maximal biconnected subgraph.

A network N is **level-k** if each block B in N contains $\leq k$ hybrid-vertices
(distinct from root ρ_B of B).



Level-1 Networks



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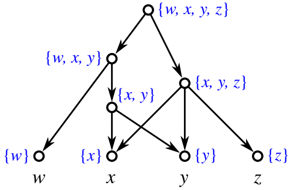

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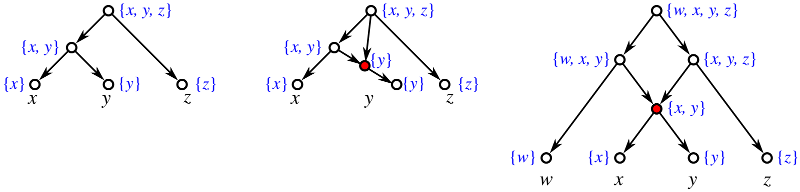

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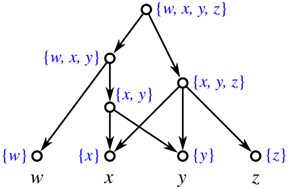

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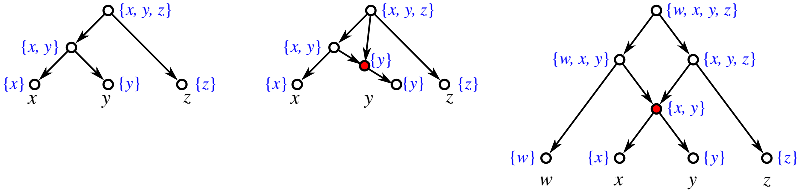

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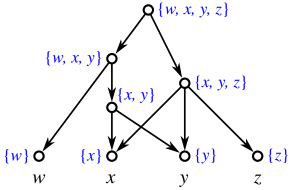

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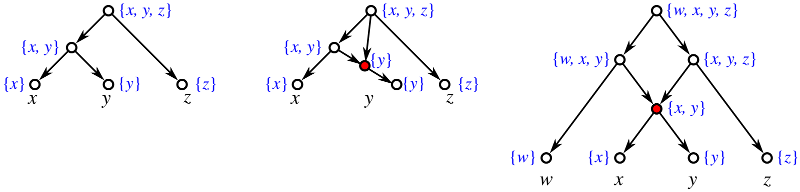

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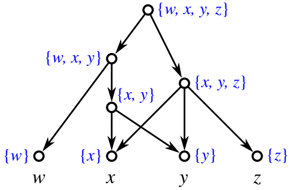

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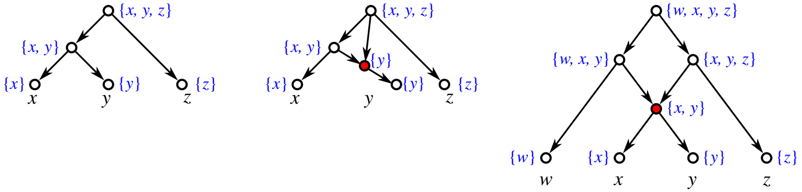

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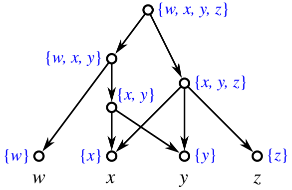

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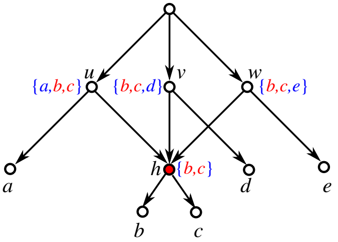
Lemma (H., Schaller, Stadler, 2023)

In level-1 networks N: u and v are $\preceq_N$-incomparable in N $\iff$ $C(u) \cap C(v) \in \{\emptyset, C(h_B)\}$ where $h_B \neq u, v$ is the unique hybrid in block B that contains u and v.

Property (L)

If $C(u)$ overlaps with $C(v)$, then $C(u) \cap C(v) = C(h_B)$ with B being the block containing u and v.

[this property cannot be observed when just looking at $\mathscr{C}_N$!]



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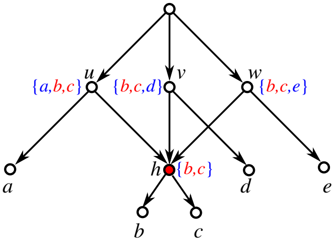

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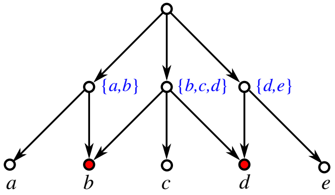

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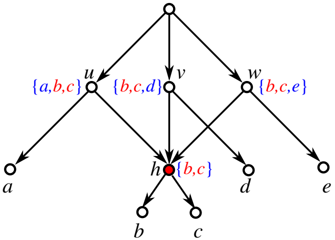

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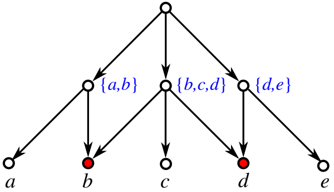

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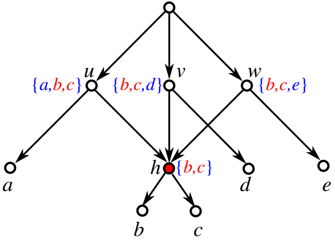

property (L)

Lemma (H., Schaller, Stadler, 2023)

For all level-1 networks N, the set $\mathscr{C}_N$ satisfies property (L).

Closed clustering systems

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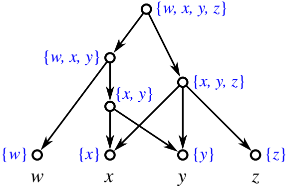

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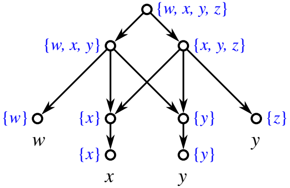

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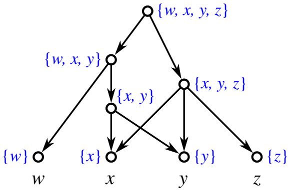

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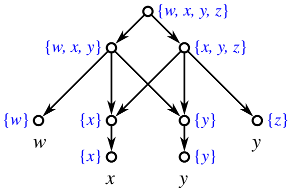

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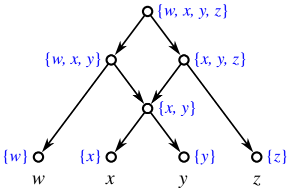

level-1 = closed

Lemma (H., Schaller, Stadler, 2023)

For level-1 networks N, the set $\mathscr{C}_N$ is always closed.

Characterization for level-1 Networks

A clustering system $\mathscr{C}$ ...

... is **closed** if $A \cap B \in \mathscr{C}$ for all $A, B \in \mathscr{C}$ with $A \cap B \neq \emptyset$.

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For $\mathscr{C}$, there is a level-1 network N with $\mathscr{C}_N = \mathscr{C} \iff \mathscr{C}$ is closed and satisfies (L).

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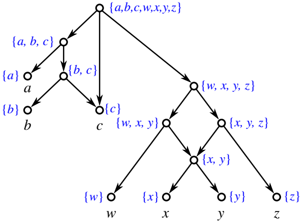
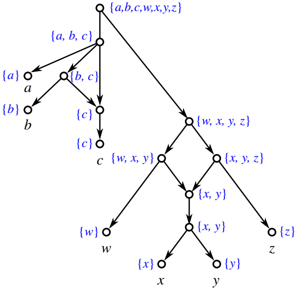
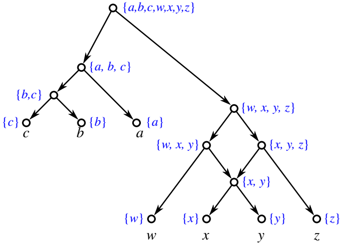
Recognition of such $\mathscr{C}$ and reconstruction of such a level-1 network can be done in polynomial time.

Uniqueness Results







Hasse diagram

In general, several different level-1 networks may represent the same $\mathscr{C}$

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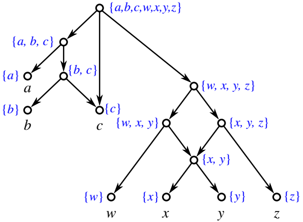

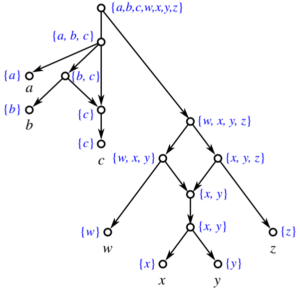

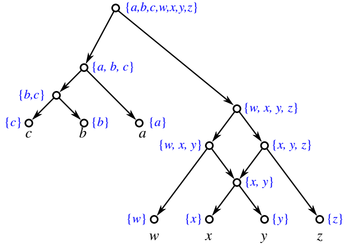

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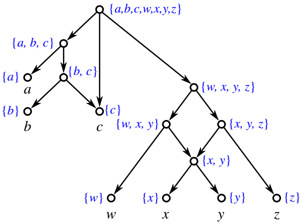

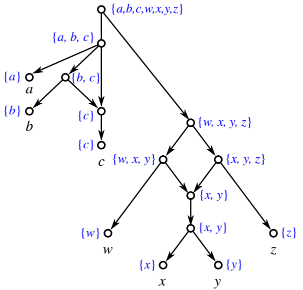

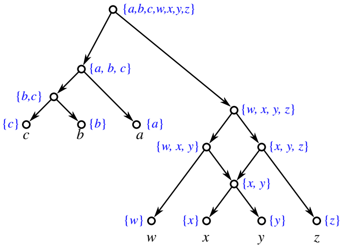

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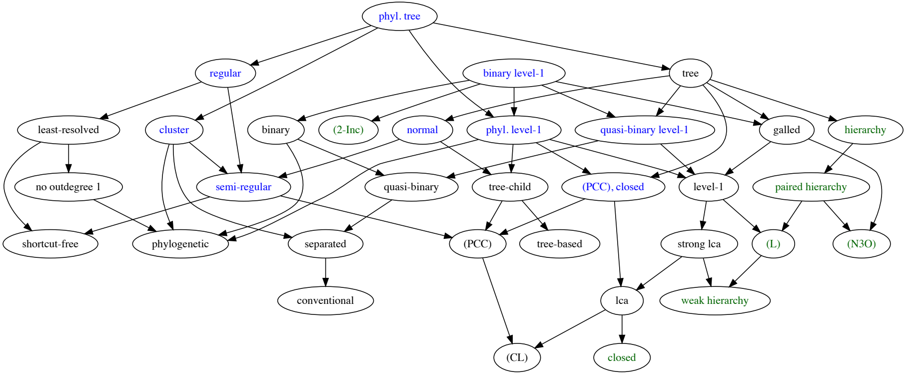
Every level-1 network N is a refinement (=adding shortcuts + expand vertices) of the Hasse-diagram of $\mathscr{C}_N$.

Other types of networks . . .



The latter results and plenty of other characterizations for dozens of other network types can be found in
*Hellmuth, Stadler, Schaller, **Clustering Systems of Phylogenetic Networks**, Theory in Biosciences (142), 301-358, 2023*

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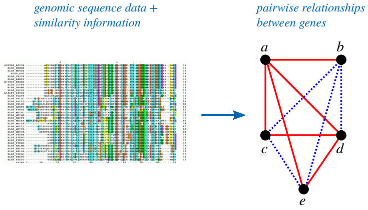

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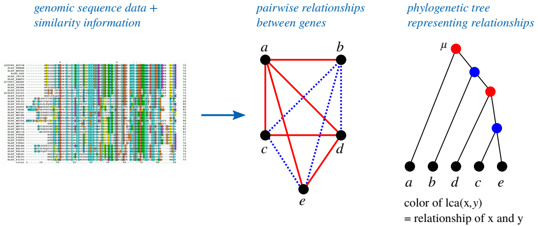

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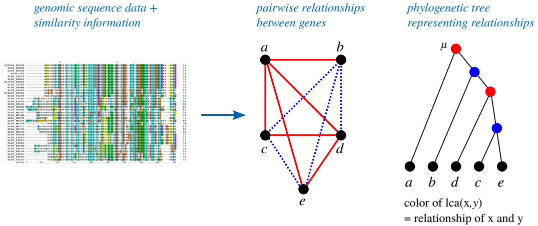

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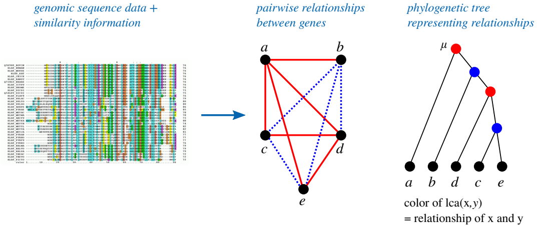

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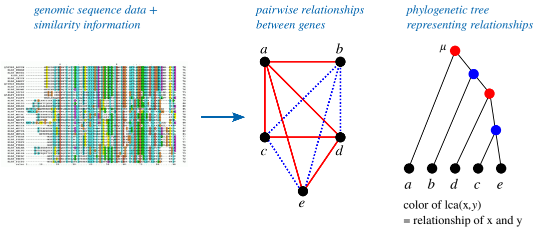

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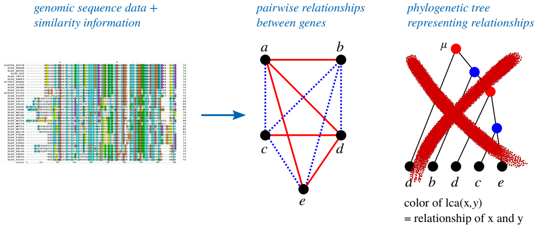

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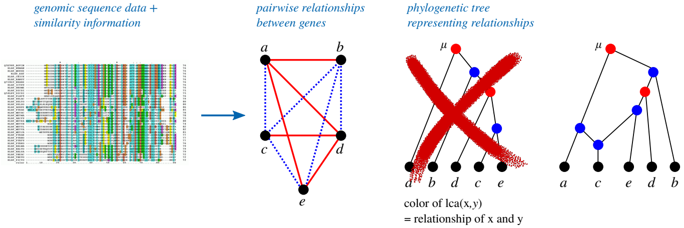

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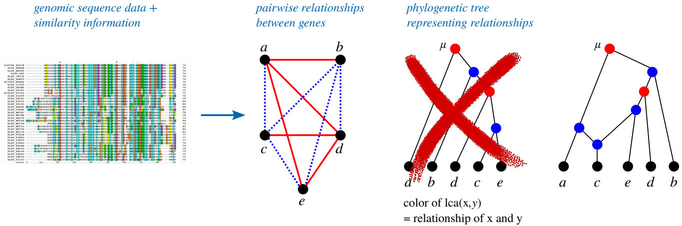

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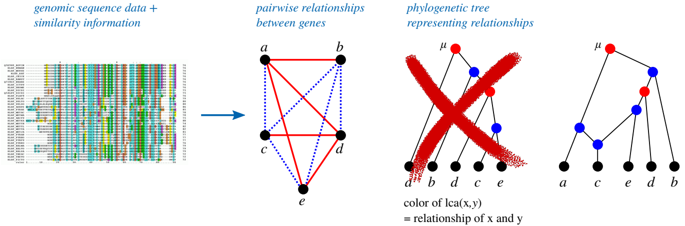

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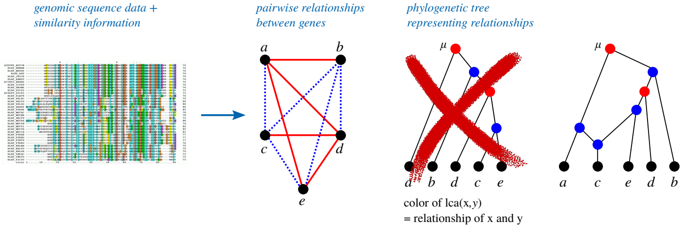

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A first starting point is provided by

Theorem (Shanavas, Changat, H., Stadler, 2024)

For $\mathscr{C}$, there is a network N with $\mathscr{C} = \mathscr{C}_N$ and pairwise lca*-property*
$\iff$ *$\mathscr{C}_N$ is pre-binary, i.e., there is a unique incl.-min. cluster $C \in \mathscr{C}$ such that $\{x,y\} \subseteq C$ for all $x,y \in L(N)$*

Shanavas, Changat, Hellmuth, Stadler ***Unique Least Common Ancestors and Clusters in Directed Acyclic Graphs****, LNCS vol 14508, 2024*

- David Schaller (Biontech, GER)
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Thanks!

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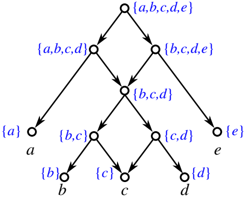

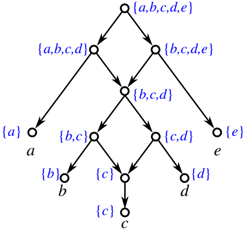

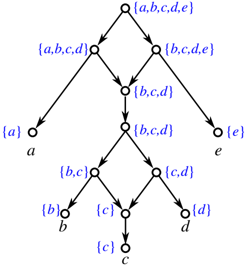

We obtain uniqueness for mild restrictions!

Theorem

Let $\mathscr{C}$ be a closed clustering system that satisfies (L). Then, there is a unique shortcut-free level-1 network N with $\mathscr{C}_N = \mathscr{C}$ that satisfies precisely one condition:

- *N contains no vertex v with outdegree 1 (=Hasse diagram)*
- *every leaf in N has indegree 1 and all vertices v with outdegree 1 are adjacent to leaves.*
- *every hybrid in N has outdegree 1 (thus all leaves have indegree 1).*

Further results

Galled tree = level-1 network where all non-trivial blocks correspond to “undirected cycles”

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*There is a galled tree N with $\mathscr{C}_N = \mathscr{C} \iff \mathscr{C}$ is **closed** and satisfies **Property (L)** and does not contain three distinct pairwise overlapping clusters.*

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Binary network = network where all non-hybrids v is either a leaf or has $\operatorname{outdeg}(v) = 2$, and every hybrid v satisfies $\operatorname{indeg}(v) = 2$ and $\operatorname{outdeg}(\mathrm{v}) = 1$.

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*$\mathscr{C}$ is compatible w.r.t. a level-1 network $N \iff \mathscr{C}$ is satisfies **Property (L)**.*

In this case, compute $A \cap B$ for all overlapping $A, B \in \mathscr{C}$ and add it to $\mathscr{C}$ if their intersection is not present. (can be done in polynomial time)

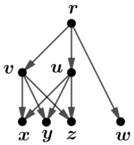

Fig. 2. Consider the DAG G with leaf set $X = L(G)$ where $\mathscr{C}_G = \{\{x\}, \{y\}, \{z\}, \{w\}, \{x, y, z\}, X\}$. Here, $\mathscr{C}_G$ satisfies **(KS)** and **(KC)** for $k = 2$. By definition, $\mathscr{C}_G$ is thus pre-binary. However, G is not a pairwise lca-network since $\operatorname{lca}(x, y)$, $\operatorname{lca}(x, z)$, and $\operatorname{lca}(y, z)$ are not defined. Moreover, $\mathscr{C}_G$ also satisfies **(KC)** for $k = 3$ but G is not a *3*-lca-network since $\operatorname{lca}(x, y, z)$ is not defined.

pre-binary clustering system but not pairwise lca-property