Higher Separation Axioms in Generalized Closure Spaces

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Abstract

The hierarchy of separation axioms that is familiar from topological spaces generalizes to spaces with an isotone and expansive closure function. Neither additivity nor idempotence of the closure function must be assumed.

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1. Introduction

Closure functions that are more general than the topological ones have been studied already by Hausdorff [18] and Day [5]. A thorough discussion is due to Hammer, see e.g., [16, 17], and more recently Gniłka [13, 14].

There has been increased interest in applications of generalized topological spaces, in particular in pattern recognition, image analysis, and related areas. Although the usual definitions of topology are generally not suited to the analysis of digital pictures, they are easily modified and yield generalized closure or neighborhood functions that fit the requirements of *digital topology*, see e.g. [24, 6, 10].

The search spaces in combinatorial chemistry as well as the sequence spaces underlying (molecular) evolution are conventionally thought of as graphs. Recombination, however, implies a non-graphical structure of the underlying spaces [12]. A unified view of combinatorial search spaces, fitness landscapes, evolutionary trajectories, and artificial chemistries based on generalized topologies is discussed in [8, 9, 28, 29, 27].

In this contribution we consider the higher separation axioms T3 to T5 and the associated concepts of Urysohn functions, regularity, and normality, which have so far not been studied in much detail in the framework of extended topologies. A discussion of higher separation axioms in the realm of semi-uniform convergence spaces can be found in [23]. Our main result is that the hierarchy of separation axioms that is familiar from topological spaces generalizes to extended topologies with isotonic, expansive closure functions, so-called neighborhood spaces; neither additivity nor idempotence of the closure function are necessary.

2. Preliminaries: Generalized Topologies

Let $\mathsf{cl} : \mathcal{P}(X) \to \mathcal{P}(X)$ be a set-valued set function which we call the *closure function*. Its conjugate is the *interior function* $\mathsf{int} : \mathcal{X} \to \mathcal{X}$ defined by

$$\mathsf{int}(A) = X \setminus \mathsf{cl}(X \setminus A)\,. \tag{1}$$

The associated *neighborhood function* $\mathcal{N} : X \to \mathcal{P}(\mathcal{P}(X))$ is defined by

$$\mathcal{N}(x) = \{N \subseteq X | x \in \mathsf{int}(N)\} \tag{2}$$

It is not hard to show that closure, interior, and neighborhood can be used to define each other. For instance, we have

$$x \in \mathsf{cl}(A) \iff (X \setminus A) \notin \mathcal{N}(x)\,. \tag{3}$$

A set A is *closed* if $A = \mathsf{cl}(A)$ and *open* if $A = \mathsf{int}(A)$. If A is closed then $X \setminus A$ is open and vice versa.

Table 1. Closure Axioms

	closure	neighborhood
K0	$\mathsf{cl}(\emptyset) = \emptyset$	$X \in \mathcal{N}(x)$
K1 isotone	$A \subseteq B \implies \mathsf{cl}(A) \subseteq \mathsf{cl}(B)$ $\mathsf{cl}(A \cap B) \subseteq \mathsf{cl}(A) \cap \mathsf{cl}(B)$ $\mathsf{cl}(A) \cup \mathsf{cl}(B) \subseteq \mathsf{cl}(A \cup B)$	$N \in \mathcal{N}(x)\,, N \subseteq N' \implies N' \in \mathcal{N}(x)$
K2 expansive	$A \subseteq \mathsf{cl}(A)$	$N \in \mathcal{N}(x) \Rightarrow x \in N$
K3 sub-linear	$\mathsf{cl}(A \cup B) \subseteq \mathsf{cl}(A) \cup \mathsf{cl}(B)$	$N', N'' \in \mathcal{N}(x) \implies N' \cap N'' \in \mathcal{N}(x)$
K4 idempotent	$\mathsf{cl}(\mathsf{cl}(A)) = \mathsf{cl}(A)$	$N \in \mathcal{N}(x) \iff \mathsf{int}(N) \in \mathcal{N}(x)$

The basic axioms for closure functions are compiled in Table 1 together with their neighborhood counterparts. The equivalence of closure and neighborhood versions of these conditions is well-known, see e.g. [11]. For instance, the various expressions for isotony listed in Table 1 can be found in [16, Lem.10]. For completeness we show here that the two formulations of (K4) are equivalent:

Proof. We have in general $x \in \mathsf{int}(\mathsf{int}(A))$ iff $\mathsf{int}(A) \in \mathcal{N}(x)$. Thus $\mathsf{int}(\mathsf{int}(A)) = \mathsf{int}(A)$ for all A if and only if $x \in \mathsf{int}(A)$ is equivalent to $\mathsf{int}(A) \in \mathcal{N}(x)$. On the other hand, we have $x \in \mathsf{int}(A)$ if and only if $A \in \mathcal{N}(x)$. Combining these two conditions we see that the closure function is idempotent if and only if $A \in \mathcal{N}(x)$ is equivalent to $\mathsf{int}(A) \in \mathcal{N}(x)$. $\square$

We say that (X, cl) is an *isotone space* if (K0) and (K1) is satisfied. If in addition (K2) holds then (X, cl) is a *neighborhood space*. Neighborhood spaces satisfying (K3) are the *pretopological spaces* studied in detail in [2]. An isotone and idempotent closure function corresponds to the intersection structures studied e.g. in lattice theory [26, 4]. In this case the intersection of an arbitrary number of closed sets is again a closed set. A neighborhood space with idempotent closure corresponds to a so-called topped intersection structure and is sometimes called a *convex closure space*. Finally, a pretopological space with idempotent closure is a *topological space* in the usual sense.

In the following we will need a few more basic results on generalized topological spaces which we compile here for later reference:

If (K1) holds we have the following expression for the closure function in terms of neighborhoods [5, Thm.3.1,Cor.3.2]

$$\mathsf{cl}(A) = \{x \in X | \forall N \in \mathcal{N}(x) : A \cap N \neq \emptyset\} \tag{4}$$

The notion of a neighborhood for an individual point can be extended naturally to sets: V is a neighborhood of A, in symbols $V \in \mathcal{N}(A)$, if $V \in \mathcal{N}(x)$ for all $x \in A$. One easily verifies for arbitrary closure functions that

$$V \in \mathcal{N}(A) \iff A \subseteq \mathsf{int}(V) \tag{5}$$

Let (X, cl) and (Y, cl) be two sets with arbitrary closure functions and let $f : X \to Y$. Then $f : X \to Y$ is *continuous* if the following equivalent conditions are satisfied:

(i) $\mathsf{cl}(f^{-1}(B)) \subseteq f^{-1}(\mathsf{cl}(B))$ for all $B \in \mathcal{P}(Y)$.
(ii) $f^{-1}(\mathsf{int}(B)) \subseteq \mathsf{int}(f^{-1}(B))$ for all $B \in \mathcal{P}(Y)$.
(iii) $B \in \mathcal{N}(f(x))$ implies $f^{-1}(B) \in \mathcal{N}(x)$ for all $B \in \mathcal{P}(Y)$ and all $x \in X$.

The equivalence of (i), (ii), and (iii) is stated (without the simple proof), e.g. in [15, Thm.3.1.]. The function $f : X \to Y$ is *closure preserving* if

(CP) for all $A \in \mathcal{P}(X)$ holds $f(\mathsf{cl}(A)) \subseteq \mathsf{cl}(f(A))$.

In isotone spaces, closure preservation and continuity are equivalent.

We will also need the "lower separation axioms":

(T0) For all $x, y \in X$, $x \neq y$ there is $N' \in \mathcal{N}(x)$ such that $y \notin N'$ or there is $N'' \in \mathcal{N}(y)$ such that $x \notin N''$.
(T0') $x \neq y$: $y \notin \mathsf{cl}(\{x\})$ or $x \notin \mathsf{cl}(\{y\})$
(T1) For all $x, y \in X$, $x \neq y$ there is $N' \in \mathcal{N}(x)$ and $N'' \in \mathcal{N}(y)$ such that $x \notin N''$ and $y \notin N'$.
(T1') $\mathsf{cl}(x) \subseteq \{x\}$
(T2) If $x \neq y$ then there is $N' \in \mathcal{N}(x)$ and $N'' \in \mathcal{N}(y)$ such that $N' \cap N'' = \emptyset$.
(T2$\frac{1}{2}$) If $x \neq y$ then there is $N' \in \mathcal{N}(x)$ and $N'' \in \mathcal{N}(y)$ such that $\mathsf{cl}(N') \cap \mathsf{cl}(N'') = \emptyset$.

Here (T0) and (T0') as well as (T1) and (T1') are equivalent in isotone spaces. In a neighborhood space (T2$\frac{1}{2}$) $\Longrightarrow$ (T2) $\Longrightarrow$ (T1) $\Longrightarrow$ (T0).

Furthermore, the following symmetry-like axioms are of interest:

(R0) If x is contained in each neighborhood of y then y is contained in each neighborhood of x.

(R0') If $x \in \mathsf{cl}(\{y\})$ then $y \in \mathsf{cl}(\{x\})$.
(S) If x is contained in each neighborhood of y then $\mathcal{N}(x) = \mathcal{N}(y)$.
(S') $\mathsf{cl}(A) \cap \mathsf{cl}(\{x\}) \neq \emptyset$ implies $x \in \mathsf{cl}(A)$.
(RE) $N_x \cap N_y \neq \emptyset$ for all $N_x \in \mathcal{N}(x)$ and all $N_y \in \mathcal{N}(y)$ implies $\mathcal{N}(x) = \mathcal{N}(y)$

The (R0) axiom was introduced by Šanin [25]. Čech proved that a pretopological space is semi-uniformizable if and only if it satisfies (R0') [2, Thm.23.B.3]. In [20] it is shown that (R0) is equivalent to "weak uniformizability" of convergence spaces. (R0) and (R0') are equivalent in all isotone spaces. The equivalence of (S) and (S') is shown in [2, Thm.29.A.3] for pretopological spaces. The argument easily extends to neighborhood spaces. Reciprocal spaces characterized by axiom (RE) were considered in [19], where this property was termed "axiom P". Finally, it is well known that (R0) and (S) are equivalent in topological spaces, see e.g. [22]. In isotone spaces we have (T1) $\iff$ (T0 and R0). In neighborhood spaces we have

$$\text{(RE)} \implies \text{(S)} \implies \text{(R0)} \qquad \text{and} \qquad \text{(T2)} \iff \text{(T0 and RE)} \tag{6}$$

Let (X, cl) be a closure space and $Y \subseteq X$. Then $c_Y : \mathcal{P}(Y) \to \mathcal{P}(Y)$, $A \mapsto Y \cap \mathsf{cl}(A)$ is the *relativization* of cl to Y. The pair (Y, c_Y) is a *subspace* of (X, cl). The closure c_Y is the finest closure function of Y such that the canonical embedding $j : Y \to (X, \mathsf{cl}), x \mapsto x$ is continuous. If $A \subseteq Y$ then the *relative interior* is given by

$$\mathsf{int}_Y(A) := Y \setminus c_Y(Y \setminus A) = Y \cap \mathsf{int}(A \cup (X \setminus Y)) \tag{7}$$

and the *relative neighborhoods* of A are

$$\mathcal{N}_Y(A) = \{N \cap Y | N \in \mathcal{N}(A)\}\,. \tag{8}$$

This can be seen e.g. following the lines of [2, 17.A].

A property $\mathfrak{P}$ of space (X, cl) is *hereditary* if every subspace (Y, c_Y) also has property $\mathfrak{P}$. It is trivial to verify that (K0), (K1), (K2), and (K3) are hereditary in arbitrary closure spaces. Furthermore, idempotency of the closure (K4) is hereditary in neighborhood spaces:

$$c_Y(c_Y(A)) \subseteq c_Y(\mathsf{cl}(A)) = Y \cap \mathsf{cl}(\mathsf{cl}(A)) = Y \cap \mathsf{cl}(A) = c_Y(A) \subseteq c_Y(c_Y(A))$$

Note that we have used (K1) and (K2), respectively, for the two inclusions.

Lemma 1. *The lower separation axioms* (T0)*,* (T1)*,* (T2)*,* (T2$\frac{1}{2}$)*, and the symmetry axioms* (R0)*, and* (S') *are hereditary in isotone spaces.*

Proof. For the axioms (T0), (T1), (R0), and (T2) we simply notice that $N \in \mathcal{N}(x)$ implies $N \cap Y \in \mathcal{N}_Y(x)$.
Suppose $x, y \in Y$, $x \neq y$ and (T2$\frac{1}{2}$) holds in (X, cl). Then there are neighborhoods $N' \in \mathcal{N}(x)$ and $N'' \in \mathcal{N}(y)$ such that $\mathsf{cl}(N') \cap \mathsf{cl}(N'') = \emptyset$. The sets $\tilde{N}' = N' \cap Y$ and $\tilde{N}'' = N'' \cap Y$ are neighborhoods of x and y in Y, respectively. $c_Y(\tilde{N}') = Y \cap \mathsf{cl}(\tilde{N}') \subseteq \mathsf{cl}(\tilde{N}') \subseteq \mathsf{cl}(N')$ and $c_Y(\tilde{N}'') \subseteq \mathsf{cl}(N')$, i.e., $\tilde{N}' \cap \tilde{N}'' = \emptyset$. Thus (T2$\frac{1}{2}$) holds in (Y, c_Y).
For (S') we argue as follows: Let $x \in Y$, $A \subseteq Y$ and $x \notin c_Y(A) = \mathsf{cl}(A) \cap Y$. Then $x \notin \mathsf{cl}(A)$ and (S') on X implies $\mathsf{cl}(x) \cap \mathsf{cl}(A) = \emptyset$ and hence $c_Y(x) \cap c_Y(A) = \emptyset$, i.e. (S') holds on Y as well. $\square$

3. Semi-Separated and Separated Sets

Lemma 2. *In an isotone space* (X, cl) *the following two conditions are equivalent for all* $A, B \subseteq X$*.*

(SS) $\mathsf{cl}(A) \cap B = A \cap \mathsf{cl}(B) = \emptyset$.
(SS') *There is* $U \in \mathcal{N}(A)$ *and* $V \in \mathcal{N}(B)$ *such that* $A \cap V = U \cap B = \emptyset$.

Proof. If (SS') holds then $\exists U, V$ such that $A \subseteq \mathsf{int}(U)$, $B \subseteq \mathsf{int}(V)$, $A \cap V = \emptyset$, and $U \cap B = \emptyset$. We argue $A \cap V = \emptyset$ implies $V \subseteq X \setminus A$ implies $B \subseteq \mathsf{int}(V) \subseteq \mathsf{int}(X \setminus A)$ by isotony, hence $X \setminus \mathsf{int}(X \setminus A) = \mathsf{cl}(A) \subseteq X \setminus B$ and thus $\mathsf{cl}(A) \cap B = \emptyset$. The same argument yields $A \cap \mathsf{cl}(B) = \emptyset$.
Now suppose (SS). We have $\mathsf{cl}(A) \cap B = \emptyset$ iff $B \subseteq X \setminus \mathsf{cl}(A) = \mathsf{int}(X \setminus A)$, i.e., $X \setminus A \in \mathcal{N}(B)$. Thus there is $V \in \mathcal{N}(B)$ such that $A \cap V = \emptyset$. Analogously, we have $A \cap \mathsf{cl}(B) = \emptyset$ iff $X \setminus B \in \mathcal{N}(A)$, i.e., there is $U \in \mathcal{N}(A)$ with $U \cap B = \emptyset$. □

Two subsets $A, B \subseteq X$ satisfying (SS) are called *semi-separated*.

Lemma 3. *Let* $A, B \subseteq Y \subseteq X$*. Then* A *and* B *are semi-separated in* (X, cl) *if and only if* A *and* B *are semi-separated in* (Y, c_Y)*.*

Proof. $A \cap \mathsf{cl}(B) = \emptyset$ implies $A \cap \mathsf{cl}(B) \cap Y = A \cap c_Y(B) = \emptyset$ and $\mathsf{cl}(A) \cap B = \emptyset$ implies $c_Y(A) \cap B = \emptyset$.
Conversely, assume $A \cap c_Y(B) = \emptyset$. We have $A \cap c_Y(B) = A \cap Y \cap \mathsf{cl}(B) = A \cap \mathsf{cl}(B)$ since $A \subseteq Y$. It follows that $A \cap \mathsf{cl}(B) = \mathsf{cl}(A) \cap B = \emptyset$. □

Two sets A and B are called *separated* if there is $U \in \mathcal{N}(A)$ and $V \in \mathcal{N}(B)$ such that $U \cap V = \emptyset$.

It is clear that if A and B are separated in (X, cl) and $A, B \subseteq Y$ then A and B are separated in (Y, c_Y).

4. Urysohn Functions

In the following we will denote by $\mathbb{R}$ and $[0, 1]$ the real numbers and the closed unit interval with the usual topology, resp.

Definition 4. *The function* $\upsilon : X \to [0, 1]$ *is an Urysohn function separating* A *and* B *if* υ *is continuous,* $\upsilon(A) \subseteq \{0\}$ *and* $\upsilon(B) \subseteq \{1\}$*. The sets* A *and* B *are* Urysohn-separated *if there is an Urysohn function separating* A *and* B*. We write* $A \|_\upsilon B$ *and* $A \| B$*, respectively.*

We have $A \|_\upsilon B$ if and only if $B \|_{1-\upsilon} A$. Thus $\|$ is a symmetric relation. Furthermore, $A \| \emptyset$ because $\emptyset \|_1 A$, where $1 : X \to [0, 1]$ is the constant function $1(x) = 1$.

Definition 5. *A set* A *is* completely within B*,* $A \Subset B$*, if there is a continuous function* $\upsilon : X \to [0, 1]$ *such that* $\upsilon(A) \subseteq \{0\}$ *and* $\upsilon(X \setminus B) \subseteq \{1\}$*.*

By definition we have $A \Subset B$ iff $A \| X \setminus B$. Clearly, $A \Subset B$ implies $X \setminus B \Subset X \setminus A$ and $A \subseteq B$.

Theorem 6. *If* (X, cl) *is an isotone space then* $A \Subset B$ *implies* $A \cup \mathsf{cl}(A) \Subset \mathsf{int}(B) \cap B$*.*

Proof. If $A = \emptyset$ or $B = X$ the claim is obviously true since we assume $\mathsf{cl}(\emptyset) = \emptyset$. In the following we may therefore use $A \neq \emptyset$ and $B \neq X$.
By assumption there is a function υ such that $\upsilon(A) = \{0\}$, $\upsilon(X \setminus B) = \{1\}$, and υ is continuous, i.e., for every $\epsilon > 0$ and every $x \in X$ there is $N \in \mathcal{N}(x)$ such that $\upsilon(N) \subseteq I_\epsilon(\upsilon(x)) := (\upsilon(x)-\epsilon, \upsilon(x)+\epsilon) \cap [0,1]$. Now suppose $x \in \mathsf{cl}(A)$, i.e., $N \cap A \neq \emptyset$ for all $N \in \mathcal{N}(x)$. (Here we use that (X, cl) is isotone). Hence $\upsilon(A) = \{0\}$ implies $0 \in \upsilon(N) \subseteq I_\epsilon(\upsilon(x))$ and consequently $0 \leq \upsilon(x) < \epsilon$. Thus $\upsilon(x) = 0$. Analogously, if $y \in \mathsf{cl}(X \setminus B)$, i.e., $N \cap X \setminus B \neq \emptyset$ for all $N \in \mathcal{N}(y)$, we have $1 \in \upsilon(N) \subseteq I_\epsilon(\upsilon(y))$ and hence $1 - \epsilon < \upsilon(y) \leq 1$ for all $\epsilon > 0$, hence $\upsilon(y) = 1$. Thus $\upsilon(A \cup \mathsf{cl}(A)) = \{0\}$ and $\upsilon(X \setminus B \cup \mathsf{cl}(X \setminus B)) = \{1\}$, i.e., $A \cup \mathsf{cl}(A) \,\|_\upsilon\, (X \setminus B) \cup \mathsf{cl}(X \setminus B)$. Therefore we have $A \cup \mathsf{cl}(A) \Subset X \setminus [(X \setminus B) \cup \mathsf{cl}(X \setminus B)] = [X \setminus (X \setminus B)] \cap [X \setminus \mathsf{cl}(X \setminus B)] = B \cap \mathsf{int}(B)$. ☐

Let $Y \subseteq X$ and suppose $A, B \subseteq Y$ are separated by the Urysohn function υ in X. The restriction $\upsilon_Y : Y \to [0,1], x \mapsto \upsilon(x)$ is an Urysohn function on the subspace (Y, c_Y). We only have to verify that υ_Y is continuous. To see this let $H \subseteq Y$ and consider $\upsilon_Y(c_Y(H)) = \upsilon(\mathsf{cl}(H) \cap Y) \subseteq \upsilon(\mathsf{cl}(H)) \subseteq \overline{\upsilon(H)} = \overline{\upsilon_Y(H)}$ where $\overline{T}$ denotes the standard topological closure in $[0,1]$.

Definition 7. *A closure space* (X, cl) *is an* Urysohn-*space, or (T2U) if for any two distinct points* $x \neq y \in X$ *are Urysohn separated,* $\{x\} \| \{y\}$.

It is clear from the discussion above that (T2U) is a hereditary property.

Lemma 8. *Every isotone Urysohn space satisfies (T2$\frac{1}{2}$).*

Proof. By assumption there is a continuous function $\upsilon : X \to [0,1]$ with $\upsilon(x) = 0$ and $\upsilon(y) = 1$. Thus for each $\epsilon > 0$ there are neighborhoods $N_x \in \mathcal{N}(x)$ and $N_y \in \mathcal{N}(y)$ such that $\upsilon(N_x) \subseteq [0, \epsilon)$ and $\upsilon(N_y) \subseteq (1-\epsilon, 1]$. Thus $\upsilon(\mathsf{cl}(N_x)) \subseteq [0, \epsilon]$ and $\upsilon(\mathsf{cl}(N_y)) \subseteq [1-\epsilon, 1]$, i.e., $\mathsf{cl}(N_x) \cap \mathsf{cl}(N_y) = \emptyset$ for $\epsilon < 1/2$. ☐

5. Regular and Completely Regular Spaces

Theorem 9. *In an isotone space the following conditions are equivalent:*

- (R) *For all* $x \in X$ *and all* $N \in \mathcal{N}(x)$ *there is* $U \in \mathcal{N}(x)$ *such that* $\mathsf{cl}(U) \subseteq N$.
- (R') *For all* $x \in X$ *and all non-empty* $A \in \mathcal{P}(X)$ *such that* $x \notin \mathsf{cl}(A)$ *there is* $U \in \mathcal{N}(x)$ *and* $V \in \mathcal{N}(A)$ *such that* $U \cap V = \emptyset$.

Proof. Suppose (R') holds. Choose an arbitrary $x \in X$ and $N \in \mathcal{N}(x)$ and set $A = -N$. We have $x \notin \mathsf{cl}(A)$ iff $-A = N \in \mathcal{N}(x)$. Now we have $U \cap V = \emptyset$, hence $U \subseteq -V$ and by isotony $\mathsf{cl}(U) \subseteq \mathsf{cl}(-V)$ and finally $\mathsf{int}(V) = -\mathsf{cl}(-V) \subseteq -\mathsf{cl}(U)$. By eq. 5 we have $A \subseteq \mathsf{int}(V)$, thus $A \subseteq -\mathsf{cl}(U)$, and $\mathsf{cl}(U) \subseteq -A = N$. Thus (R) is satisfied.
Conversely assume (R), and let $N \in \mathcal{N}(x)$. Then there is $U \in \mathcal{N}(x)$ such that $\mathsf{cl}(U) \subseteq N$. Set $V = -U$ and $A = -N$. Then $-N = A \subseteq -\mathsf{cl}(U) = -\mathsf{cl}(-V) = \mathsf{int}(V)$, i.e., $V \in \mathcal{N}(A)$ and $U \cap V = \emptyset$. Observing again that $N \in \mathcal{N}(x)$ if and only if $x \notin \mathsf{cl}(A)$ completes the proof. ☐

Definition 10. *An isotone space is* regular *if it satisfies one of the conditions* (R) *or* (R') *in theorem 9.*

It is worth noting that condition (R) naturally appears in the theory of generalized convergence spaces (see e.g. [7, 3]), while (R') is the straightforward generalization of the usual regularity axiom in topological spaces.

Regularity is hereditary in isotone spaces. This can be verified by observing that the arguments in [2, 27.B.8] are valid in all isotone spaces.

A stronger property is

(tR) For all $x \in X$ and all non-empty closed sets $\emptyset \neq A = \mathsf{cl}(A) \in \mathcal{P}(X)$ there is $U \in \mathcal{N}(x)$ and $V \in \mathcal{N}(A)$ such that $U \cap V = \emptyset$.

It is clear that (tR) implies (R). Obviously (R) and (tR) are equivalent if cl is idempotent.

Definition 11. *A closure space is* (T3) *if it satisfies* (R) *and* (T0)*.*

Obviously, (T3) is a hereditary property in isotone spaces.

Lemma 12. *If* (X, cl) *is a neighborhood space then* (T3) $\Longrightarrow$ (T2$\frac{1}{2}$) $\Longrightarrow$ (T2)*.*

Proof. We start with (R') and set $A = \{y\}$. Since (R) implies (R0) we know that a (T3) space is (T1), hence $\mathsf{cl}(A) = \mathsf{cl}(\{y\}) = \{y\}$. Thus (R') reduces to the existence of $N' \in \mathcal{N}(x)$ and $N'' \in \mathcal{N}(y)$ such that $N' \cap N'' = \emptyset$, i.e., to (T2). Now we can use (R) to obtain $U' \in \mathcal{N}(x)$ and $U'' \in \mathcal{N}(y)$ with $\mathsf{cl}(U') \subseteq N'$ and $\mathsf{cl}(U'') \subseteq N''$. Clearly $\mathsf{cl}(U') \cap \mathsf{cl}(U'') = \emptyset$, i.e., (T2$\frac{1}{2}$) is satisfied. □

Definition 13. *An isotone space is* completely regular *if for all* $x \in X$ *and all* $N \in \mathcal{N}(x)$ *there is* $N' \in \mathcal{N}(x)$ *such that* $N' \Subset N$*. An isotone space is* (T3$\frac{1}{2}$) *if it is* (T1) *and completely regular.*

Completely regular filter convergence spaces are discussed in detail in [1].

It follows directly from the definition that a (T3$\frac{1}{2}$) space is an Urysohn space. Thm. 6 and (R) together immediately imply that a completely regular isotone space is regular. Hence (T3$\frac{1}{2}$) implies (T3) in isotone spaces.

Lemma 14. *Complete regularity, and hence* (T3$\frac{1}{2}$)*, are hereditary properties in isotone spaces.*

Proof. $N' \Subset N$ means $N' \|_{\upsilon} X \setminus N$ such that $\upsilon(N') \subseteq \{0\}$, $\upsilon(N) \subseteq \{1\}$. Now consider $N' \cap Y \subseteq N'$, and $(X \setminus N) \cap Y = Y \setminus (N \cap Y) \subseteq X \setminus N$. Thus $\upsilon(N' \cap Y) \subseteq \{0\}$, and $\upsilon(Y \setminus (N \cap Y)) \subseteq \{1\}$. Since the restriction of υ to Y is continuous this implies $N' \cap Y \Subset N \cap Y$ on Y, i.e., the subspace (Y, c_Y) is completely regular. □

Theorem 15. *A completely regular neighborhood space has idempotent closure.*

Proof. $N' \Subset N$ implies $N' \subseteq \mathsf{int}(N)$ by theorem 6 and by (K1) $N' \in \mathcal{N}$ implies $\mathsf{int}(N) \in \mathcal{N}$. By (K2) $\mathsf{int}(N) \subseteq N$, hence $N \in \mathcal{N}(x)$ if and only if $\mathsf{int}(N) \in \mathcal{N}(x)$. This is equivalent to the idempotence of the closure. □

It follows immediately that a completely regular neighborhood space satisfies (tR).

6. Normal Spaces

Definition 16. *A closure space* (X, cl) *is*

(tN) t-normal *if any two non-empty disjoint closed sets are separated.*
(QN) quasi-normal *if, for all non-empty sets* $A, B \in \mathcal{P}(X)$ *satisfying* $\mathsf{cl}(A) \cap \mathsf{cl}(B) = \emptyset$*, there are neighborhoods* $U \in \mathcal{N}(A)$ *and* $V \in \mathcal{N}(B)$ *such that* $U \cap V = \emptyset$*;*
(N) normal *if, for all non-empty sets* $A, B \in \mathcal{P}(X)$ *satisfying* $\mathsf{cl}(A) \cap \mathsf{cl}(B) = \emptyset$*, there are neighborhoods* $U \in \mathcal{N}(\mathsf{cl}(A))$ *and* $V \in \mathcal{N}(\mathsf{cl}(B))$ *such that* $U \cap V = \emptyset$*;*
(UN) Urysohn-normal *if, for all non-empty sets* $A, B \in \mathcal{P}(X)$ *satisfying* $\mathsf{cl}(A) \cap \mathsf{cl}(B) = \emptyset$*, there is an Urysohn function separating* A *and* B*, i.e.* $A \| B$*.*

The condition for quasi-normality appears in Čech's book [2, 29.A.]. There a pretopological space is called "normal" if it satisfies (S) and (QN).

Normal neighborhood und Urysohn-normal spaces (there termed "completely normal") are considered by Thampuran [30]. In [21] a notion of normality for filter converge spaces is considered that corresponds to the axiom (tN) above.

Theorem 17. *If* (X, cl) *is an isotone space then* (UN) $\implies$ (N) $\implies$ (tN) *and* (QN) $\implies$ (tN)*.*
If (X, cl) *is a neighborhood space then* (UN) $\implies$ (N) $\implies$ (QN) $\implies$ (tN)*.*

Proof. We will first show that Urysohn-normal implies normal. By assumption, for each $x \in \mathsf{cl}(A)$ there is $N_x \in \mathcal{N}(x)$ such that $\upsilon(N_x) < \varepsilon$. Then $U = \cup_{x \in \mathsf{cl}(A)} N_x$ is a neighborhood of $\mathsf{cl}(A)$) and $\upsilon(U) \leq \varepsilon$. Analogously, we can find a neighborhood $V \in \mathcal{N}(\mathsf{cl}(B))$ such that $\upsilon(V) \geq 1 - \varepsilon$. Setting $\varepsilon < 1/2$ implies that $U \cap V = \emptyset$, hence $\mathsf{cl}(A)$ and $\mathsf{cl}(B)$ are separated. The implications (N) $\implies$ (tN) and (QN) $\implies$ (tN) are obvious.
Now suppose that (X, cl) is a neighborhood space. We have $U \in \mathcal{N}(\mathsf{cl}(A)) \iff \mathsf{cl}(A) \subseteq \mathsf{int}(U)$. From (K2) we conclude $A \subseteq \mathsf{cl}(A) \subseteq \mathsf{int}(U)$ and hence $U \in \mathcal{N}(A)$. Thus (N) $\implies$ (QN). □

Lemma 18. *If* (X, cl) *is an neighborhood space with an idempotent closure (i.e., a convex closure space) then* (N)*,* (QN)*, and* (tN) *are equivalent.*

Proof. It suffices to show that for idempotent closures (tN) implies (N). If (K4) holds then $\mathsf{cl}(A)$ and $\mathsf{cl}(B)$ are closed sets. Hence if $\mathsf{cl}(A) \cap \mathsf{cl}(B) = \emptyset$ they are separated by (tN) and hence (N) holds. □

An important result for pretopological spaces is the following

Proposition 19. [2, 29.A.4] *If* (X, cl) *is a pretopological space then* (QN and S) $\iff$ (K4 and TN and R0)*.*

The following result is the famous Urysohn lemma which was generalized to pretopological spaces by Thampuran:

Proposition 20. [30] *If* (X, cl) *is a pretopological space then it is Urysohn-normal if and only if it is normal.*

Urysohn's lemma fails in more general neighborhood spaces. An example is *Thampuran's Line* $\mathbb{T}$ which is normal but not Urysohn-normal.

Example. For any real number x, let $\mathcal{R}(x)$ be the family of all subsets N of the real numbers $\mathbb{R}$ such that $\{y : y < v\} \subseteq N$ for some $v > x$ or $\{y : u < y\} \subseteq N$ for some $u < x$. For a subset A of $\mathbb{R}$, let

$$\mathsf{cl}(A) = \{y \in \mathbb{R} | N \cap A \neq \emptyset \text{ for all } N \in \mathcal{R}(y)\} \tag{9}$$

The closure space $\mathbb{T} = (\mathbb{R}, \mathsf{cl})$ is called *Thampuran's Line.*

We have $z \in \mathsf{cl}(A)$ iff for all $N \in \mathcal{R}(z)$ we have $N \cap A \neq \emptyset$, thus

$$\mathsf{cl}(A) = \begin{cases} [\inf A, \sup A] & \text{if} \quad \inf A \in \mathbb{R} \quad \text{and} \quad \sup A \in \mathbb{R} \\ (-\infty, \sup A] & \text{if} \quad \nexists \inf A \quad \text{and} \quad \sup A \in \mathbb{R} \\ [\inf A, +\infty) & \text{if} \quad \inf A \in \mathbb{R} \quad \text{and} \quad \nexists \sup A \\ \mathbb{R} & \text{if} \quad \nexists \inf A \quad \text{and} \quad \nexists \sup A \\ \emptyset & \text{if} \quad A = \emptyset \end{cases} \tag{10}$$

Note that $\mathcal{R}(x)$ is a neighborhood basis of x.

It is clear therefore that (K0), (K1), and (K2) are satisfied. Furthermore, cl is idempotent. However, the closure is not additive as the example of a set with two points $A = \{x, y\}$ shows. We have $\mathsf{cl}(A) = [x, y]$ but $\mathsf{cl}(\{x\}) = \{x\}$ and $\mathsf{cl}(\{y\}) = \{y\}$. Thampuran's line is therefore a convex closure space.

Let A and B be two disjoint non-empty closed sets. Then $\sup A, \inf B \in \mathbb{R}$. We set $\alpha = (\sup A + \inf B)/2$, $N' = \{z | z < \alpha\}$, and $N'' = \{z | z < \alpha\}$. We have $\mathsf{int}(N') = \mathbb{R} \setminus \mathsf{cl}(\mathbb{R} \setminus N') = \mathbb{R} \setminus \mathsf{cl}(\{z | z \geq \alpha\}) = \mathbb{R} \setminus \{z | z \geq \alpha\} = \{z | z < \alpha\} = N'$. Since $A \subseteq \mathsf{int}(N')$ we have $N' \in \mathcal{N}(A)$ and analogously $N'' \in \mathcal{N}(B)$. By construction $N' \cap N'' = \emptyset$, i.e., Thampuran's line satisfies (tN). From Lemma 18 we see that $\mathbb{T}$ is normal.

On the other hand, there is no Urysohn function separating sets A and B as above. Suppose $\upsilon : \mathbb{T} \to [0, 1]$ (with the standard topology) is continuous and $\upsilon(A) = 0$, $\upsilon(B) = 1$. If $\upsilon(x) = y$ with $y \neq 0, 1$ then, for all $\epsilon > 0$, there is a neighborhood N of x such that $\upsilon(N) \subseteq (y - \epsilon, y + \epsilon)$. By definition each neighborhood contains a set from $\mathcal{R}$ thus either A or B is a subset of N, i.e., either $y = 0$ or $y = 1$. Now suppose $\upsilon(x) = 0$ and for all $\eta > 0$ there is a $y \in (x - \eta, x + \eta)$ such that $\upsilon(y) = 1$. Since $N = (-\infty, x + \eta)$ is a neighborhood of x we see that $\upsilon(N) < \epsilon$ for sufficiently small η and any given $\epsilon > 0$, and hence even $\upsilon(N) = 1$. Thus a continuous function υ cannot "switch" from 0 to 1 anywhere, i.e., there is no Urysohn function separating sets A and B.

$\triangleleft$

It is shown in [30] that a neighborhood space (X, cl) is normal if and only if for any two nonempty sets A, B with disjoint closures $\mathsf{cl}(A) \cap \mathsf{cl}(B) = \emptyset$ there is a continuous function $\tau : (X, \mathsf{cl}) \to \mathbb{T} \cap [0, 1]$ such that $\tau(A) = \{0\}$ and $\tau(B) = \{1\}$.

7. Completely Normal Spaces

Definition 21. *A closure space* (X, cl) *is*

(CN) completely normal *if any two semi-separated sets are separated.*

(T5) *if it is (T1) and completely normal.*
(CNU) completely Urysohn-normal *if each pair of semi-separated sets is Urysohn-separated.*
(T5U) *if it is (T1) and completely Urysohn- normal.*

Both (CN) and (CNU), and therefore also (T5) and (T5U), are hereditary.

Lemma 22. (CNU) $\implies$ (CN) $\implies$ (QN)*, and* (UN) $\implies$ (N).

Proof. The implications (CNU) $\implies$ (CN) and (CN) $\implies$ (QN) follow directly from the definitions. In order to see (CNU) $\implies$ (N) we recall that $A\|B$ implies $\mathsf{cl}(A)\|\mathsf{cl}(B)$ which in turn implies that A and B are separated. $\square$

Every subspace of a completely normal neighborhood space is normal. In topological spaces the converse it true as well, i.e., if every subspace of (X, cl) is normal then (X, cl) is completely normal. This appears not to be true in general neighborhood spaces because the proof of this result, e.g. in [2, 30.A.4], depends on the observation that $Y = X \setminus (\mathsf{cl}(A) \cap \mathsf{cl}(B))$ is an open set in X but not on the additivity of the closure. Thus we have

Lemma 23. *A convex closure space is hereditarily normal if and only if it is completely normal.*

8. Summary

In all neighborhood spaces (i.e., generalized topologies with an expansive closure function) the following implications hold:

$$\begin{array}{ccccccccccccc}
\mathrm{T5U} & \Longrightarrow & \mathrm{T4U} & \Longrightarrow & \mathrm{T3\tfrac{1}{2}} & \Longrightarrow & \mathrm{T2U} & & & & & & \\
\Downarrow & & \Downarrow & & \Downarrow & & \Downarrow & & & & & & \\
\mathrm{T5} & \Longrightarrow & \mathrm{T4} & \Longrightarrow & \mathrm{T3} & \Longrightarrow & \mathrm{T2\tfrac{1}{2}} & \Longrightarrow & \mathrm{T2} & \Longrightarrow & \mathrm{T1} & \Longrightarrow & \mathrm{T0}
\end{array}$$

$$\begin{array}{ccccccc}
\mathrm{CNU} & \Longrightarrow & \mathrm{UN} & \Longrightarrow & \mathrm{N} & & \\
\Downarrow & & & & \Downarrow & & \\
\mathrm{CN} & & \Longrightarrow & & \mathrm{QN} & \Longrightarrow & \mathrm{tN}
\end{array}$$

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