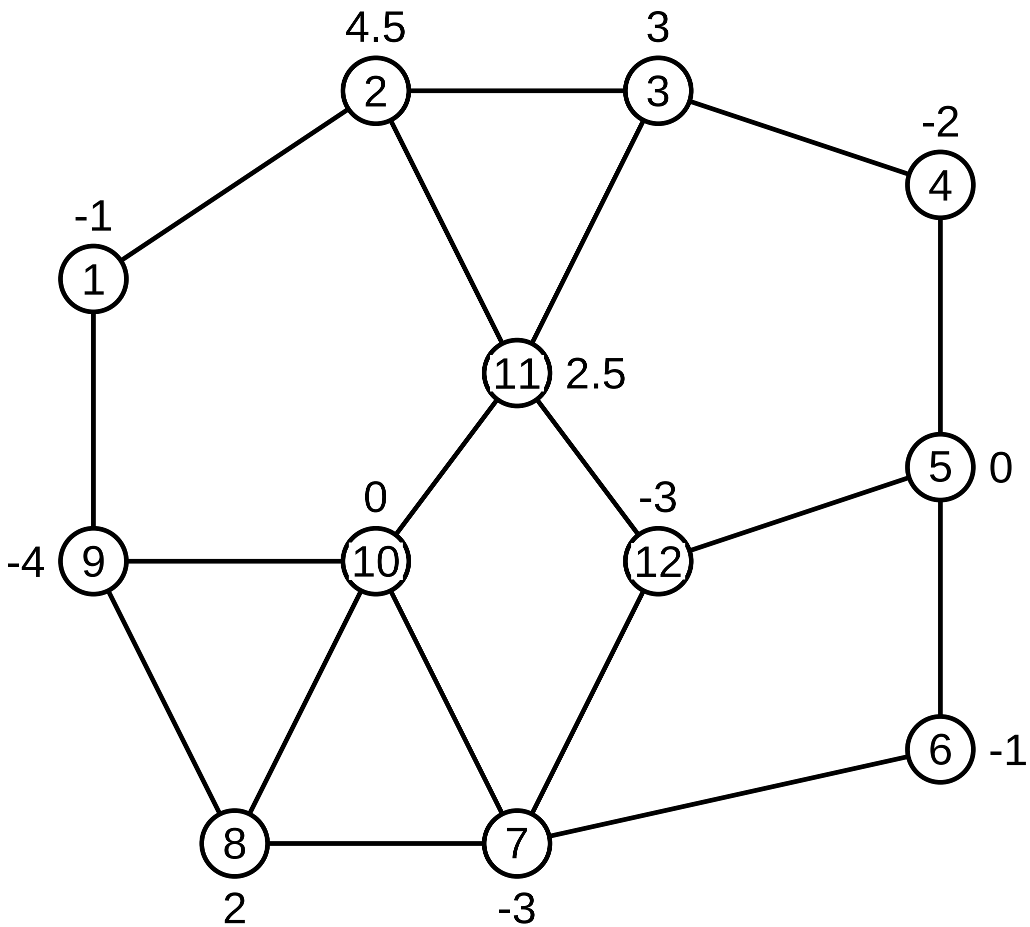
Graph Laplacians and Nodal Domains

Türker Bıyıkoğlu

Overview

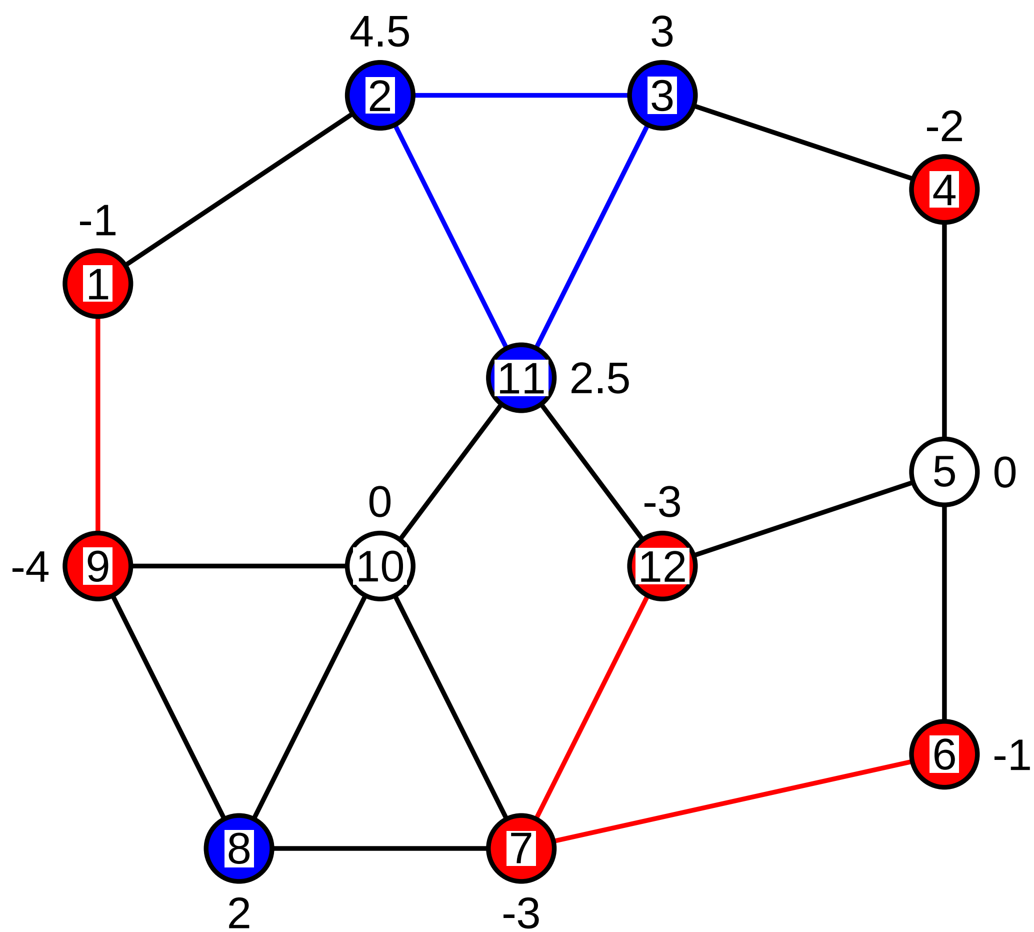
- what's a nodal domain?
- discrete nodal domain theorem
- (possible) applications of nodal domains
- The number of nodal domains of
  - trees
  - hypercubes
  - cographs

Nodal Domains



x=(-1,4.5,3,-2,0,-1,-3,2,-4,0,2.5,-3)

Nodal Domains



x=(-1,4.5,3,-2,0,-1,-3,2,-4,0,2.5,-3)

Applications of Eigenvectors and Nodal Domains

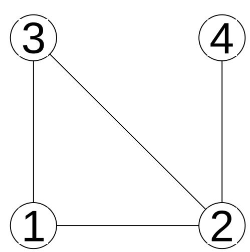
- Graph Drawing
- Graph Partitioning
- Graph Coloring
- Cross-correlations of World Financial Indices

Generalized Laplacian Matrix

generalized Laplacian matrix L of a graph is symmetric

$$L_{xy} = \begin{cases} \text{arbitrary} & \text{if} \quad x = y \\ 0 & \text{if} \quad x \text{ and } y \text{ are not adjacent} \\ \text{negative} & \text{if} \quad x \text{ and } y \text{ are adjacent} \end{cases}$$

Exp.



$$L = \begin{pmatrix} 8 & -4 & -6.5 & 0 \\ -4 & 0 & -1 & -5 \\ -6.5 & -1 & -9 & 0 \\ 0 & -5 & 0 & 7 \end{pmatrix}$$

Babel Tower of Laplacians

generalized Laplacian is also called:
(Discrete) Schrödinger operator

important generalized Laplacian matrices:

- -Adjacency matrix
- Laplacian matrix

Discrete Nodal Domain Theory

Eigenvalues of generalized Laplacian in non-decreasing order:

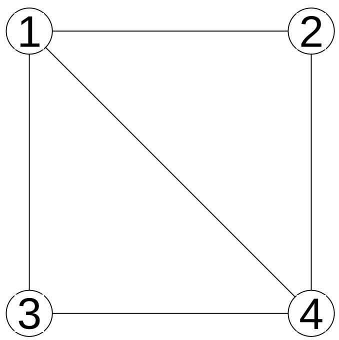
$$\lambda_1 \le \cdots \le \lambda_{k-1} < \lambda_k = \lambda_{k+1} = \cdots = \lambda_{k+r-1} < \lambda_{k+r} \le \cdots \le \lambda_n$$

Discrete nodal domain theory
(Davies, Gladwell, Leydold, Stadler 2001)

Each eigenvector of λ_k with multiplicity r has at most $k + r - 1$ nodal domains.

(Discrete version of Courant's nodal domain theory for Riemannian manifolds)

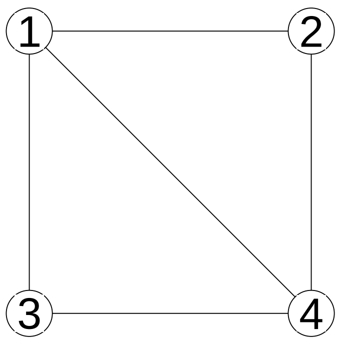
Example



$$L = \begin{pmatrix} 3 & -1 & -1 & -1 \\ -1 & 2 & 0 & -1 \\ -1 & 0 & 2 & -1 \\ -1 & -1 & -1 & 3 \end{pmatrix}$$

$$\begin{array}{lcccc} \text{Eigenvalues:} & 0 & 2 & 4 & 4 \\ \text{Eigenvectors:} & \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} & \begin{pmatrix} 0 \\ 1 \\ -1 \\ 0 \end{pmatrix} & \begin{pmatrix} -3 \\ 1 \\ 1 \\ 1 \end{pmatrix} & \begin{pmatrix} 1 \\ 0 \\ 0 \\ -1 \end{pmatrix} \end{array}$$

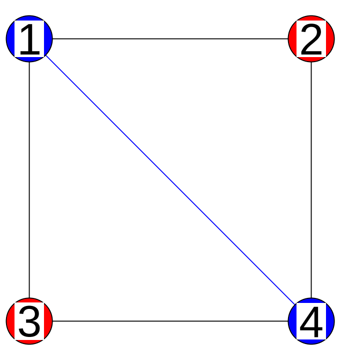
Example



$$L = \begin{pmatrix} 3 & -1 & -1 & -1 \\ -1 & 2 & 0 & -1 \\ -1 & 0 & 2 & -1 \\ -1 & -1 & -1 & 3 \end{pmatrix}$$

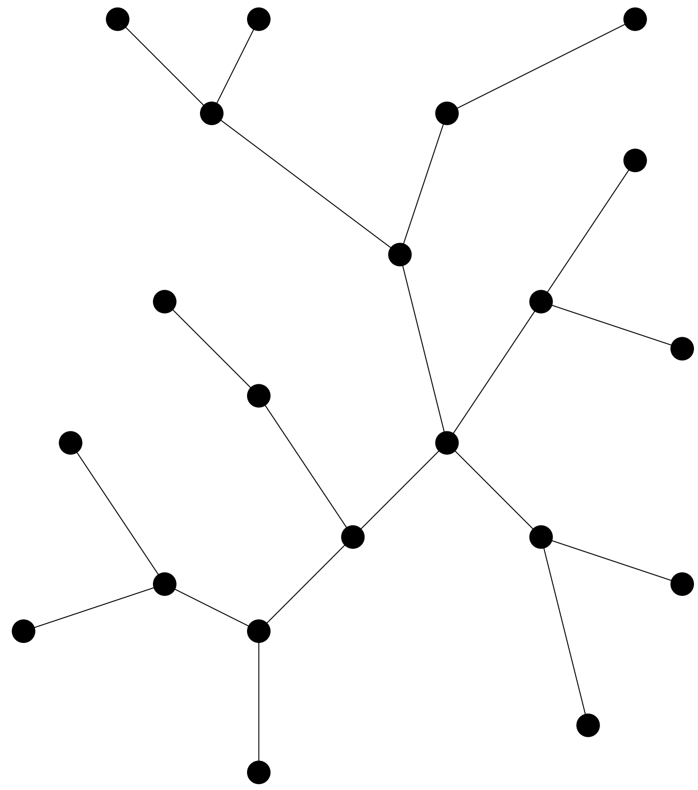
Eigenvalues: $0 \quad 2 \quad 4 \quad 4$

Eigenvectors: $\begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} \quad \begin{pmatrix} 0 \\ 1 \\ -1 \\ 0 \end{pmatrix} \quad \begin{pmatrix} -3 \\ 1 \\ 1 \\ 1 \end{pmatrix} \quad \begin{pmatrix} 1 \\ 0 \\ 0 \\ -1 \end{pmatrix}$



$$\begin{pmatrix} -1 \\ 1 \\ 1 \\ -1 \end{pmatrix} = \begin{pmatrix} -3 \\ 1 \\ 1 \\ 1 \end{pmatrix} + 2 \begin{pmatrix} 1 \\ 0 \\ 0 \\ -1 \end{pmatrix}$$

Tree



Max Nodal Domains of a Tree

Input: Tree with n vertices,
generalized Laplacian matrix of the tree,
and eigenbasis of eigenvalue λ.

Output: An eigenvector of λ
with maximum number of nodal domains.

Max Nodal Domains of a Tree

Input: Tree with n vertices,
generalized Laplacian matrix of the tree,
and eigenbasis of eigenvalue λ.

Output: An eigenvector of λ
with maximum number of nodal domains.

We can find such an eigenvector in $O(n^2)$ time.

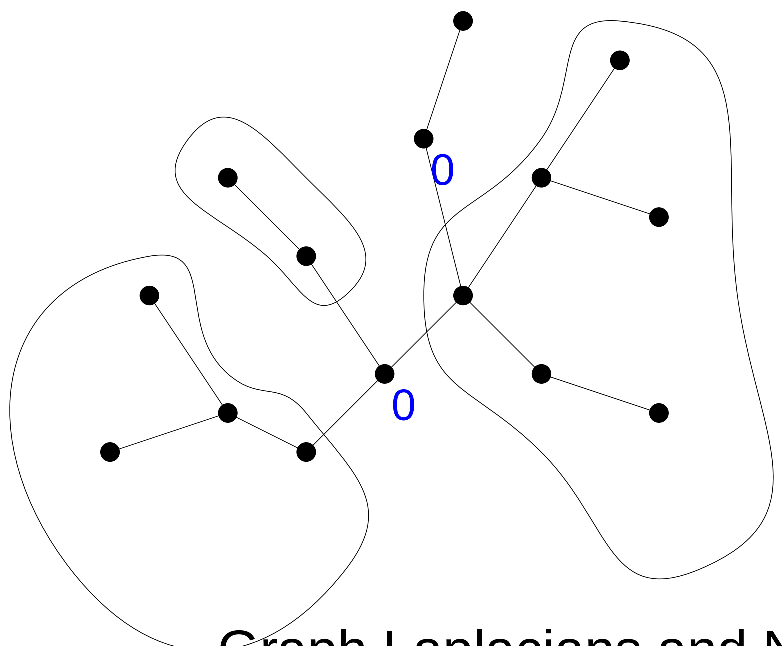
Main Idea

First a special case:
If λ_k has an eigenvector x without vanishing coordinate
$\Rightarrow \lambda_k$ is a simple and x has exactly k nodal domains

Main Idea

First a special case:
If λ_k has an eigenvector x without vanishing coordinate
$\Rightarrow \lambda_k$ is a simple and x has exactly k nodal domains

Second: λ with multiplicity $r \geq 2$,
$\Rightarrow$ Eigenbasis $e_1, \ldots, e_r$ has form:



Min Nodal Domains of Tree

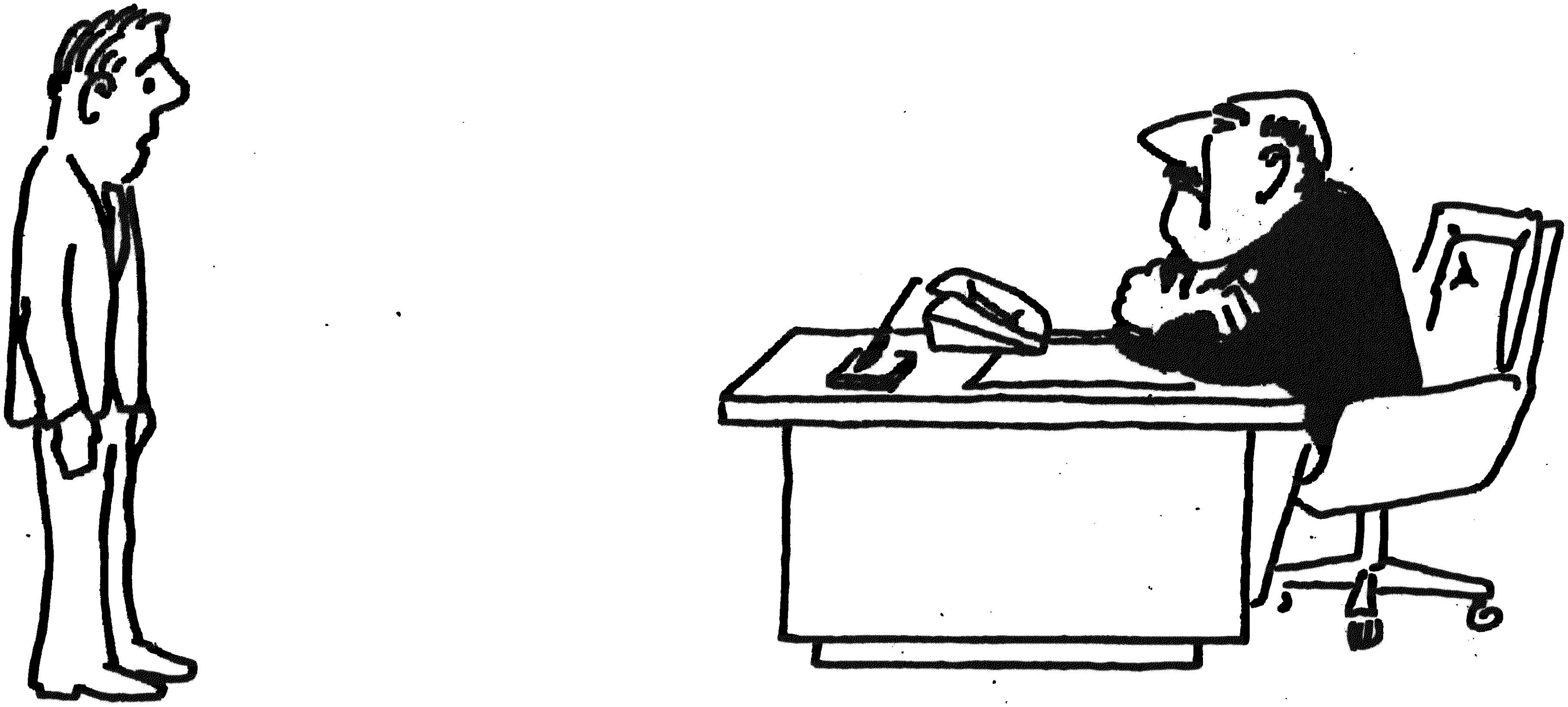
What about the minimum number of nodal domains of a tree?

Min Nodal Domains of Tree

What about the minimum number of nodal domains of a tree?

Finding an eigenvector of the eigenvalue λ with minimum number of nodal domains is NP-complete.

NP-complete



"I can't find an efficient algorithm, I guess I'm just too dumb."

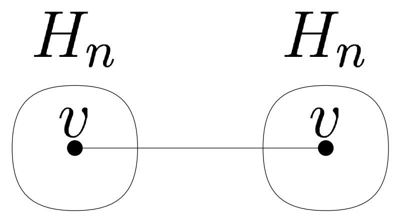
NP-complete

"I can't find an efficient algorithm, but neither can all these famous people."

Hypercubes (H_n)

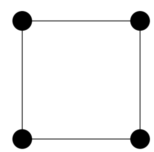
$$H_{n+1} = H_n \square K_2$$

H_{n+1} :

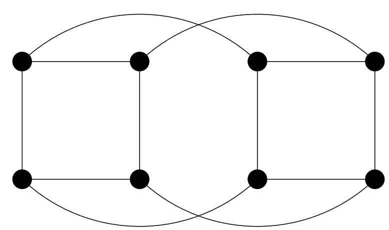


H_1 :

H_2 :



H_3 :



Min Nodal Domains of Hypercubes

The first half of the eigenvalues of the hypercube H_n has an eigenvector with two nodal domains.

Idea: The partition of vertices into two sets A and B of equal size such that each set

induces a connected $n/2$-regular subgraph

Conjecturem

Conjecture:
All eigenvalues (except largest and second largest) of hypercube H_n have an eigenvector with two nodal domains.

numerical experiments

some IQ-Test Sequences

Upper bounds on minimum number of nodal domains of second largest eigenvalue:

n	2	3	4	5	6	7	8	9	10	11	12
ND	2	3	4	8	14	24	44	84	160	314	??

numerical experiments

some IQ-Test Sequences

Upper bounds on minimum number of nodal domains of second largest eigenvalue:

n	2	3	4	5	6	7	8	9	10	11	12
ND	2	3	4	8	14	24	44	84	160	314	??

Lower bounds on maximum number of nodal domains:

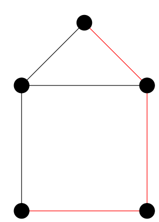
n	2	3	4	5	6	7	8	9	10
$ND_{\lambda=2n-2}$	2	4	8	16	32	64	128	261	??
$ND_{\lambda=2n-4}$	2	2	4	10	18	34	57	72	??

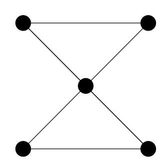
For which graphs are the nodal domains easy?

For which graphs are the nodal domains easy?

Nodal Domains are easy for Cographs

Graph cograph if it has no induced path P_4





not cograph   cograph