Tree Like Structures

What is a tree?

A tree has endpoints

A tree is dismantlable

A tree has a center

A tree has a unique shortest path between any two of its vertices

Any three vertices x, y, z of a tree have a unique median, that is a vertex that is on shortest paths between any two of the vertices x, y, z

A tree is determined by the distances between its endvertices

A tree is determined by splits (generated by the removal of edges)

One can define ends for infinite graphs and add them to the tree as points at infinity

Then the splits correspond to a neighborhood basis for the (metrizable) topology of the infinite tree median graphs =: graphs with a unique median for any three vertices

example: trees, hypercubes Q_n

properties: bipartite, isometric subgraphs of hypercubes

median graphs =: retracts of hypercubes (retracts are images of retractions, retractions are idempotent endomophisms)

Dress, Bandelt, Moulton use them to describe phylogenies

Recognition complexity for median graphs on m edges and n vertices:

$O(m\sqrt{n})$

$O((md)^{1.41})$, where $d \leq \log n$

Remark $O(m^{1.41})$ is the current complexity for the recognition of triangle-free graphs (Alon) partial cubes =: isometric subgraphs of hypercubes

W_{ab}, Θ, U_{ab}

partial cubes == bipartite graphs with transitive Θ (Winkler)

recognition complexity $O(mn)$, withstood all attempts to improve it in general

Remark: For subgraphs of hypercubes $m \leq n \log n$

A periphery =: a U_{ab} that is equal to W_{ab}, corresponds to endpoints in trees

A median graph is dismantlable

There exist many other classes of partial cubes with recognition complexity better than $O(mn)$

Acyclic Cubical Complexes =: dismantlable, invariant under removal of peripheries, every periphery is a hypercube

recognition complexity: linear

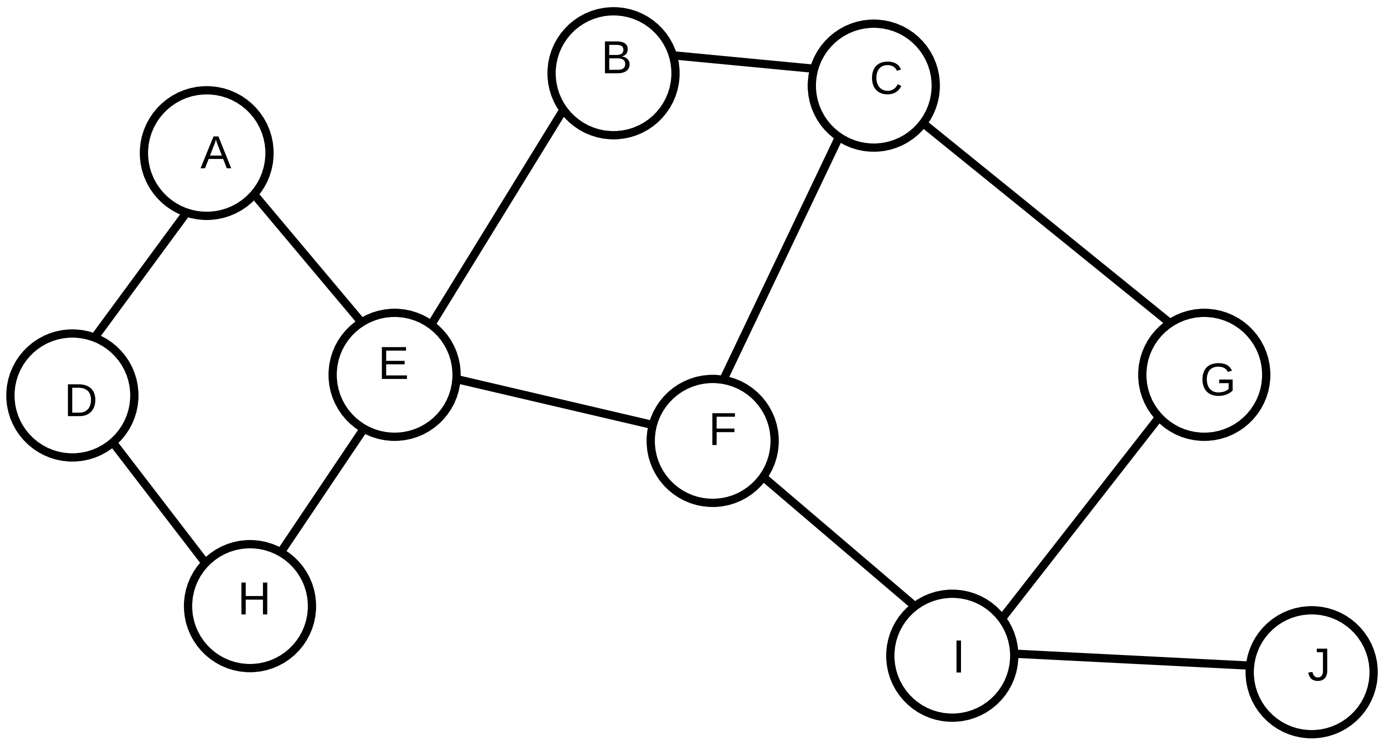
algorithm uses ideas of linear algorithm for the prime factorization of graphs with respect to the Cartesian product

Dimension of the hypercube into that a partial cube can be (isoerically) embedded = the number of equivalence classes with respect to Θ

For a tree this is the number of edges

A path of length n needs a hypercube of dimension $n-1$

But if one embeds into the integer lattice of dimension k, dimension 1 sufices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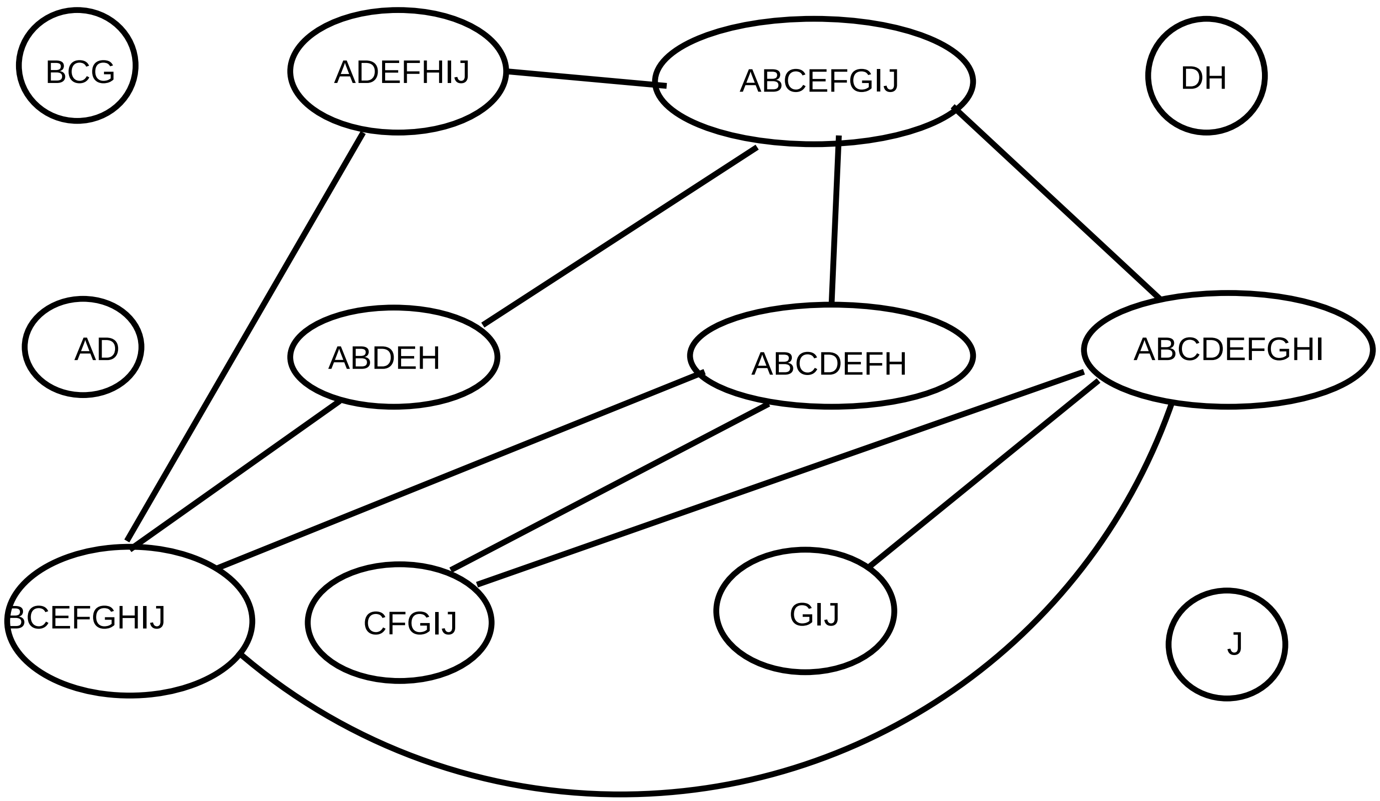
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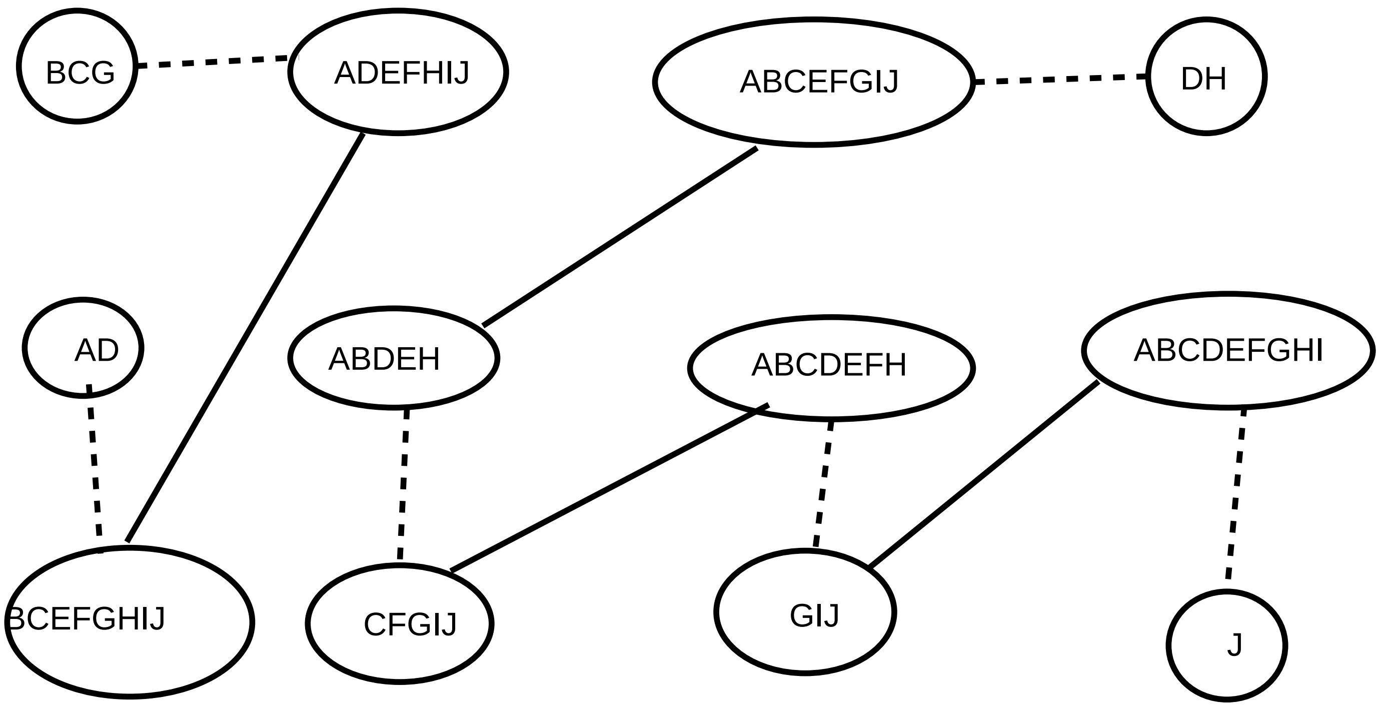
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What is the smallest lattice dimension into that a partial cube can be embedded?

Eppstein: For a partial cube G this dimension is the number of equivalence classes τ of Θ minus the number of edges in a maximal matching M of the semicube graph of G

How fast can it be found?

Eppstein: $O(mn + n\tau^2)$

Conjecure: $O(mn)$ for partial cubes with connected U_{ab}-s