Why Relations are Topologies

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Genotype Spaces

Given:

a set X of possible genotypes
a set A of realized genotypes
a fixed collection of genetic operators
[such as mutation, recombination, gene-rearrangement]

define the set A' of genotypes accessible from A.

Properties

(i) No spontaneous creation, i.e, $\emptyset' = \emptyset$.

(ii) A more diverse population produces more diverse offsprings:
$A \subseteq B$ implies $A' \subseteq B'$

(iii) All parental genotypes are also accessible in the next time step
$A \subseteq A'$.

In the case of mutation as the only source of diversity:

haploid populations, no sex, no recombination, etc

(iv) Diversity of offsprings depends only on the parent:
$A' = \bigcup_{x \in A} \{x\}'$

Set-Valued Set-Functions

Let X be a set, $\mathcal{P}(X)$ its power set (i.e., theset of all subsets of X)
Let $\mathrm{cl} : \mathcal{P}(X) \to \mathcal{P}(X)$ be an arbitrary function.
We call $\mathrm{cl}(A)$ the *closure* of the set A.
The dual of the closure function is the *interior function* $\mathrm{int} : \mathcal{P}(X) \to \mathcal{P}(X)$ defined by

$$\mathrm{int}(A) = X \setminus \mathrm{cl}(X \setminus A)$$

Given the interior function, we can recover the closure:

$$\mathrm{cl}(A) = X \setminus (\mathrm{int}(X \setminus A))$$

Neighborhoods

Let cl and int be a closure function and its dual interior function on X. Then the *neighborhood function* $\mathcal{N} : X \to \mathcal{P}(\mathcal{P}(X))$

$$\mathcal{N}(x) = \{ N \in \mathcal{P}(X) \big| x \in \mathrm{int}(N) \}$$

of its *neighborhoods*.
Closure and neighborhood are equivalent:

$$x \in \mathrm{cl}(A) \iff (X \setminus A) \notin \mathcal{N}(x) \quad \text{and} \quad x \in \mathrm{int}(A) \iff A \in \mathcal{N}(x)$$

Axioms for Generalized Closure Spaces

	closure	interior	neighborhood
K0'	$\exists A : x \notin \text{cl}(A)$	$\exists A : x \in \text{int}(A)$	$\mathcal{N}(x) \neq \varnothing$
K0	$\text{cl}(\emptyset) = \emptyset$	$\text{int}(X) = X$	$X \in \mathcal{N}(x)$
K1 isotonic, monotone	$A \subseteq B \implies \text{cl}(A) \subseteq \text{cl}(B)$ $\text{cl}(A \cap B) \subseteq \text{cl}(A) \cap \text{cl}(B)$ $\text{cl}(A) \cup \text{cl}(B) \subseteq \text{cl}(A \cup B)$	$A \subseteq B \implies \text{int}(A) \subseteq \text{int}(B)$ $\text{int}(A) \cup \text{int}(B) \subseteq \text{int}(A \cup B)$ $\text{int}(A \cap B) \subseteq \text{int}(A) \cap \text{int}(B)$	$N \in \mathcal{N}(x)$ and $N \subseteq N'$ $\implies$ $N' \in \mathcal{N}(x)$
KA	$\text{cl}(X) = X$	$\text{int}(\emptyset) = \emptyset$	$\emptyset \notin \mathcal{N}(x)$
KB	$A \cup B = X \implies$ $\text{cl}(A) \cup \text{cl}(B) = X$	$A \cap B = \emptyset \implies$ $\text{int}(A) \cap \text{int}(B) = \emptyset$	$N', N'' \in \mathcal{N}(x) \implies$ $N' \cap N'' \neq \emptyset$
K2 expansive	$A \subseteq \text{cl}(A)$	$\text{int}(A) \subseteq \text{int}(A)$	$N \in \mathcal{N}(x) \implies x \in N$
K3 sub-linear	$\text{cl}(A \cup B) \subseteq \text{cl}(A) \cup \text{cl}(B)$	$\text{int}(A) \cap \text{int}(B) \subseteq \text{int}(A \cup B)$	$N', N'' \in \mathcal{N}(x) \implies$ $N' \cap N'' \in \mathcal{N}(x)$
K4 idempotent	$\text{cl}(\text{cl}(A)) = \text{cl}(A)$	$\text{int}(\text{int}(A)) = \text{int}(A)$	$N \in \mathcal{N}(x) \iff$ $\text{int}(N) \in \mathcal{N}(x)$
K5 additive	$\bigcup_{i \in I} \text{cl}(A_i) = \text{cl}\left(\bigcup_{i \in I} A_i\right)$	$\bigcap_{i \in I} \text{int}(A_i) = \text{int}\left(\bigcap_{i \in I} A_i\right)$	$\mathcal{N}(x) = \emptyset$ or $\exists N(x)$: $N \in \mathcal{N}(x)$ $\iff N(x) \subseteq N$

Isotonic Spaces

(K1) $A \subseteq B$ implies $\mathrm{cl}(A) \subseteq \mathrm{cl}(B)$ for all $A, B \in \mathcal{P}(X)$.

(K1') $\mathrm{cl}(A) \cup \mathrm{cl}(B) \subseteq \mathrm{cl}(A \cup B)$ for all $A, B \in \mathcal{P}(X)$.

(K1") $\mathrm{cl}(A \cap B) \subseteq \mathrm{cl}(A) \cap \mathrm{cl}(B)$

A (not necessarily non-empty) collection $\mathcal{F} \subseteq \mathcal{P}(X)$ is a *stack* if $F \in \mathcal{F}$and $F \subseteq G$ implies $G \in \mathcal{F}$. The closure function cl is isotonic if and only if $\mathcal{N}(x)$ is a stack for all $x \in X$.
Isotony (K1) is necessary and sufficient to express the closure in terms of neighborhoods *in the usual way*:

$$c(A) = \{x \in X | \forall N \in \mathcal{N}(x) : A \cap N \neq \emptyset\}$$

Binary Relations

Let $\mathfrak{R}$ be a binary relation on a (not necessarily finite) set X, i.e., $\mathfrak{R} \subseteq X \times X$. We write $x\mathfrak{R}y$ or $(x, y) \in \mathfrak{R}$ to mean that x “is in relation $\mathfrak{R}$ to y´´.
We define:

$$\mathfrak{R}x = \{z \in X | z\mathfrak{R}x\}$$
$$x\mathfrak{R} = \{z \in X | x\mathfrak{R}z\}$$

Furthermore, we define:

$$\mathrm{dom}\,[\mathfrak{R}] = \{x \in X | \exists z \in X : \; x\mathfrak{R}z\}$$
$$\mathrm{img}\,[\mathfrak{R}] = \{y \in X | \exists z \in X : \; z\mathfrak{R}y\}$$

Then $\mathrm{dom}\,[\mathfrak{R}] = \bigcup_x \mathfrak{R}x$ and $\mathrm{img}\,[\mathfrak{R}] = \bigcup_x x\mathfrak{R}$.

Topology of a Binary Relation

Let $\mathfrak{R}$ be a relation of X and consider a subset $A \subseteq X$. A natural way of defining the interior of A is to consider all points $x \in X$ that are *isolated* from the complement of A in the sense that there is no point $y \in X \setminus A$ for which $y\mathfrak{R}x$. We have:

$$\text{int}(A) = \{x \in X | \not\exists y \in X \setminus A : y\mathfrak{R}x\}$$

Equivalently, $x \in X \setminus \text{int}(X \setminus A)$ if $\exists y \in A : y\mathfrak{R}x$, i.e.,

$$\text{cl}(A) = \bigcup_{y \in A} \{x | y\mathfrak{R}x\} = \bigcup_{x \in A} x\mathfrak{R}$$

It follows immediately that cl is additive and satisfies (K0).

Binary Relation from Totally Additive Closures

Conversely, consider an additive closure function c satisfying (K0). Then there is a unique relation $\mathfrak{R}_c$ defined by

$$x \mathfrak{R}_c y \iff y \in c(\{x\}).$$

Now construct the closure function $\mathrm{cl}_{\mathfrak{R}_c}$ associated with relation $\mathfrak{R}_c$. We see:
$c(\{x\}) = \mathrm{cl}_{\mathfrak{R}_c}(\{x\})$ for all $x \in X$.
Additivity of c now implies $c(A) = \mathrm{cl}_{\mathfrak{R}_c}(A)$ for all $A \in \mathcal{P}(X)$.
Hence additive closure spaces satisfying (K0) are equivalent to binary relations.

Vicinities

The most important property of totally additive (K0) spaces is that there is a smallest neighborhood (*vicinity*) for each point:

$$\mathrm{vc}(x) = \bigcap \left\{ N \middle| N \in \mathcal{N}(x) \right\} \in \mathcal{N}(x)$$

We have $\mathrm{cl}(x) = x\mathfrak{R}$ and $\mathrm{vc}(x) = \mathfrak{R}x$, i.e., the vicinity is the closure of the transposed relation $\mathfrak{R}^+$.
Thus: $x \in \mathrm{cl}(y) \iff y \in \mathrm{vc}(x)$
(R0) cl is symmetric if $x \in \mathrm{vc}(y) \iff y \in \mathrm{vc}(x)$.
Result: cl is symmetric $\mathfrak{R}$ is symmetric.

Separation Axioms

- (T0) $\forall x, y \exists N' \in \mathcal{N}(x)$ or $N'' \in (y)$ such that $y \notin N'$ or $x \notin N''$
  (K0+K5) spaces: $x \neq y \implies x \notin \text{vc}(y)$ or $y \notin \text{vc}(x)$, i.e., $y \notin \text{cl}(x)$ or $x \notin \text{cl}(y)$.
  Equivalently: If $x \neq y$ then $x \mathfrak{R} y$ implies $y \not\mathfrak{R} x$.
  Thus $(T0)$ is equivalent to antisymmetry of the relation.
- (T1) $\forall x, y \exists N \in \mathcal{N}(x)$ such that $y \notin N$
  (K0+K5) spaces: $x \neq y \implies x \notin \text{vc}(y)$
  $x \notin \text{cl}(y) \iff \text{cl}(x) \subseteq \{x\}$, i.e., there are no "off-diagonal elements" in $\mathfrak{R}$.

<u>Lemma</u> For isotonic spaces holds (R0) and (T0) $\iff$ (T1)

Separation Axioms

- (T2) $\forall x \neq y \, \exists N' \in \mathcal{N}(x)$ and $N'' \in \mathcal{N}(y)$ such that $N' \cap N'' = 0$
  (K0+K5) spaces: $\mathrm{vc}(x) \cap \mathrm{vc}(y) = 0$,i.e.,
  $z \in \mathrm{vc}(x) \implies z \notin \mathrm{vc}(y)$, i.e., $x \in \mathrm{cl}(z) \implies y \notin \mathrm{cl}(z)$, i.e., $|\mathrm{cl}(z)| \leq 1$.

A (K0+K5+T2) space corresponds to a function on X:

$$\psi : \mathrm{dom}\mathfrak{R} \to \mathrm{img}\mathfrak{R} : x \mapsto \psi(x) \quad \text{where} \quad \mathrm{cl}(x) = \{\psi(x)\}$$

Transitive Relations and Topologies

- Def. $\mathfrak{R}$ is reflexive if $x \mathfrak{R} x$ for all $x \in X$.
  This is equivalent to $A \in \mathrm{cl}(A)$ (enlarging, K2)
- Def. $\mathfrak{R}$ is transitive if $x \mathfrak{R} y$ and $y \mathfrak{R} z$ implies $x \mathfrak{R} z$
  $x \in \mathrm{cl}(y)$ and $y \in \mathrm{cl}(z)$ implies $x \in \mathrm{cl}(z)$.
  This is equivalent to $\mathrm{cl}(\mathrm{cl}(A)) \subseteq \mathrm{cl}(A)$.
- Pre-Order relation = reflexive and transitive.
  This implies: $A \in \mathrm{cl}(A)$ and $\mathrm{cl}(\mathrm{cl}(A)) \subseteq \mathrm{cl}(A)$, i.e.,
  $\mathrm{cl}(\mathrm{cl}(A)) = \mathrm{cl}(A)$ (idempotent closure).
- Thm. (K0,K5)-space is topological if and only if the corresponding relation $\mathfrak{R}$ is a pre-order.
- In particular, finite topologies and finite pre-order relations are the same thing.

So Long, and Thanks for all the fish!