Geometric Nodal Domains and Trees with Minimal Algebraic Connectivity

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Outline

- Graph with boundary and Dirichlet matrix.
- Concept of geometric nodal domain.
- Give condition on trees with prescribed degree sequence with minimal algebraic connectivity.

Graph Laplacian

The **graph Laplacian** of a simple graph $G = (V, E)$ with weights w_{uv} is defined by

$$\boxed{L(G) = D(G) - A(G)}$$

$A(G)$ ... adjacency matrix of G
$D(G)$ ... degree matrix with $D_{vv} = \sum_{u \sim v} w_{uv}$

with eigenvalues $0 = \lambda_1 \leq \lambda_2 < \cdots < \lambda_n$.

λ_2 is called the **algebraic connectivity** of G.

$$\boxed{\alpha(G) = \lambda_2}$$

Graph with Boundary

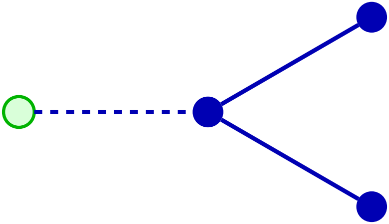
A **graph with boundary** is a graph $G^\circ = (V^\circ \cup \partial V, E^\circ \cup \partial E)$ where

V° … interior vertices

∂V … boundary vertices

E° … interior edges $\subseteq V^\circ \times V^\circ$

∂E … boundary edges $\subseteq V^\circ \times \partial V$



Dirichlet Matrix

The **Dirichlet matrix** is the graph Laplacian restricted to the interior vertices of a graph with boundary:

$$L^\circ(G) = D^\circ(G) - A^\circ(G)$$

$A^\circ(G)$ ... adjacency matrix of graph induced by V°
$D^\circ(G)$ ... degree matrix $D(G)$ restricted to V°

Hence $L^\circ(G)$ is the Laplacian $L(G)$ restricted to V°.

We denote the first Dirichlet eigenvalue by $\nu(G)$.

Dirichlet Matrix

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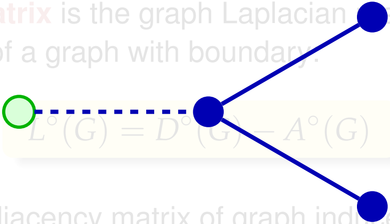

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$$L\ (T) = \begin{pmatrix} 1 & -1 & 0 & 0 \\ -1 & 3 & -1 & -1 \\ 0 & -1 & 1 & 0 \\ 0 & -1 & 0 & 1 \end{pmatrix}$$

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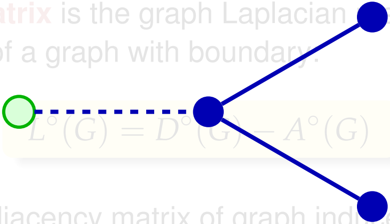

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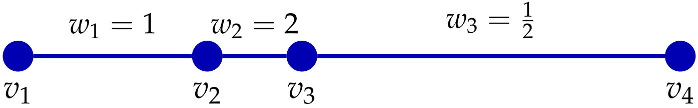
$$L^\circ(T) = \begin{pmatrix} 3 & -1 & -1 \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{pmatrix}$$

Geometric Realization

The **geometric realization** $\mathcal{G}$ of a graph G with weights w_{uv} is a metric space where the vertices are points and the edges uv correspond to arcs of length $1/w_{uv}$ that connects the incident vertices u and v.



Define two measures

$$\mu_V(\mathcal{G}) = |V| \qquad \ldots \text{ number of vertices}$$

$$\mu_E(\mathcal{G}) = \sum_{uv \in E} \frac{1}{w_{uv}} \qquad \ldots \text{ cumulated length of edges}$$

(Lebesgue measure of $\mathcal{G}$)

Rayleigh Quotient

For a vector f on G:

$$\mathcal{R}_L(f) = \frac{\langle f, Lf \rangle}{\langle f, f \rangle} = \frac{\sum_{uv \in E} w_{uv} (f(u) - f(v))^2}{\sum_{v \in V} f(v)^2}$$

For a continuous piecewise differentiable function ϕ on $\mathcal{G}$:

$$\mathcal{R}_{\mathcal{L}}(\phi) = \frac{\int_{\mathcal{G}} |\nabla \phi|^2 \, d\mu_E}{\int_{\mathcal{G}} |\phi|^2 \, d\mu_V}$$

The latter defines a continuous version of the graph Laplacian on $\mathcal{G}$: **geometric Laplacian $\mathcal{L}$**

The Geometric Laplacian

The eigenvalues of the geometric Laplacian $\mathcal{L}$ and the graph Laplacian G coincide.

The eigenfunctions of $\mathcal{L}$ are piecewise linear (on the edges of E). Their restrictions to V are exactly the eigenvectors of G.

[Friedman, 1993]

Nodal Domains

Let f be an eigenvector of G. We call the components of the two graphs induced by the vertices of non-negative and non-positive valuations the **weak nodal domains** of f. (Perron components)

$$G[\{v : f(v) \geq 0\}] \qquad \text{and} \qquad G[\{v : f(v) \leq 0\}]$$

Geometric Nodal Domains

Let ϕ be the eigenfunction on $\mathcal{G}$ corresponding to eigenvector f on G.

- Insert new vertices where ϕ changes sign on an edge xy (and thus subdivide edges).
- Use arc lengths $\frac{|\phi(x)|}{|\phi(x)-\phi(y)|}$ and $\frac{|\phi(y)|}{|\phi(x)-\phi(y)|}$, resp.
  ϕ is eigenfunction of the new graph with same eigenvalue.
- All vertices where ϕ vanishes but have non-vanishing neighbors are considered as boundary vertices.
- Split all boundary vertices such that each component has vertices with non-zero valuation (all of same sign).

We call these components the **geometric nodal domains** of f.

Geometric Nodal Domains

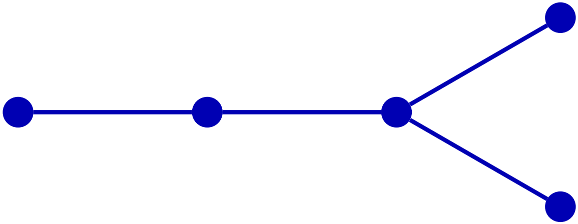
Let f be an eigenvector of G corresponding to eigenvalue λ.

The first Dirichlet eigenvalue at each of these geometric nodal domains coincides with λ.

f restricted to a geometric nodal domain is an eigenvector to the first Dirichlet eigenvalue [Bıyıkoğlu et al., 2007].

(This idea is related to the bottleneck matrix introduced in [Kirkland et al., 1996].)

Example



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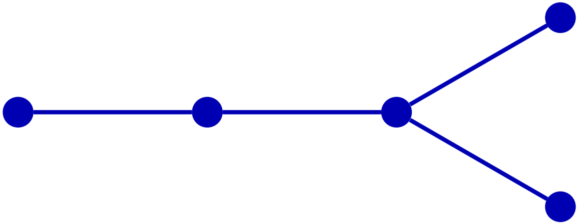
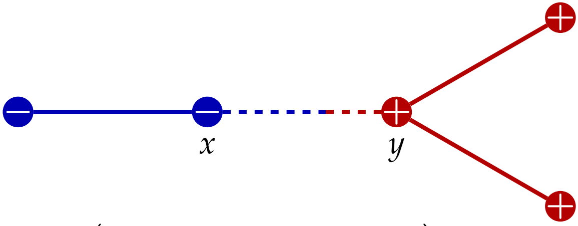


$$L(T) = \begin{pmatrix} 1 & -1 & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & 0 \\ 0 & -1 & 3 & -1 & -1 \\ 0 & 0 & -1 & 1 & 0 \\ 0 & 0 & -1 & 0 & 1 \end{pmatrix}$$

Example – Fiedler Vector



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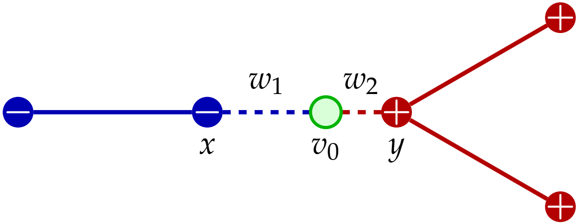
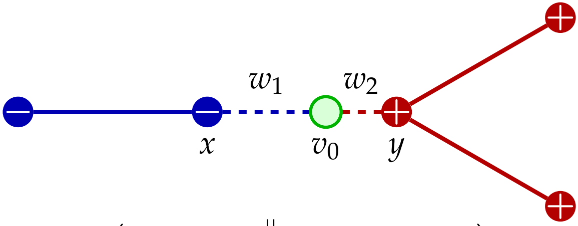
Example – Split



$$w_1 = |f(y) - f(x)|/|f(x)|$$
$$w_2 = |f(y) - f(x)|/|f(y)|$$

Example – Split



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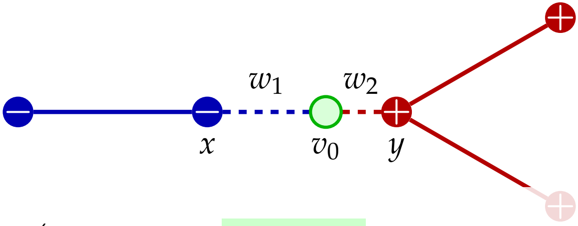
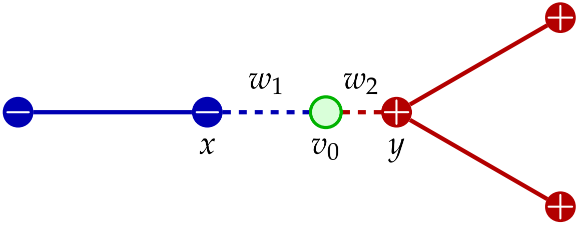


$$L(T') = \begin{pmatrix} 1 & -1 & 0 & 0 & 0 & 0 \\ -1 & 1+w_1 & -w_1 & 0 & 0 & 0 \\ 0 & -w_1 & w_1+w_2 & -w_2 & 0 & 0 \\ 0 & 0 & -w_2 & 2+w_2 & -1 & -1 \\ 0 & 0 & 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & -1 & 0 & 1 \end{pmatrix}$$

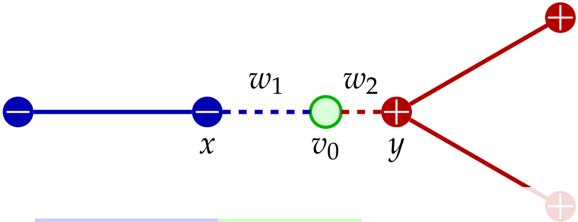
Example – Split



The algebraic connectivities of T and T' coincide.

$$\alpha(T) = \alpha(T')$$

Example – Nodal Domains



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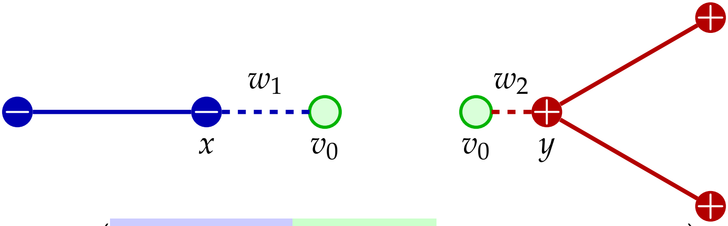

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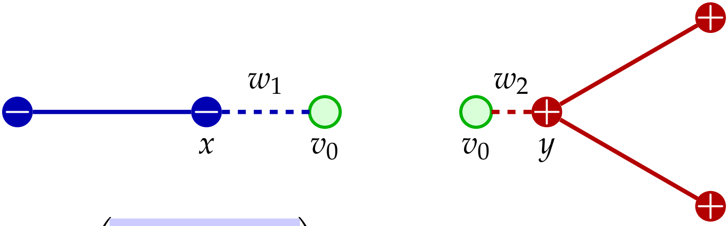
Example – Dirichlet Matrix



$$L^\circ(T_n) = \begin{pmatrix} 1 & -1 \\ -1 & 1+w_1 \end{pmatrix}$$

$$L^\circ(T_p) = \begin{pmatrix} 2+w_2 & -1 & -1 \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{pmatrix}$$

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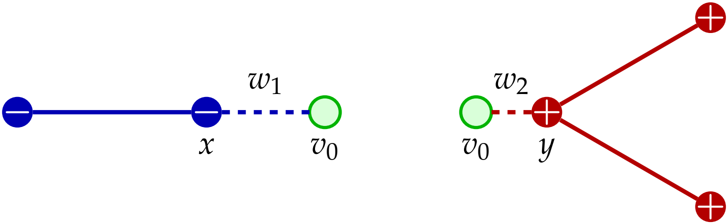


The algebraic connectivity of T and
the first Dirichlet eigenvalues of T_n and T_p
coincide.

$$\alpha(T) = \nu(T_n) = \nu(T_p)$$

Fiedler Vector and Algebraic Connectivity

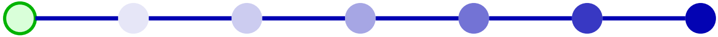
The second eigenvalue $\alpha(G)$ is greater than 0 whenever G is connected. It is called **algebraic connectivity** [Fiedler, 1973]. A corresponding eigenvector is called **Fiedler vector**.

A Fiedler vector f has exactly **two** nodal domains [Fiedler, 1975].

If G is some tree T then these are separated by either

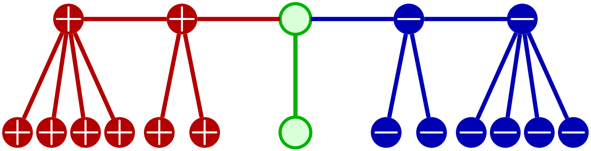
- a **characteristic edge** (where f changes sign), or
- a **characteristic vertex** (where f vanishes).

Then on every path starting at the characteristic set f is either strictly **increasing**, decreasing or constant zero.



Theorem

Let T be a tree that has minimal algebraic connectivity among all trees with given degree sequence $\pi = (d_0, \dots, d_{n-1})$.
Then T is a caterpillar.
Moreover, if P is the path induced by all non-pendant vertices of T with non-negative (non-positive) valuation, then its degree sequence is monotone with a minimum at the characteristic vertex or edge.



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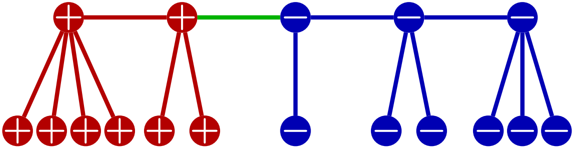

Outline of Proof

- Get a Fiedler vector f for tree T.
- Split T into the geometric nodal domains T_n and T_p of f.
- For each nodal domain the **first Dirichlet eigenvalue** must be minimal amongst all trees with one boundary vertex and the same degree sequence. Otherwise we could glue the extremal trees T'_n and T'_p with boundary together and obtain a tree T' with smaller algebraic connectivity, since

$$\alpha(T') \leq \max(\nu(T'_n), \nu(T'_p))$$

The inequality is strict if $\nu(T'_n) \neq \nu(T'_p)$.

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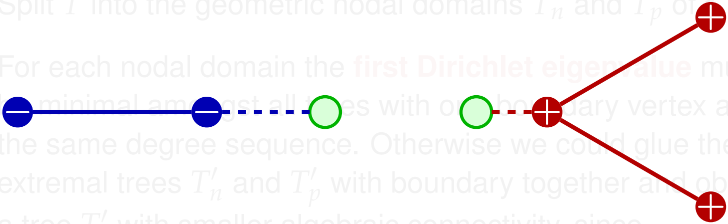

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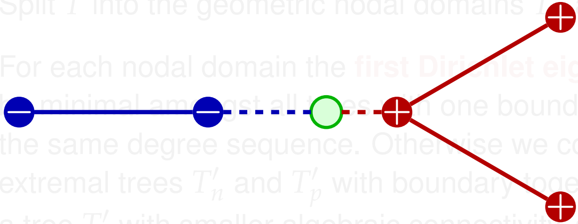

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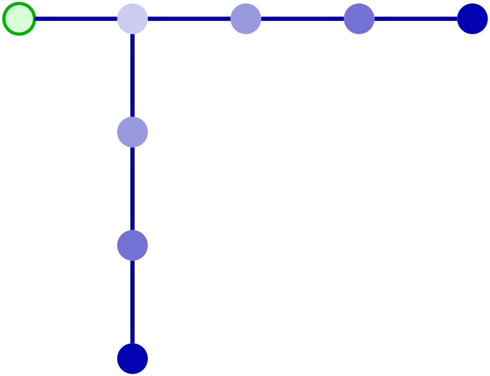
- Each nodal domain of an extremal tree must be a caterpillar. Otherwise shift branches (but leave pendant vertex) and thus decrease the Rayleigh quotient.



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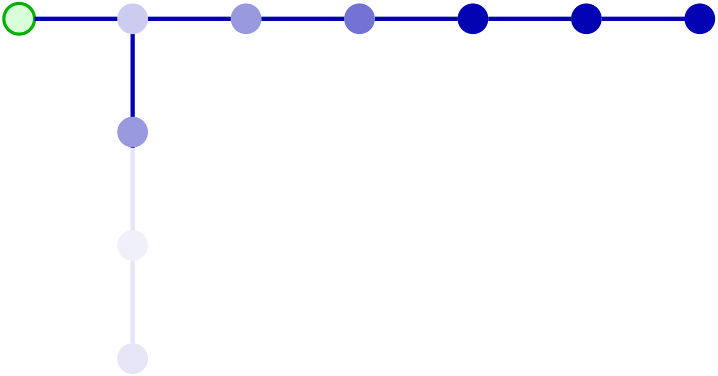

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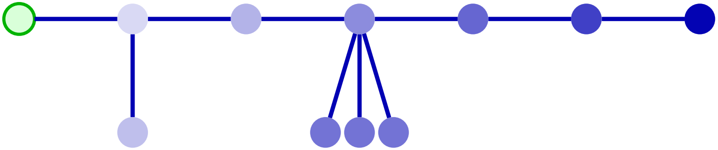

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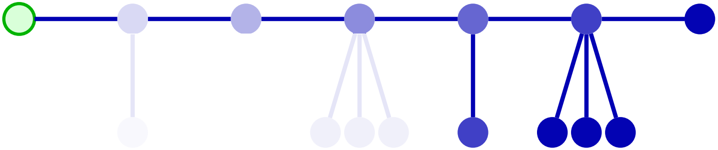

Remark

It is not known how the degree sequence has to be split for the two nodal domains.

Thank You

References I

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