Robust cycle bases

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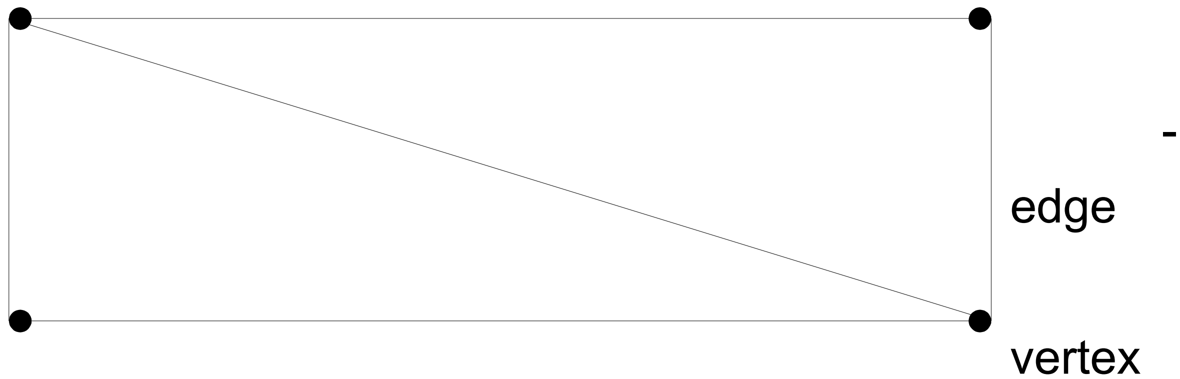
- $G=(V,E)$ graph, V vertex set, E edge set



- subgraph of G: a graph $G'=(V',E')$ with $V' \subseteq V$, $E' \subseteq E$

- cycle: an Eulerian subgraph in which every vertex degree is even

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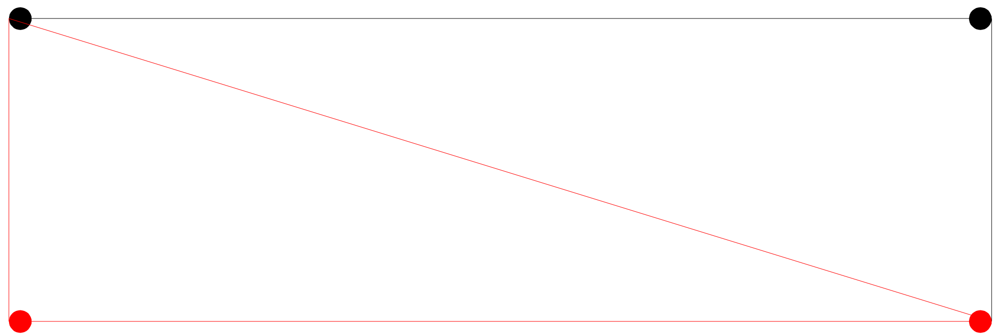

- subgraph of G: a graph $G'=(V',E')$ with $V' \subseteq V$, $E' \subseteq E$

- cycle: an Eulerian subgraph in which every vertex degree is even

- circuit: a connected Eulerian subgraph in which every vertex degree is 2

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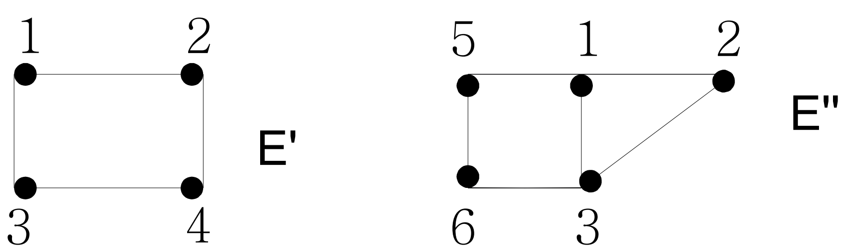

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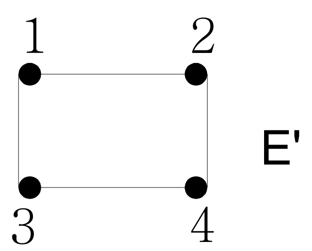

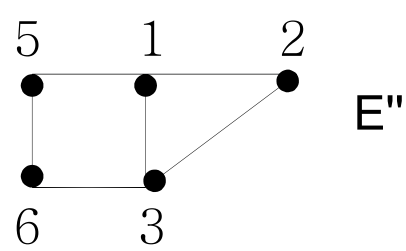

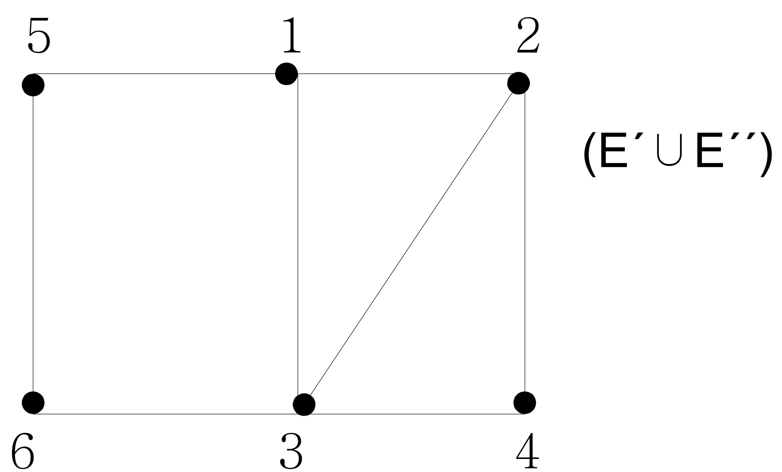

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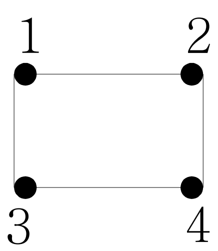
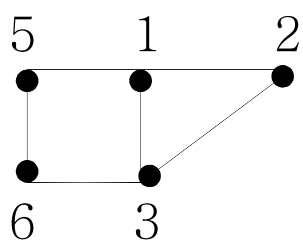
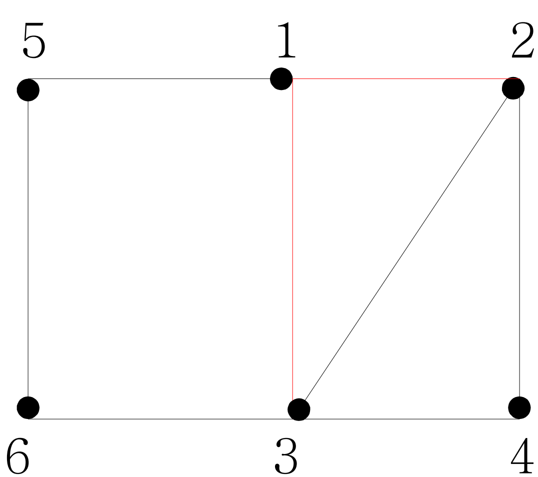


E'



E''



$(E' \cap E'')$

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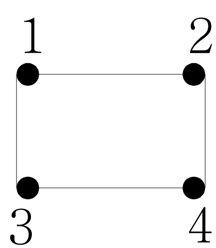

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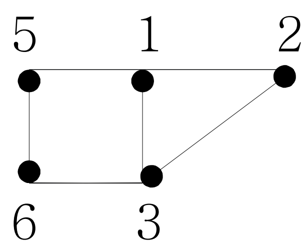

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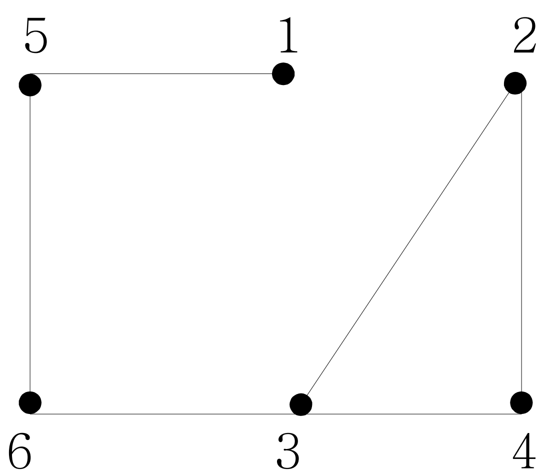


$(E' \cup E'') \setminus (E' \cap E'')$

- edge space $\mathcal{E}(G)$: $\mathbb{Z}/2\mathbb{Z}$-vector space $(\mathcal{P}(E),\oplus,*)$
  - $\mathcal{P}(E)$ power set of E
  - vector addition $\oplus$ defined as above
  - scalar multiplication * : $1*P=P$, $0*P=\varnothing$, $P \in \mathcal{P}(E)$

- cycle space $\mathcal{C}(G)$: subspace of $\mathcal{E}(G)$ consisting of the cycles of G, including the "empty cycle" $\varnothing$

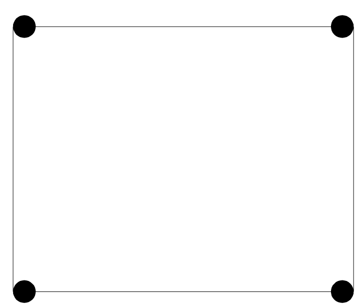
- cyclomatic number: $\mu(G):=|E|-|V|+1$

- $\dim \mathcal{C}(G)=\mu(G)$

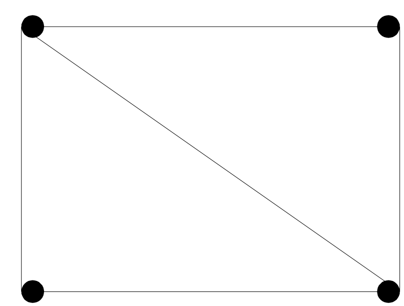
  (Intuition: when there exist more edges at a vertex, $\mu(G)$ is greater)

- for example :

$\mu(G) = 0$



$\mu(G) = 1$

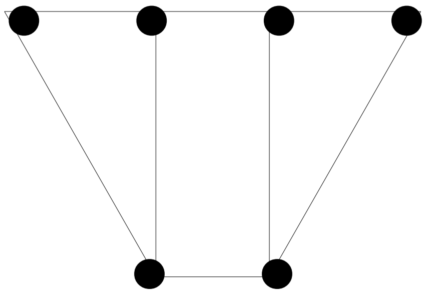


$\mu(G)=2$

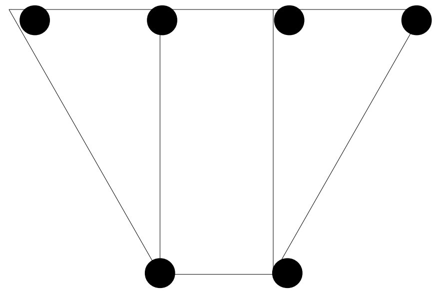
- cycle basis of G: a basis B of $\mathcal{C}$(G) consisting of circuits only, i.e. for every cycle C in G, there exists a unique subset B(C)$\subseteq$B of circuits in B such that C= $\oplus_{C' \in B(C)}$C' holds

- a sequence ($C_1, \ldots, C_k$) of circuits is defined to be cyclically well-arranged, if each partial sum $Q_j = \oplus_{i=1}^{j} C_i$ is a circuit for all j≤k

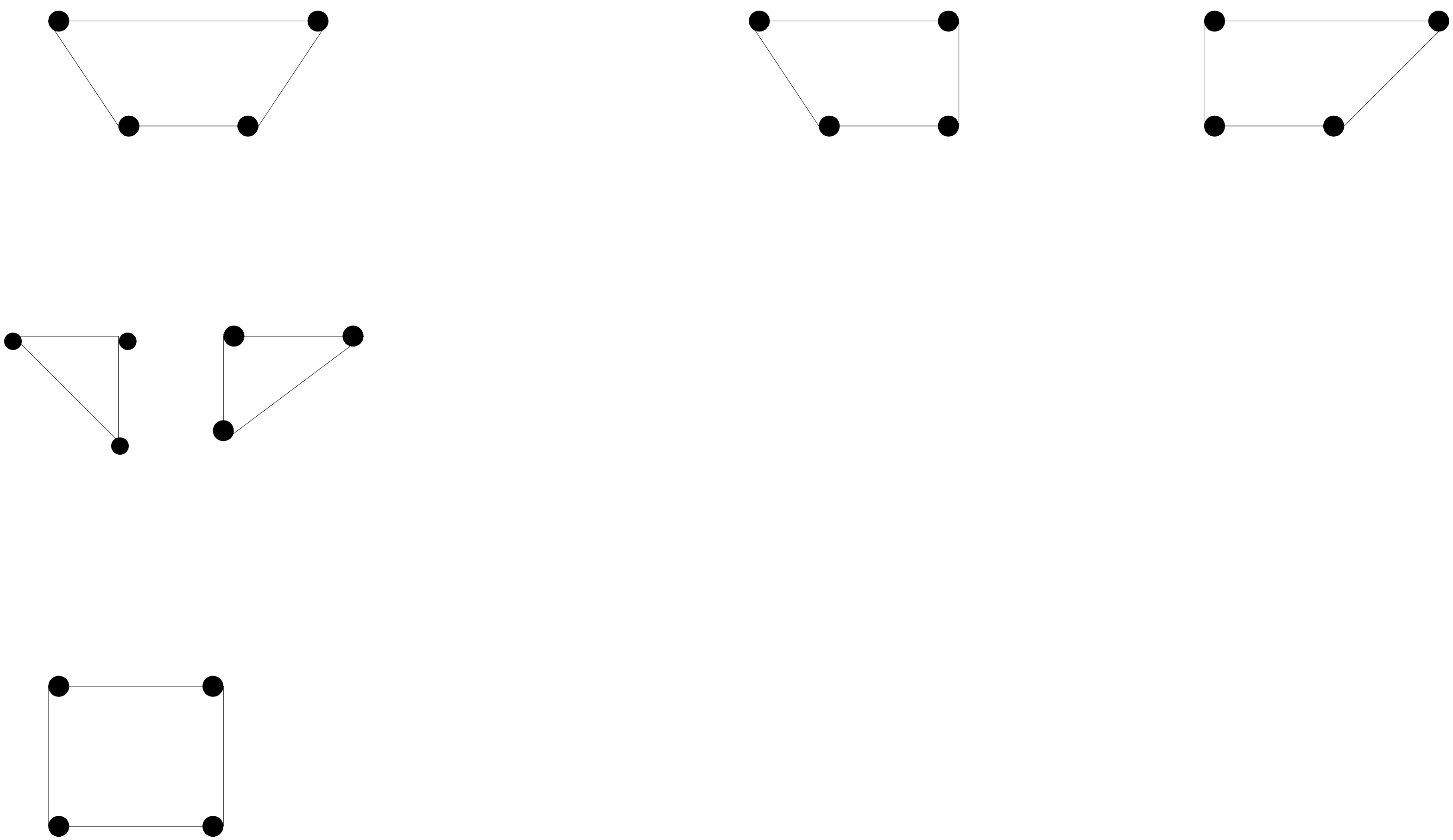
- a cycle basis B is cyclically robust, if for every circuit C, the corresponding set B(C) can be cyclically well-arranged

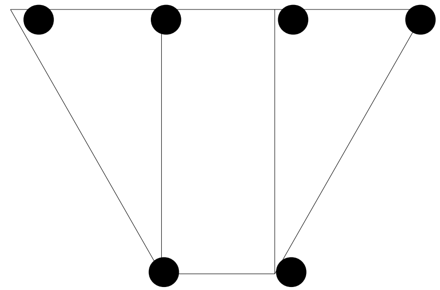
- example: 

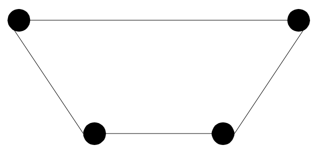
- example:

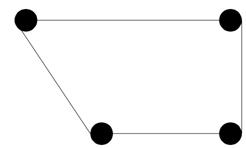


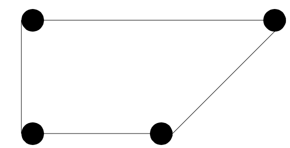
cuircuits:



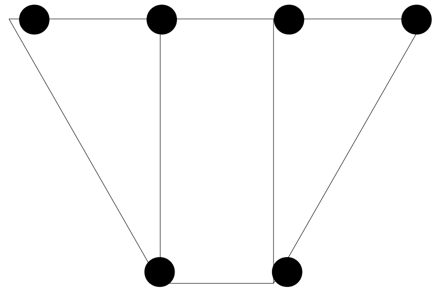
- example:  is a graph G=(V,E) with cycle basis

 C_1

 C_2

 C_3

- example:



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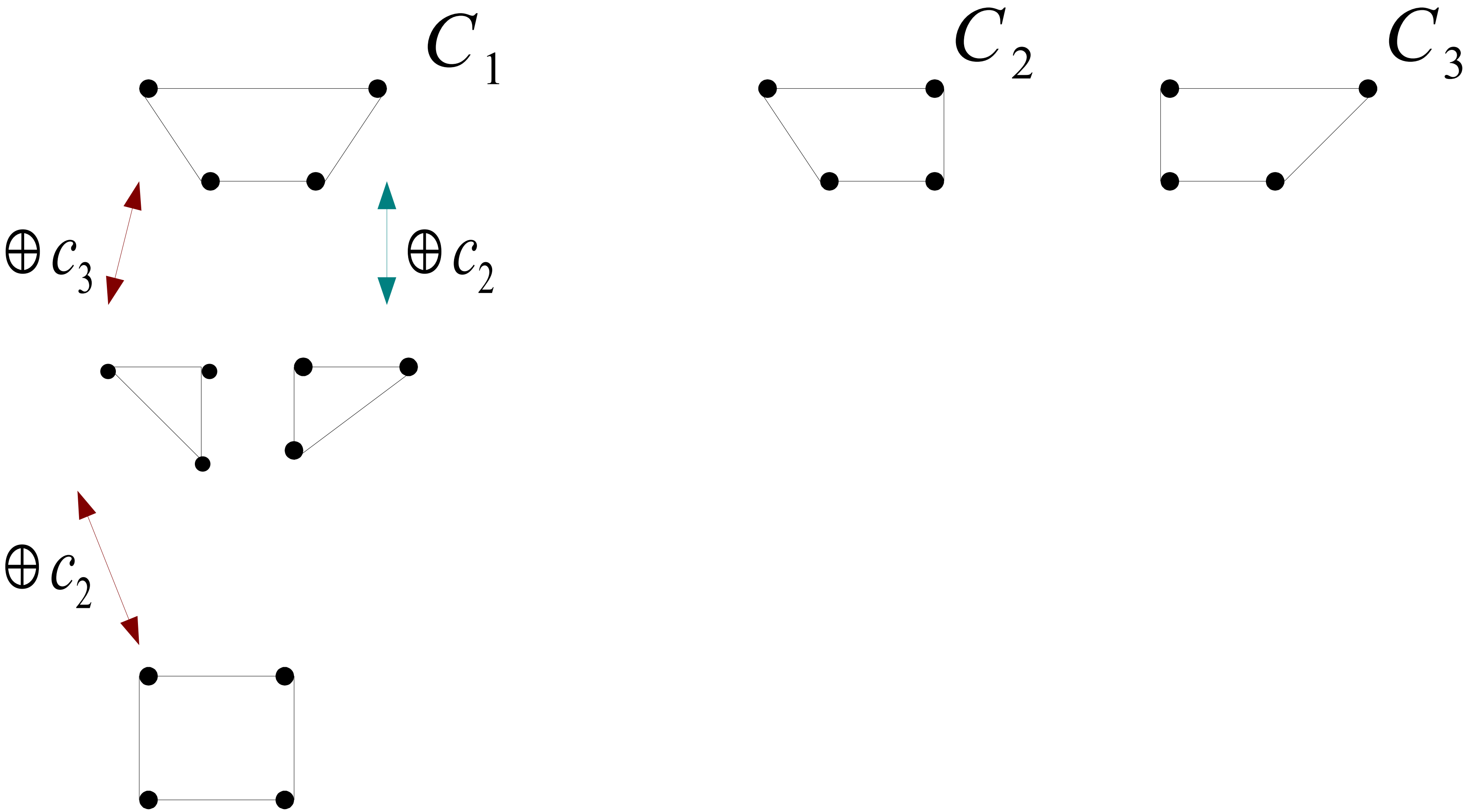
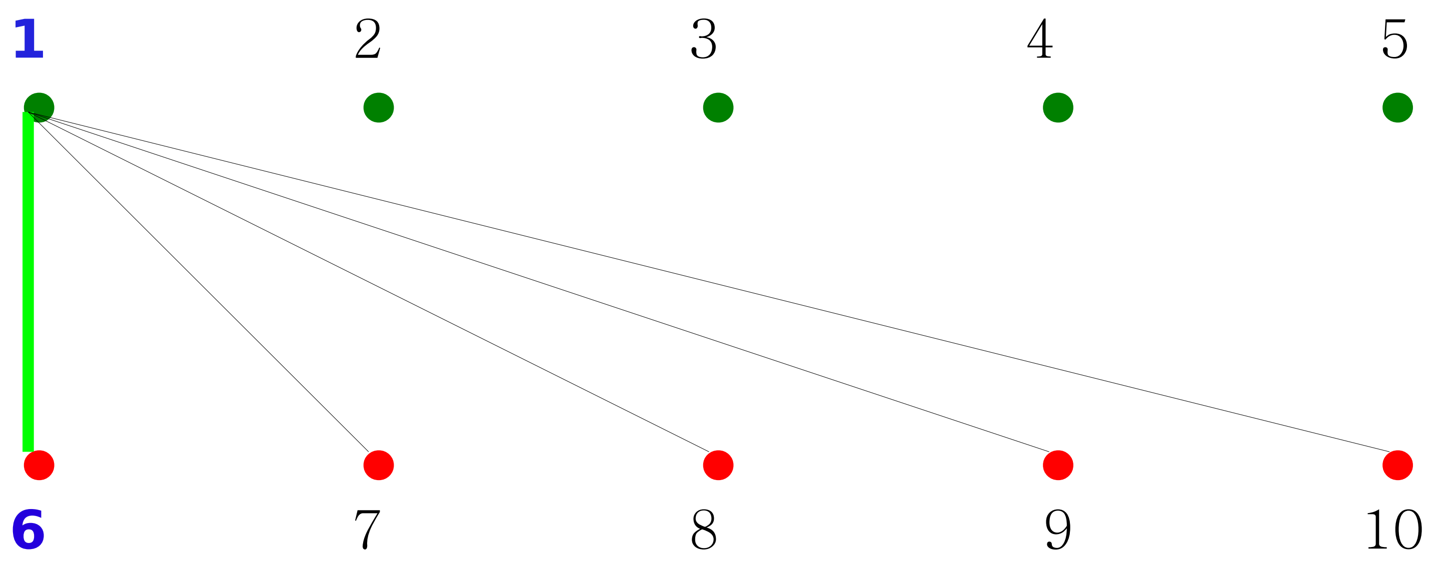


- a graph is complete if each pair of vertices has an edge connecting them
- a graph whose vertices can be divided into two disjoint sets U and V, such that every edge connects a vertex in U to one in V, that is, U and V are independend sets, is called bipartite
- a complete bipartite graph is a graph in which every two vertices from different vertex sets are connected

$K_{p,q}$ is the complete bipartite graph, p,q $\in \mathbb{N}$
|U|=q, |V|=p

- connecting each green vertex with all red vertices

$K_{5,5}$

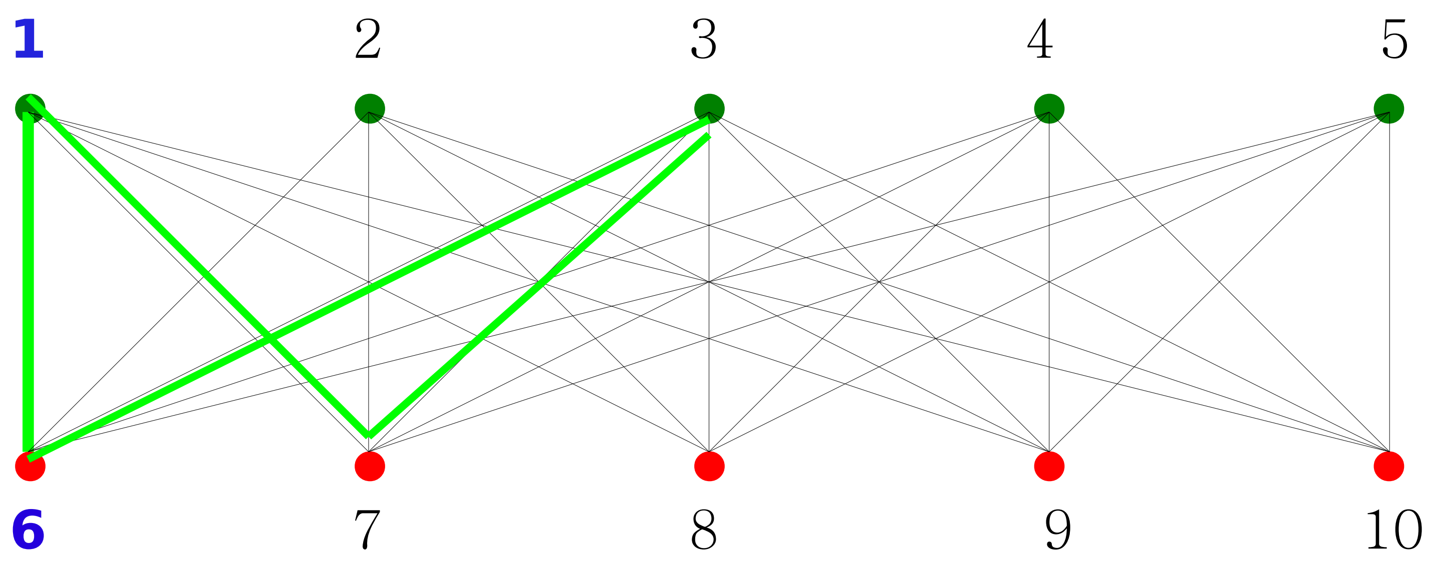


- $K_{p,q}$ is the complete bipartite graph, p,q $\in \mathbb{N}$

Paul Kainen asserts:

For every pair of postive integers p,q, $K_{p,q}$ has a robust cycle basis consisting of all **4-cycles** containing two **fixed** vertices, one of each color

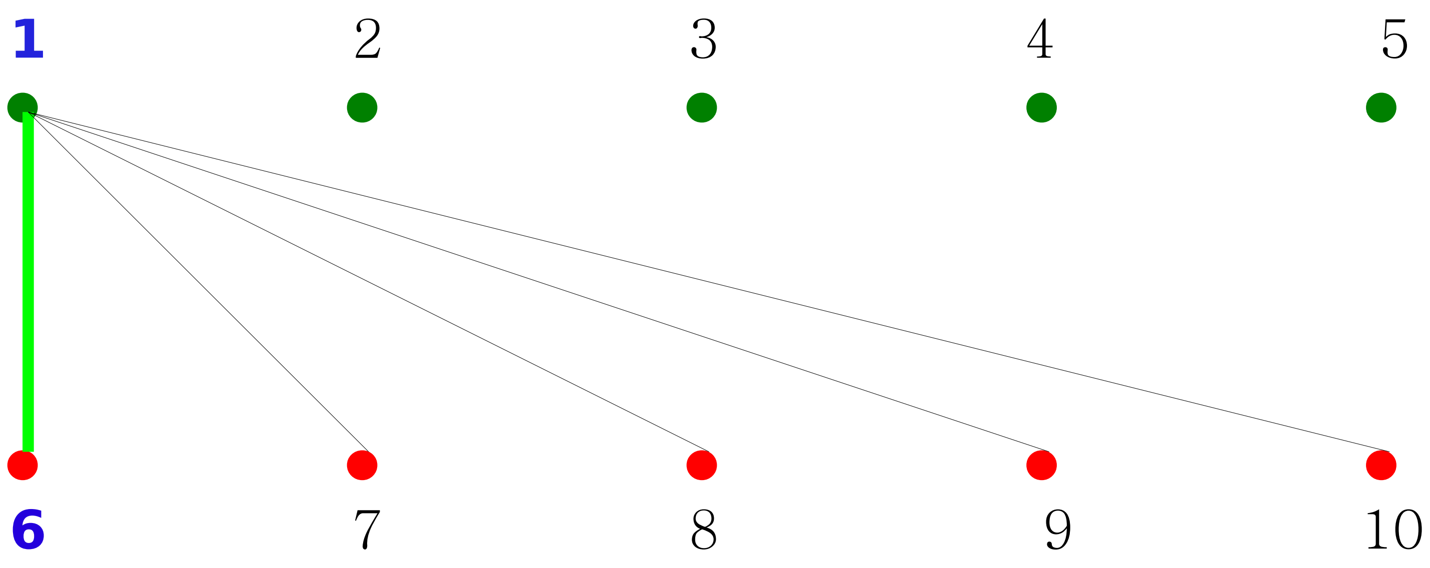
Counter example: $K_{5,5}$



- μ=25-10+1=16

- every basis element has the following form:
  (1-6- a_i- b_i-1) , i∈{1,..,16},
  (a-b) is an edge, a is a green vertex, b is a red vertex

$K_{5,5}$



- denote $\mathrm{I}=\{1,..,6\}$

  $a_1 \neq a_i \forall i \in \mathrm{I} \backslash \{1\}, a_2 \neq a_i \forall i \in \mathrm{I} \backslash \{2\}, a_3 = a_4, a_5 = a_6$

  $b_3 \neq b_i \forall i \in \mathrm{I} \backslash \{3\}, b_5 \neq b_i \forall i \in \mathrm{I} \backslash \{5\}, b_1 = b_4, b_2 = b_6$

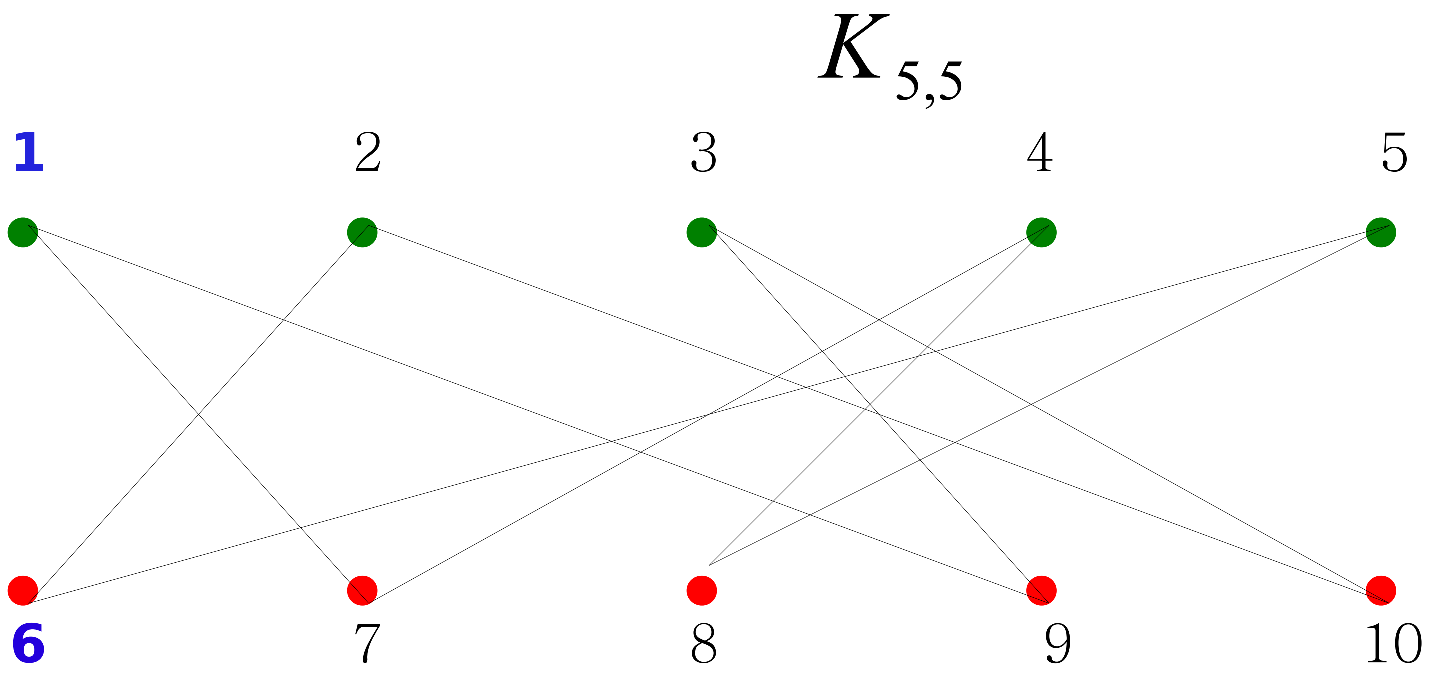
- no we construct the following circuit by symmetric difference of these 6 circuits:

$$\left(1 - b_5 - a_5 - b_2 - a_2 - 6 - a_1 - b_1 - a_3 - b_3 - 1\right)$$

- all graphs generated by 5 of those 6 circuits contain (1-6), if one computes the symmetric difference with a basis cycle which includes an a_i contained in two cycles, then the vertex degree at 6 is 4, the same holds at 1 and b_i by computing the symmetric difference with a basis cycle that contains an a_i which is not contained in another basis cycle

- for example:

  -(1-7-4-8-5-6-2-10-3-9-1)
  - compute the symm. diff. with the generating circuit :
    (1-6-3-9-1), the new graph contains the edges (1-6),(2-6),(5-6),(6-3)



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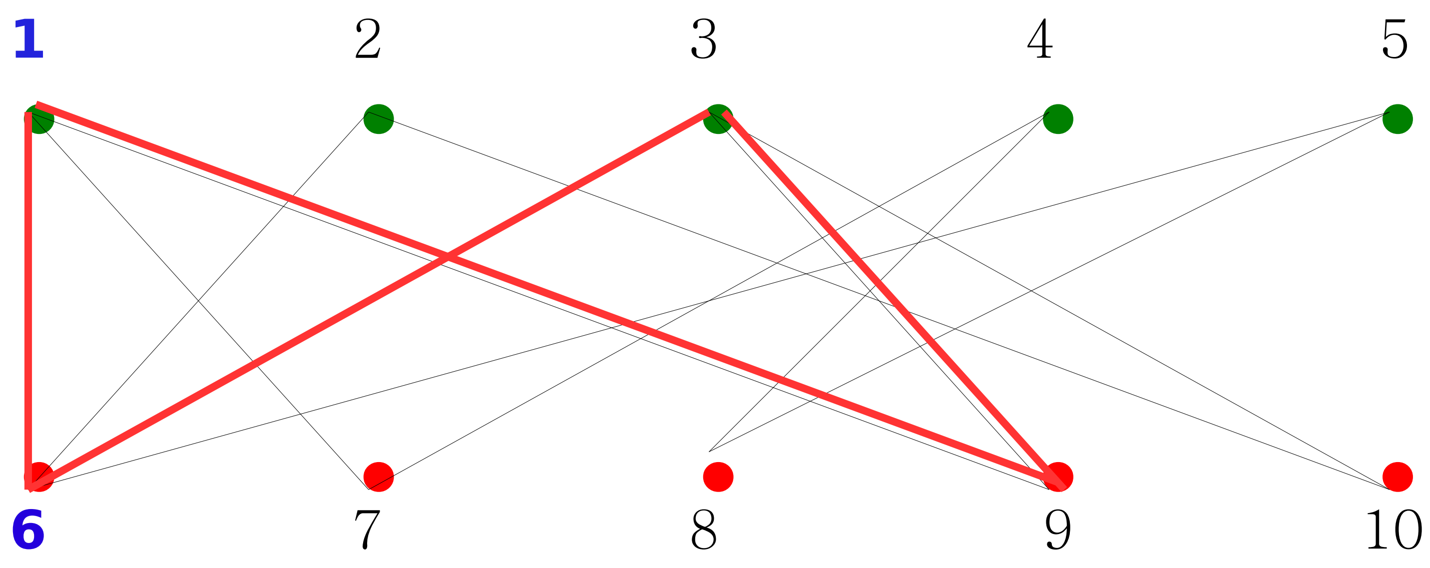
$K_{5,5}$



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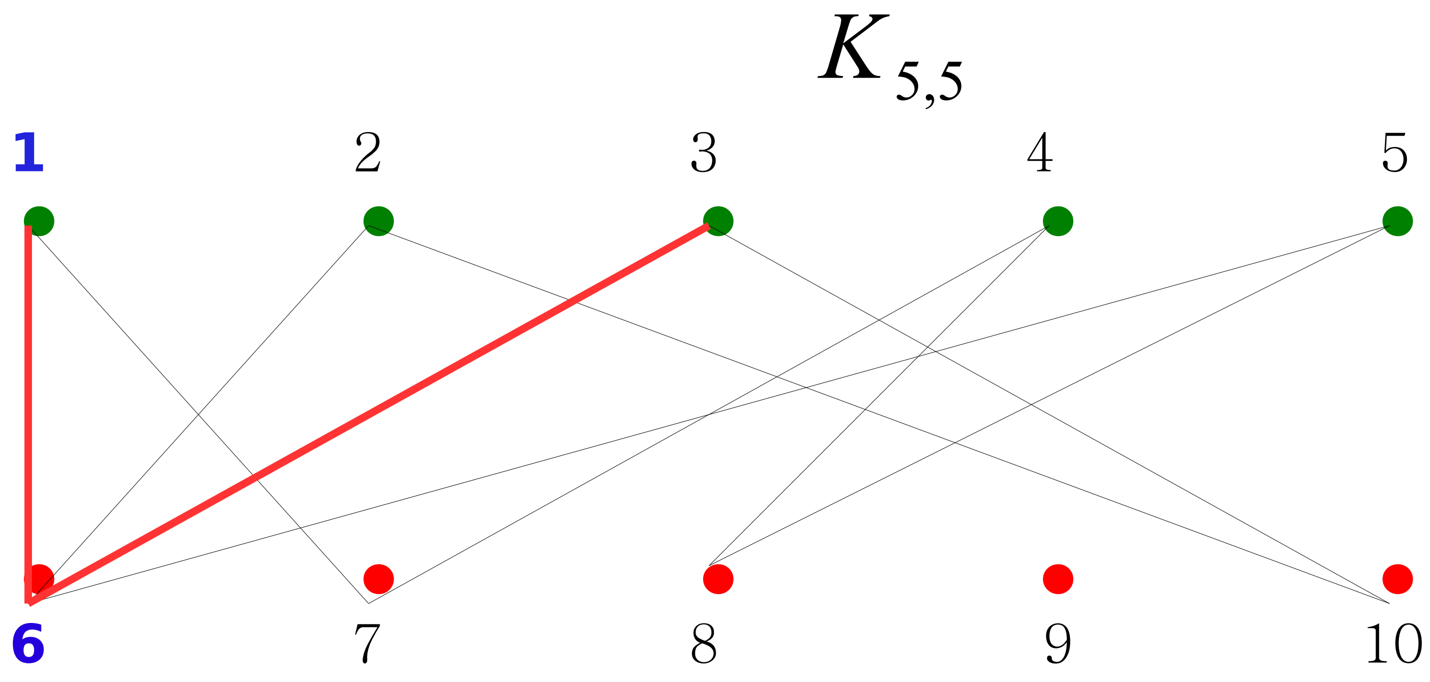

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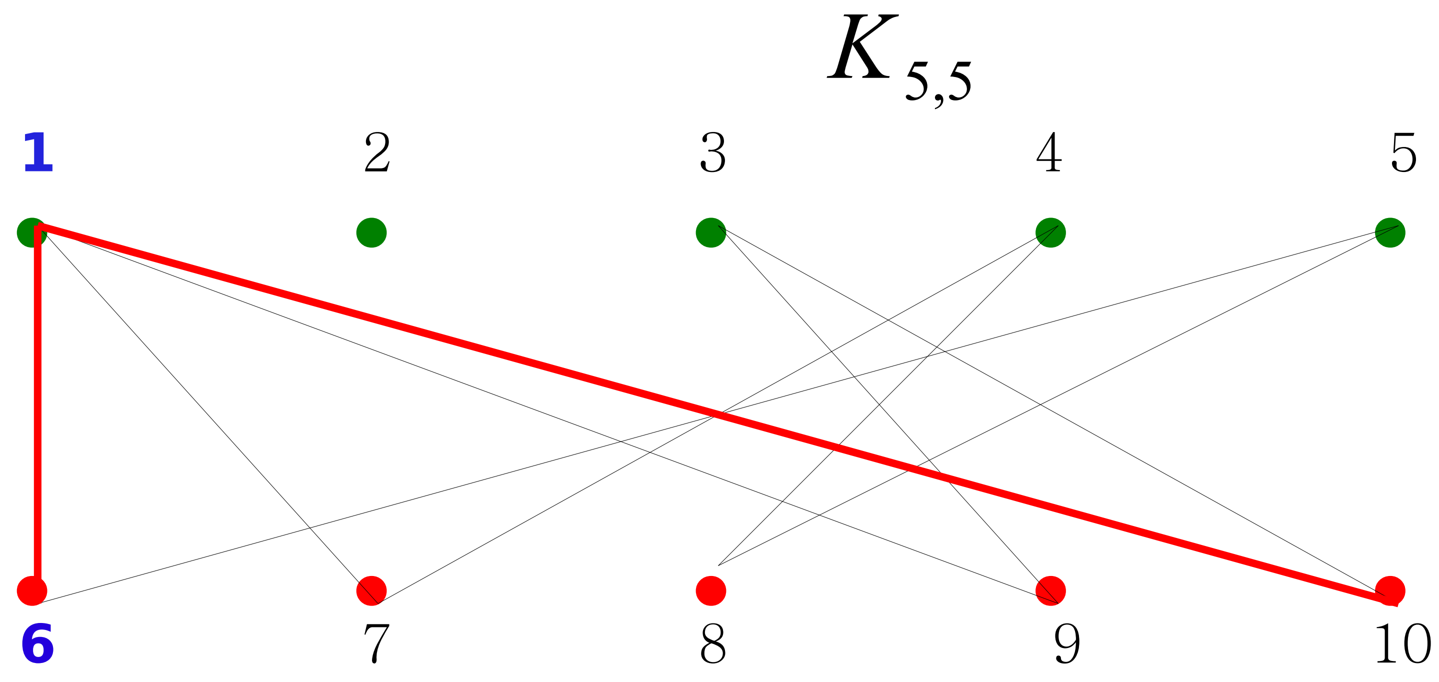

- a set of circuits F on the graph G is called quasi-robust, if and only if the following graph H=(K,L) is connected, K is the set of elementary cycles of G, L the set (C,D) of all pairs of circuits, such that there exists an $O \in F$ with $C \oplus D = O$

- the Kainen basis for $K_{5,5}$ is quasi-robust (proved by a computer programme)

Thanks for your attention !!!