QUASI-ROBUST CYCLE BASES

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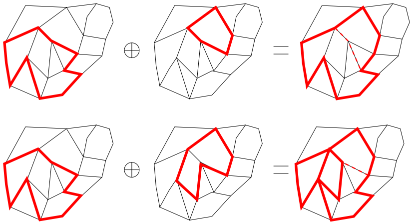
Motivation



- We want to construct the set of all circuits of a graph from a cycle basis $\mathscr{B}$ by iteratively computing the symmetric difference of a circuit and a basis cycle, subsequently retaining the result if and only if it is again a circuit. We will call this a “cycle space”-algorithm.

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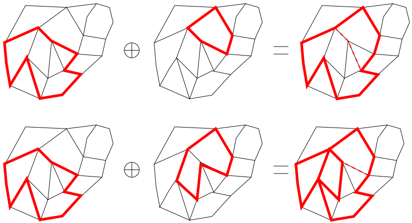

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- Problem: For which cycle bases does it work?

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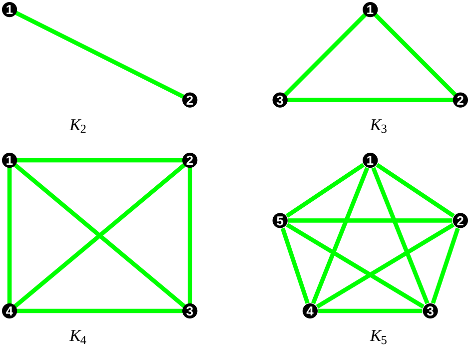

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- T is a *spanning tree* of G
  $\Leftrightarrow T$ is a tree and $V(T) = V(G), E(T) \subseteq E(G)$

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  - *symmetric difference* $X \oplus Y := (X \cup Y) \setminus (X \cap Y)$,
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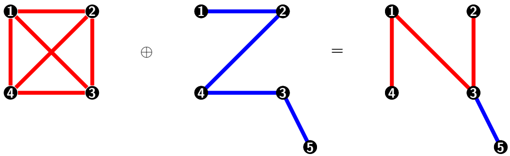

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- $F \in \mathscr{C}(G)$
  $\Leftrightarrow F$ is a disjoint union of circuits in G
  $\Leftrightarrow$ all vertex degrees of the graph $(V(F), F)$ are even

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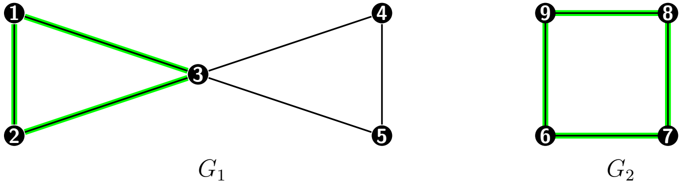
G_1 G_2

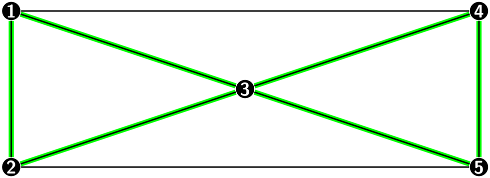
Linear independent cycles



- $\mathscr{C}(G_1 \cup G_2) = \mathscr{C}([1,2,3],[3,4,5],[6,7,8,9])$
- $|\mathscr{C}(G_1 \cup G_2| = 2^3$ and $\varnothing$ is the empty cycle.

A Problem

Let $\mathscr{S}$ be the set of all circuits in a graph.



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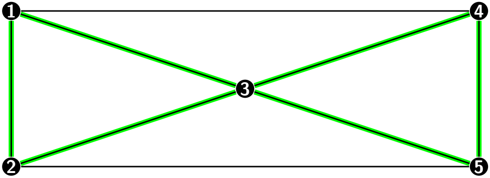

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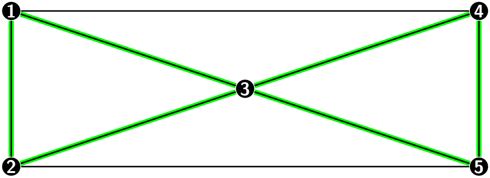

- $[1,2,3] \oplus [3,4,5] \notin \mathscr{S}$ for $[1,2,3],[3,4,5] \in \mathscr{S}$.
- $\Rightarrow$ $\mathscr{S}$ is no semigroup.
- $\Rightarrow$ Maybe the “cycle space”-algorithm cannot generate all circuits of a cycle space for a given basis.

Well-arranged sequences

- A possible way of solving this problem partially is to find a cycle basis and orderings of its elements such that the “cycle space”-algorithm can generate all circuits of a graph.

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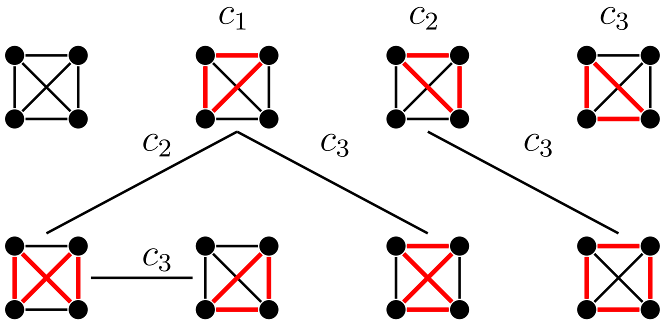
Definition

A sequence $\mathscr{S} = (C_1, C_2, \ldots, C_k)$ of cycles is *well-arranged*

$\Leftrightarrow \forall j \leq k : Q_j = \bigoplus_{i=1}^{j} C_i$ is a circuit.

Well-arranged Sequences of Basis Cycles



Definition

A cycle basis $\mathscr{B}$ is

- quasi-robust $\Leftrightarrow \forall$ circuits $C \in \mathscr{C}(G)$ $\exists$ a well-arranged sequence $\mathscr{S}_C = (C_1, C_2, \ldots, C_{k_C})$ with $C_i \in \mathscr{B}$,$1 \leq i \leq k_C - 1$, and $C_{k_C} = C$.

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- robust $\Leftrightarrow$ $\mathscr{B}$ is quasi-robust and all cycles in $\mathscr{S}_C$ are linear independent.

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Split graphs

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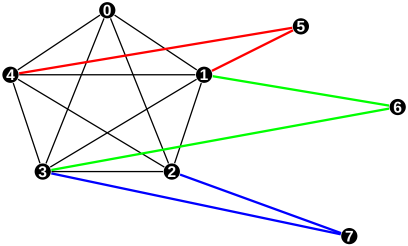

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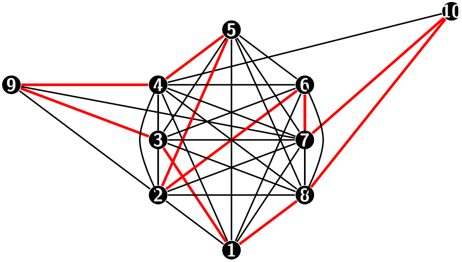
Lemma

$\mathscr{B} = \{B_{ij} | i \in V, j \in \Gamma(i) \setminus \{m(i)\}, i > 2\}$ is a (fundamental and) quasi-robust cycle basis of the split graph $G = (V, E)$.

$\mathscr{B}$ is not robust.

- $K = \{1, \dots, 8\}$ and $I = \{9, 10\}$.



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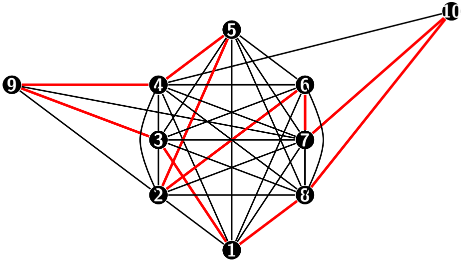
- $\mathscr{B}_C = \{B_{42}, B_{54}, B_{84}, B_{25}, B_{26}, B_{67}, B_{23}, B_{47}, B_{10,8}, B_{49}, B_{93}, B_{10,7}\}$ is cycle basis of the red cycle C.



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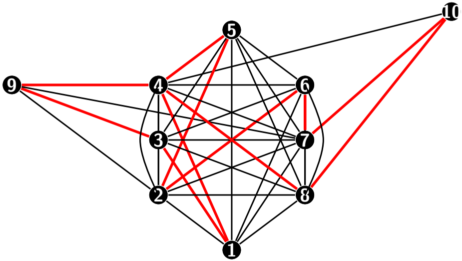
- $C \oplus \{B_{84}\}$ is no circuit holds for all $C \oplus B_{ij}$.



$\Rightarrow$ Not every triangular cycle basis is robust.

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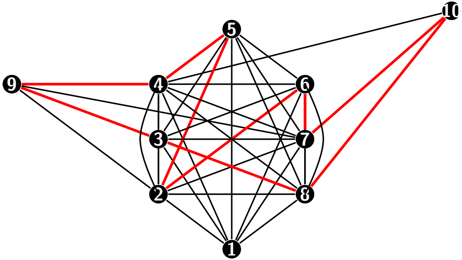
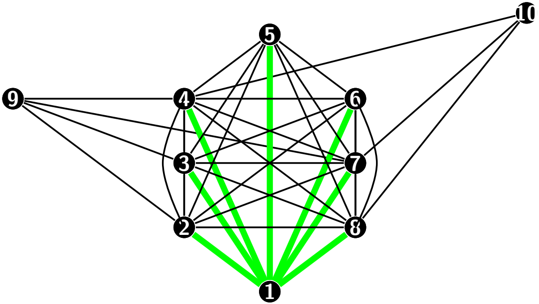
- $C \oplus \{B_{83}\}$ is a circuit with $B_{83} \notin \mathscr{B}_C$.



$\mathscr{B}$ is not strictly fundamental



The green edges form a spanning tree T of K_8.

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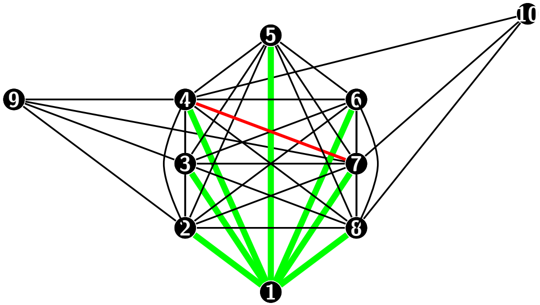

$T \cup \{\{4,7\}\}$ is not acyclic.

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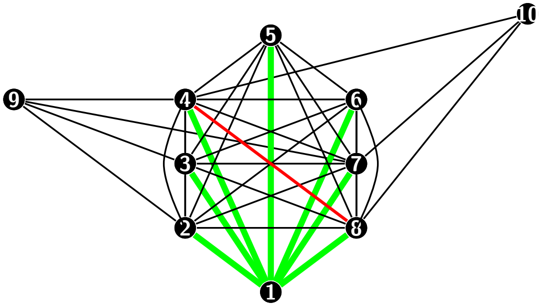

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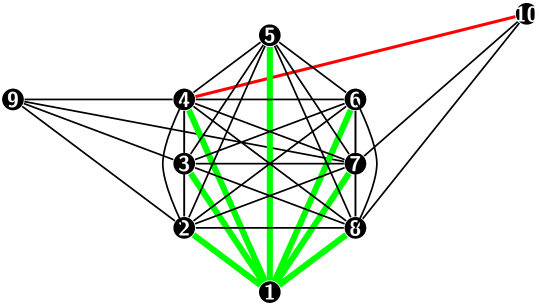

$T \cup \{\{4, 10\}\}$ would generate the circuit $[4, 10, 7, 1]$.

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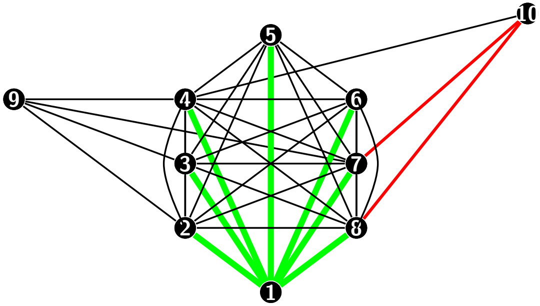
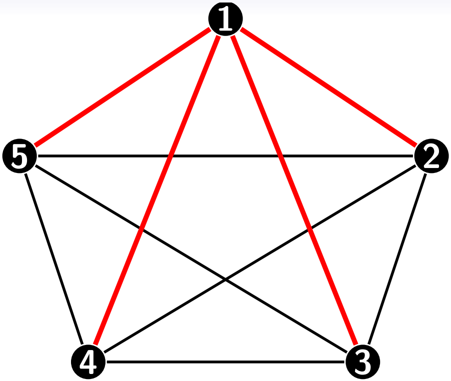


$T \cup \{\{10,8\}\}$ or $T \cup \{\{10,7\}\}$ would generate the circuit $[8,10,7,1]$.

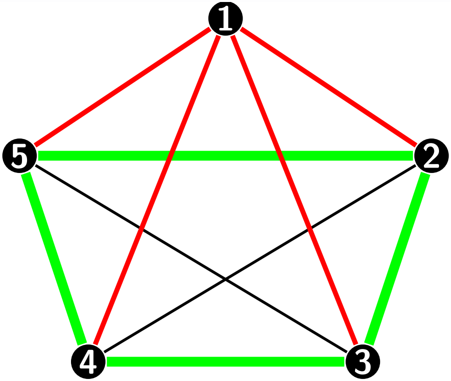
Theorem

Every triangular strictly fundamental cycle basis is strictly robust.



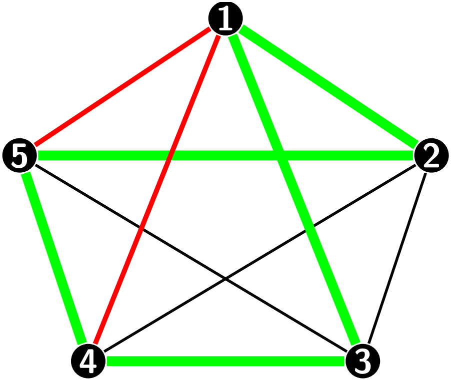
K_5

- The red spanning tree T induces a triangular basis with 6 elements.



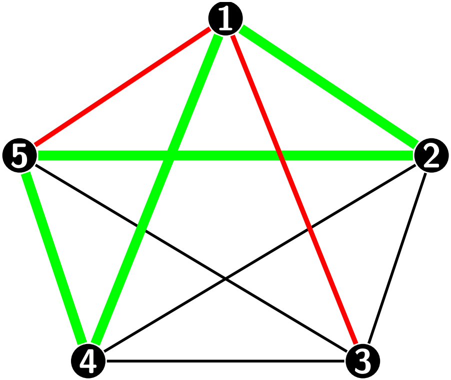
K_5

- Consider the green cycle, $C \cap T = \emptyset$.



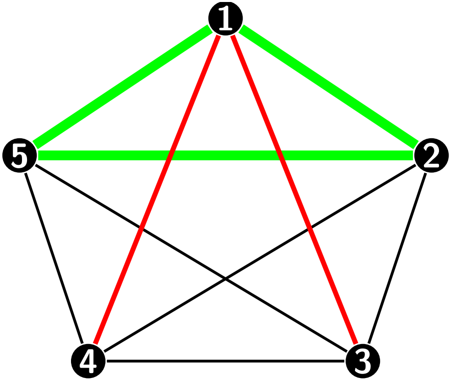
K_5

- $C \oplus [1, 2, 3]$.



K_5

- C $\oplus$ [1, 2, 3] $\oplus$ [1, 3, 4].



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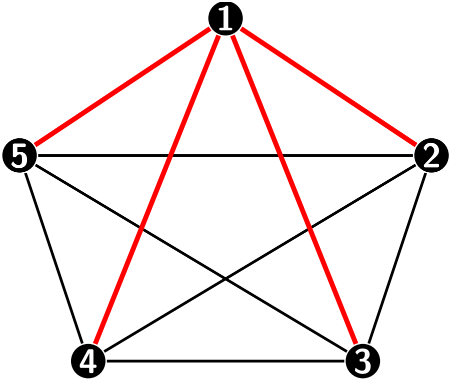
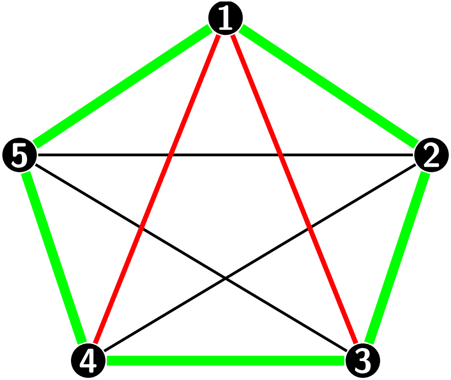
- $C \oplus [1,2,3] \oplus [1,3,4] \oplus [1,5,4]$.



K_5

- C $\oplus$ [1, 2, 3] $\oplus$ [1, 3, 4] $\oplus$ [1, 5, 4] $\oplus$ [1, 2, 5].



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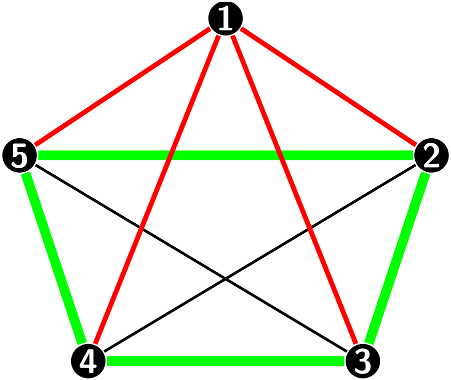
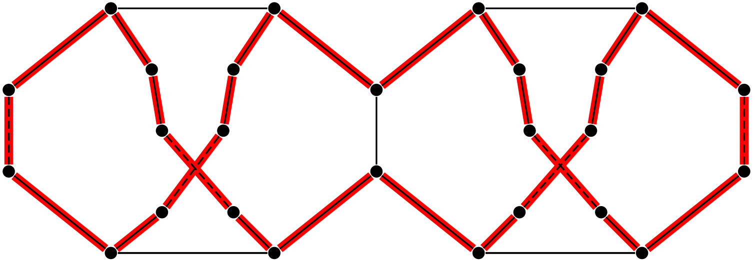
- Consider the green cycle, $C' \cap T \neq \emptyset$.



K_5

- $C' \oplus [1,2,5] = C$, now we can start as above.

Is every strictly fundamental cycle basis quasi-robust[1]?



- $T = \{\text{black edges}\}$
  $\Rightarrow B_T = \{C_1, \dots, C_6\}$
- $C = \bigoplus_{i=1}^{6} C_i$ is the red circuit.
- $C \oplus C_i,\ i = 1, \dots, 6$ is no circuit

ANSWER: No!!!

Complete Bipartite Graphs $K_{m,n}$

$V(K_{m,n}) = V_1 \cup V_2$ with $|V_1| = m, |V_2| = n$ and $\{v, v'\} \in E(K_{m,n})$ iff $v \in V_1, v' \in V_2$

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The Kainen basis $\mathscr{B}^{p,q}$

$\mathscr{B}^{p,q} = \{[p,q,x,y] \,|\, p,x \in V_1, q,y \in V_2, p,q \text{ are fixed }\}$ is a basis of $K_{m,n}$.

A spanning tree for $\mathscr{B}^{1,6}$.

Assume $p = 1, q = 6$.

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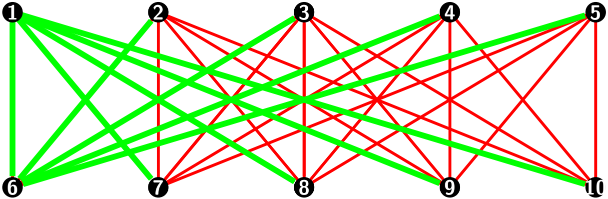

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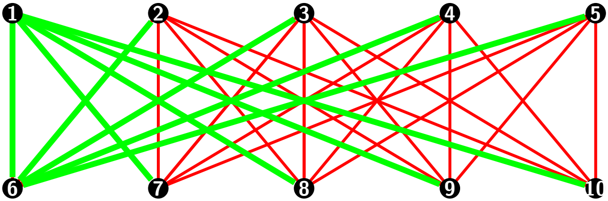
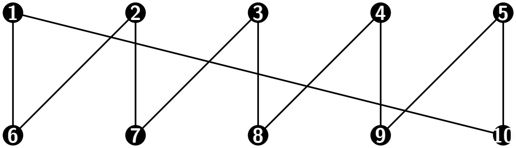


$T^{1,6} =$
$\big\{\{1,6\},\{1,y_1\},\ldots,\{1,y_{n-1}\},\{6,x_1\},\ldots,\{6,x_{m-1}\}\ |\forall x_1,\ldots,x_{m-1} \in V_1 \forall y_1,\ldots,y_{n-1} \in V_2\big\}$

The not robust cycle basis $\mathscr{B}^{1,8}$

1



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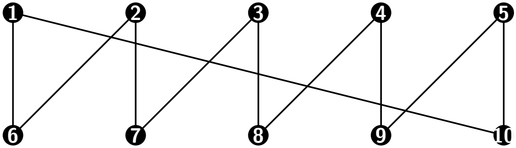

- Consider the circuit $\mathscr{CYCLE} = [1, 6, 2, 7, 3, 8, 4, 9, 5, 10]$.

The not robust cycle basis $\mathscr{B}^{1,8}$

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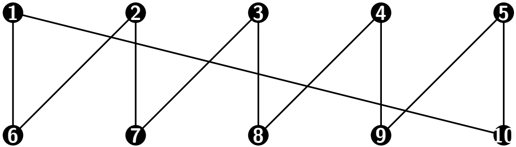

- Consider the circuit $\mathscr{CYCLE} = [1, 6, 2, 7, 3, 8, 4, 9, 5, 10]$.
- $\mathscr{B}^{1,8}_{\mathscr{CYCLE}} = \{C_{4,9}, C_{3,7}, C_{2,6}, C_{5,10}, C_{2,7}, C_{5,9}\}$, where $C_{xy} = [1, 8, x, y]$.

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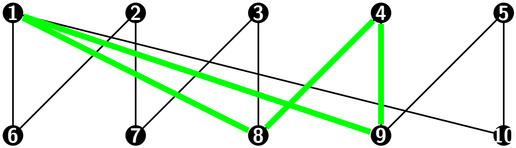

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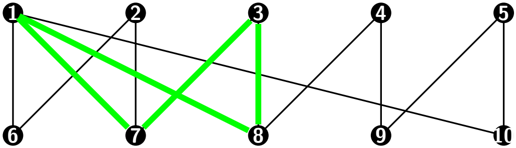

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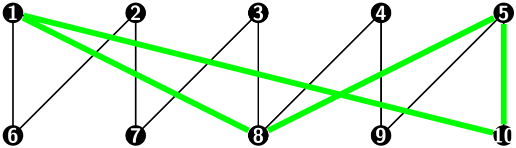
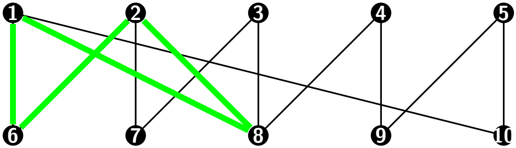
The not robust cycle basis $\mathscr{B}^{1,8}$



- Consider the circuit $\mathscr{CYCLE} = [1, 6, 2, 7, 3, 8, 4, 9, 5, 10]$.
- $\mathscr{B}^{1,8}_{\mathscr{CYCLE}} = \{C_{4,9}, C_{3,7}, C_{2,6}, C_{5,10}, C_{2,7}, C_{5,9}\}$, where $C_{xy} = [1, 8, x, y]$.
- Degree of vertex 8 is 4.

The not robust cycle basis $\mathscr{B}^{1,8}$

1



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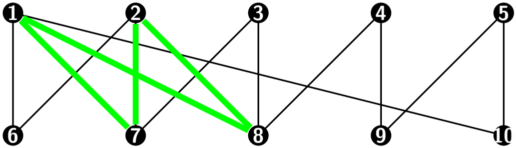
1



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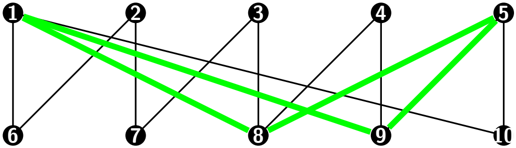
The not robust cycle basis $\mathscr{B}^{1,8}$



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The not robust cycle basis $\mathscr{B}^{1,8}$

$\Longrightarrow$

The Kainen basis is not robust.

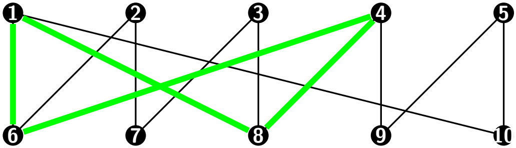
Theorem

The Kainen basis $\mathscr{K}$ of $K_{m,n}$ is quasi-robust for all m, n.

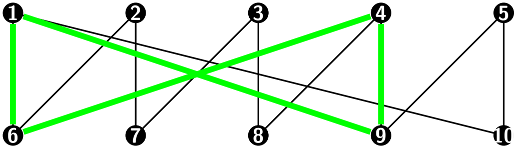
Theorem

The Kainen basis $\mathscr{K}$ of $K_{m,n}$ is quasi-robust for all m, n.

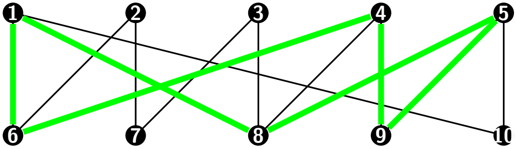
Let us construct a quasi-robust sequence $\mathscr{S}$ for $\mathscr{CYCLE} = [1, 6, 2, 7, 3, 8, 4, 9, 5, 10]$.



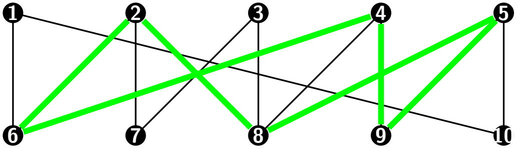
- Consider the circuit $\mathscr{CYCLE} = [1, 6, 2, 7, 3, 8, 4, 9, 5, 10]$.
- $\mathscr{S} = (C_{4,6}, C_{4,9}, C_{5,9}, C_{2,6}, C_{2,7}, C_{5,10}, C_{3,7}, C_{4,6})$, where $C_{xy} = [1, 8, x, y]$.



- Consider the circuit $\mathscr{CYCLE} = [1,6,2,7,3,8,4,9,5,10]$.
- $\mathscr{S} = (C_{4,6}, C_{4,9}, C_{5,9}, C_{2,6}, C_{2,7}, C_{5,10}, C_{3,7}, C_{4,6})$, where $C_{xy} = [1,8,x,y]$.



- Consider the circuit $\mathscr{CYCLE} = [1,6,2,7,3,8,4,9,5,10]$.
- $\mathscr{S} = ($$C_{4,6}, C_{4,9}, C_{5,9}$$, C_{2,6}, C_{2,7}, C_{5,10}, C_{3,7}, C_{4,6})$, where $C_{xy} = [1,8,x,y]$.



- Consider the circuit $\mathscr{CYCLE} = [1,6,2,7,3,8,4,9,5,10]$.
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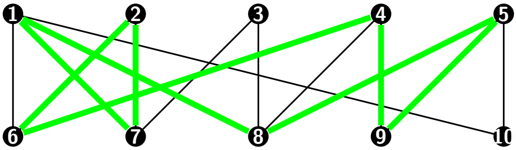

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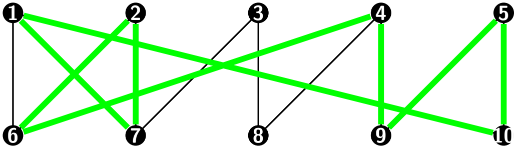

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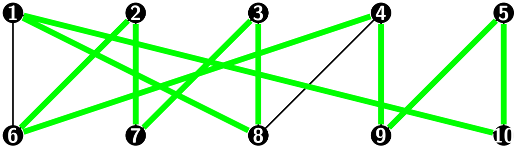

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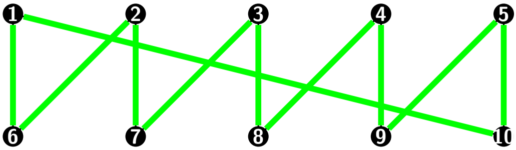

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Thanks for your attention!

Thanks to Peter, Konstantin, Marc, Josef, ...