Convex Excess and Inequalities for Partial Cubes

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- convex $\Rightarrow$ isometric $\Rightarrow$ induced

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- Hence partial cubes are precisely the graphs that admit isometric embeddings into hypercubes.

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Examples of partial cubes

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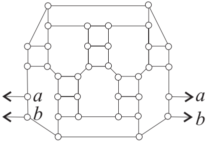
- Cartesian products of partial cubes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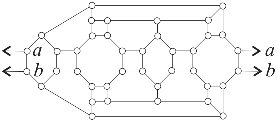
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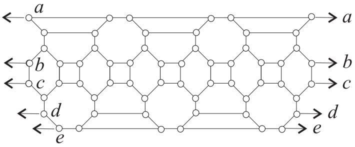
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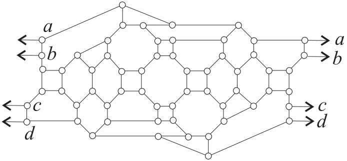
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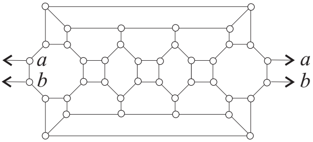
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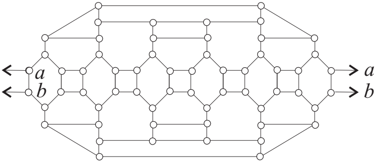
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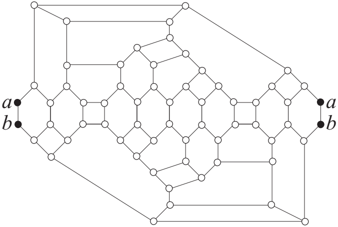
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- G graph, new vertices its acyclic orientations, orientations differing by one edge of G.
- Integer partitions: vertices = partitions, edges = increment largest value and decrement some other value (or vice versa).
- Flips of triangulations; oriented matroids; media theory.

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Theorem (Djoković, 1973)

A connected graph G is a partial cube if and only if G is bipartite and for any edge uv of G the subgraph W_{ab} is convex.

Winkler

- Edges $e = xy$ and $f = uv$ of G are in relation Θ if

$$d(x,u) + d(y,v) \neq d(x,v) + d(y,u)\,.$$

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Theorem (Winkler, 1984)

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Chepoi

- A proper cover of G: isometric subgraphs G_1 and G_2 such that $G = G_1 \cup G_2$ and $G_0 = G_1 \cap G_2 \neq \emptyset$.

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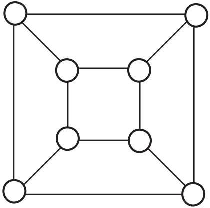
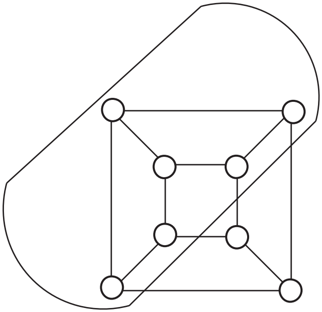
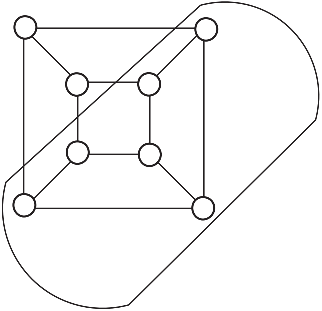
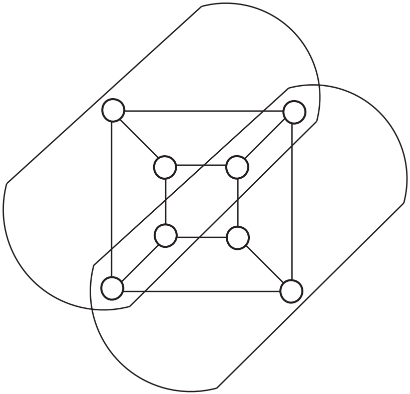
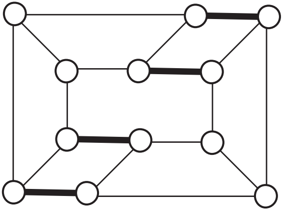
- Insert the edges $v_1 u_1$ and $v_2 u_2$ whenever $v, u \in G_1 \cap G_2$ are adjacent in G.

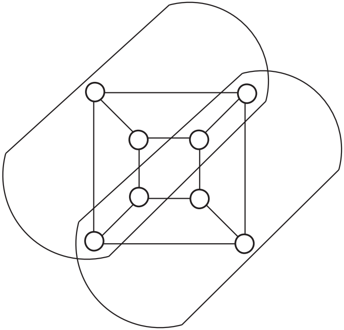


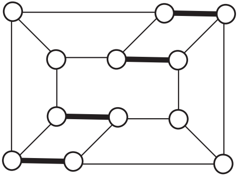












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That is, the smallest d such that G isometrically embeds into Q_d.

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- Bipartite ℓ_1-graphs.
- Bipartite graphs whose distance matrix has exactly one positive eigenvalue.

Median graphs

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Median graphs and triangle-free graphs

Theorem (Imrich, K., Mulder, 1999)

Let $M(m, n)$ be the complexity of checking whether a graph G with m edges and n vertices is median. Then the complexity of checking whether G is triangle-free is at most $O(M(m, m))$.

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Theorem (Imrich, K., Mulder, 1999)

Let $T(m, n)$ be the complexity of finding all triangles of a given graph with m edges and n vertices. Then the complexity of checking whether a graph G on n vertices and m edges is a median graph is at most $O(m \log n + T(m \log n, n))$.

Inequality for median graphs

Theorem (K., Mulder, Škrekovski, 1998)

G median graph with n vertices and m edges. Then

$$2n - m - i(G) \leq 2\,.$$

Moreover equality holds if and only if G is Q_3-free.

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Theorem (Brešar, K., Škrekovski, 2003)

Let G be a graph with n vertices and m edges that is obtained by a sequence of connected expansions from K_1. Then $2n - m - i(G) \leq 2$. Moreover equality holds if and only if G is $C_t \Box K_2$-free ($t \geq 3$) and K_4-free.

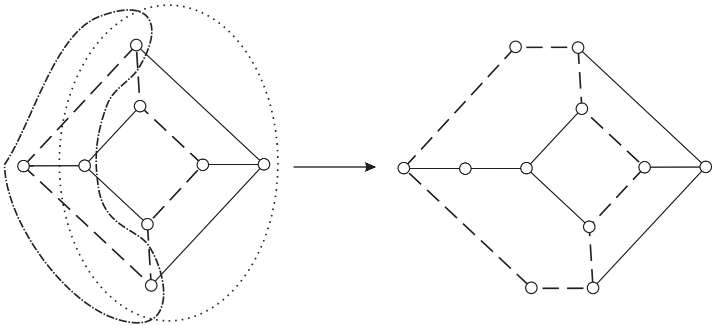
The question

Brešar, Imrich, K., Mulder, Škrekovski (JGT, 2002): Is there such an inequality for all partial cubes? In particular, does

$$2n - m - 2i(G) \leq 0$$

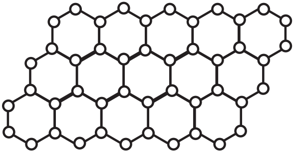
hold for any partial cube?

A reason for troubles



$2n - m - 2i(G) \leq 0$ need not hold

- $P(r, s)$, $1 \leq s \leq r$, parallelogram hexagonal graph.
- $n = (r+1)(2s+2) - 2$,
  $m = (r+1)(2s+1) - 2 + r(s+1)$,
  $i(P(r,s)) = 2r + 2s - 1$.
- $2n - m - 2i(P(r,s)) = rs - 2(r+s) + 3$.



The inequality

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$$ce(G) = \sum_{C \in \mathcal{C}(G)} \frac{|C| - 4}{2}.$$

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- Spread partial cube: all zone graphs are trees.

Theorem

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Moreover the equality holds if and only G is a spread partial cube.

Proof

Proposition

Let G be a partial cube and let $\widetilde{G}$ be the expansion of G with respect to an isometric cover G_1, G_2. If C is a convex cycle of G, then its expansion $\widetilde{C}$ is a convex cycle of $\widetilde{G}$.

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Proposition

The zone graphs of partial cubes are connected.

Proof cont'd

- $\widetilde{G}$ expansion of G with respect to G_1, G_2. By induction, $2n - m - i(G) - ce(G) \leq 2$.

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- $\widetilde{n} = n + n_0 \quad \text{and} \quad \widetilde{m} = m + n_0 + m_0.$

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- $\widetilde{n} = n + n_0 \quad \text{and} \quad \widetilde{m} = m + n_0 + m_0$. $i(\widetilde{G}) = i(G) + 1$.
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- t: the number of connected components of G_0.
- By the two propositions, $\widetilde{G}$ contains at least $t - 1$ convex cross cycles (with respect to G_1, G_2) of length at least six. So $ce(\widetilde{G}) \geq ce(G) + t - 1$.

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- $m_0 \geq n_0 - t$.

Proof cont'd

$$
\begin{aligned}
&2\widetilde{n} - \widetilde{m} - i(\widetilde{G}) - ce(\widetilde{G}) \\
&\leq 2(n + n_0) - (m + n_0 + m_0) - (i(G) + 1) - (ce(G) + t - 1) \\
&= (2n - m - i(G) - ce(G)) + (n_0 - m_0 - t) \\
&\leq 2 + (n_0 - (n_0 - t) - t) \\
&= 2\,.
\end{aligned}
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- $m_0 = n_0 - t$ (G_0 must be a forest)

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- $2n - m - i(G) - ce(G) = 2$ (the contraction G satisfies the equality)
- $m_0 = n_0 - t$ (G_0 must be a forest)
- $ce(\widetilde{G}) = ce(G) + t - 1$ (among the edges of the zone graph Z_F there are exactly $t - 1$ cycles of length at least six and, furthermore, every convex cycle of $\widetilde{G}$ contracts to a convex cycle in G)

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The last two conditions imply that Z_F is a tree.

Proof cont'd

Proposition

G spread partial cube. Then for any two different Θ-classes F and F' there is at most one convex cycle such that it is an edge in both Z_F and $Z_{F'}$.

Proof cont'd

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Corollary

G spread partial cube, C and C' different convex cycles that are edges of Z_F. Then these cycles share no edges outside F.

Proof cont'd

- G spread partial cube, F Θ-class F, G_1 and G_2 connected components of $G \setminus F$.

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Proof cont'd

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- G_{10} subgraph of G_1 induced on vertices that have a neighbor in G_2, G_{20} the isomorphic subgraph of G_2. Let $n_0 = |V(G_{10})| = |V(G_{20})|$.

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- G_{10} is a forest (it is isomorphic to a subgraph of Z_F).

Proof cont'd

- t: the number of connected components of G_{10}.

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Proof cont'd

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- $n = n_1 + n_2 \quad \text{and} \quad m = m_1 + m_2 + n_0.$
- $i(G) = 1 + i(G_1) + i(G_2) - (n_0 - 1) - \sum_{j=1}^{t-1} ce(C^{(j)}).$

Proof cont'd

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- $n = n_1 + n_2 \quad$ and $\quad m = m_1 + m_2 + n_0$.
- $i(G) = 1 + i(G_1) + i(G_2) - (n_0 - 1) - \sum_{j=1}^{t-1} ce(C^{(j)})$.
- $$\begin{aligned} \sum_{C \in E(Z_F)} \frac{|C| - 2}{2} &= \sum_{C \in E(Z_F)} (ce(C) + 1) \\ &= n_0 - 1 + \sum_{j=1}^{t-1} ce(C^{(j)}) . \end{aligned}$$

Proof cont'd

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Proof cont'd

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$$2n - m - i(G) - ce(G) = 2(n_1 + n_2) - (m_1 + m_2 + n_0)$$

$$-\Big(1 + i(G_1) + i(G_2) - (n_0 - 1) - \sum_{j=1}^{t-1} ce(C^{(j)})\Big)$$

$$-\Big(ce(G_1) + ce(G_2) + \sum_{j=1}^{t-1} ce(C^{(j)})\Big)$$

$$= (2n_1 - m_1 - i(G_1) - ce(G_1)) + (2n_2 - m_2 - i(G_2) - ce(G_2)) - 2$$

$$= 2 + 2 - 2 = 2\,.$$