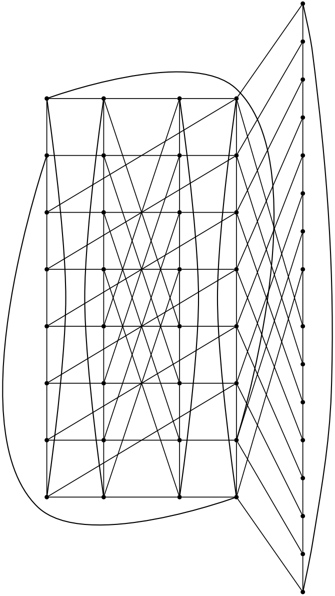
Twisted Cartesian Products

Lydia Ostermeier

Bioinformatik Leipzig

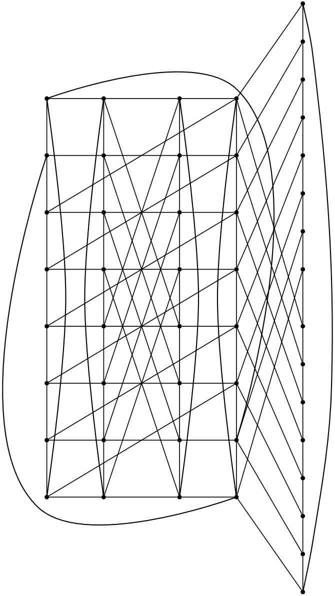
February 15, 2011

Introduction



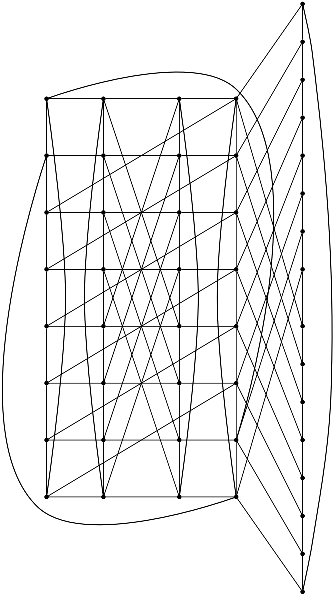
- Given: graph G similar to a Cartesian product
- G has non-trivial δ^*
- **Aim:**
  - find the underlying Cartesian product graph
  - reconstruct "approximate" factors

Introduction



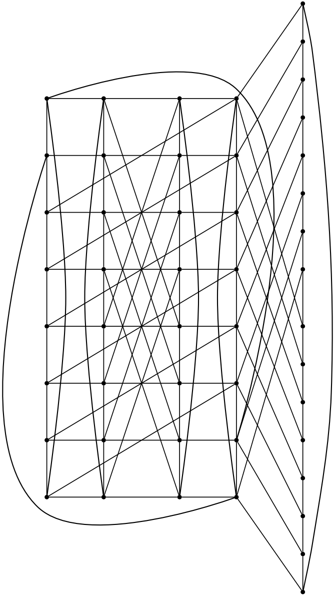
- Given: graph G similar to a Cartesian product
- G has non-trivial δ^*
- **Aim:**
  - find the underlying Cartesian product graph
  - reconstruct "approximate" factors

Introduction



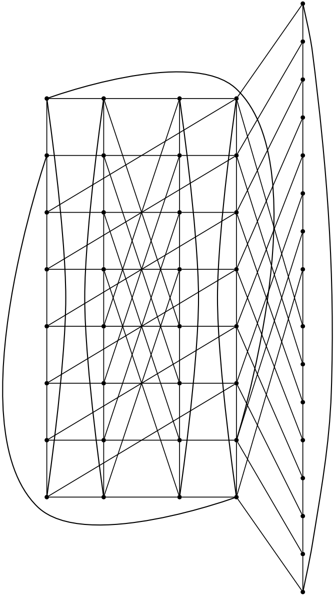
- Given: graph G similar to a Cartesian product
- G has non-trivial δ^*
- **Aim:**
  - find the underlying Cartesian product graph
  - reconstruct "approximate" factors

Introduction



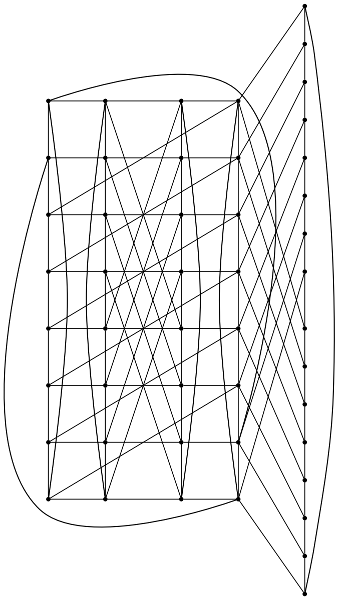
- Given: graph G similar to a Cartesian product
- G has non-trivial δ^*
- **Aim:**
  - find the underlying Cartesian product graph
  - reconstruct "approximate" factors

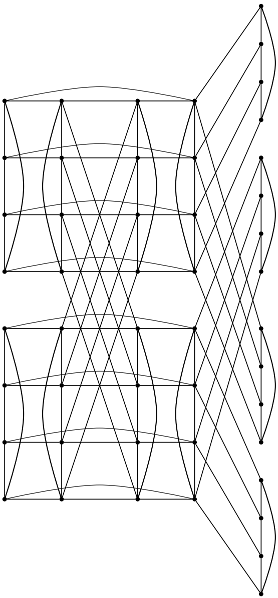
Introduction



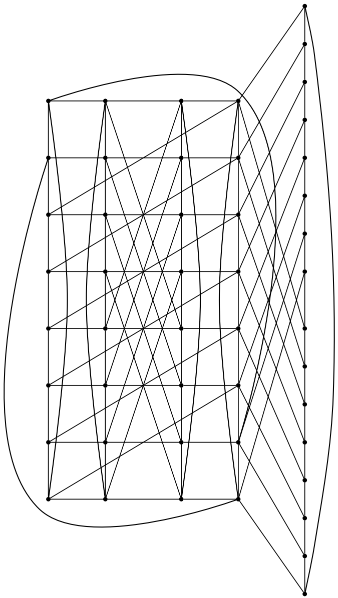
- Given: graph G similar to a Cartesian product
- G has non-trivial δ^*
- **Aim:**
  - find the underlying Cartesian product graph, or
  - reconstruct "approximate" factors

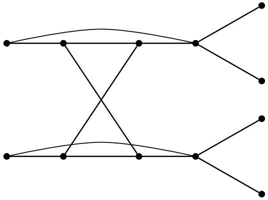
Introduction





Introduction



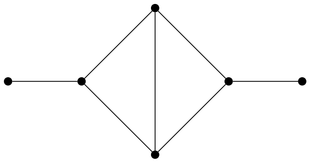


Outline

1. Basics
2. Idea
3. (proven) Facts and (unproven) Conjectures

Graphs

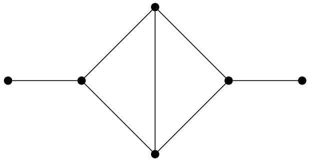
Graph $G = (V, E)$



- **here:** finite, undirected, simple graphs

Graphs

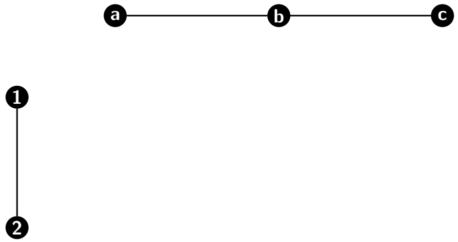
Graph $G = (V, E)$



- **here:** finite, undirected, simple graphs

The Cartesian Product

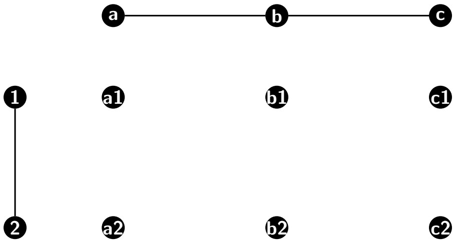
$G_1 = (V_1, E_1)$ $G_2 = (V_2, E_2)$



The Cartesian Product

$G_1 = (V_1, E_1)$ $G_2 = (V_2, E_2)$

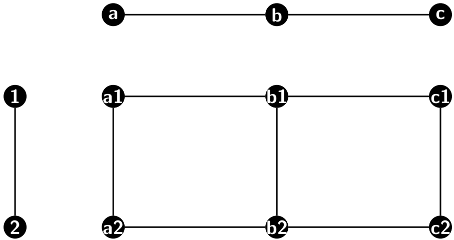
$$V(G_1 \square G_2) = V_1 \times V_2$$



The Cartesian Product

$G_1 = (V_1, E_1)$ $G_2 = (V_2, E_2)$

$$V(G_1 \Box G_2) = V_1 \times V_2$$
$$E(G_1 \Box G_2) = \{[(u_1, u_2), (v_1, v_2)] \mid [u_1, v_1] \in E_1, u_2 = v_2, \text{ or } u_1 = v_1, [u_2, v_2] \in E_2\}.$$



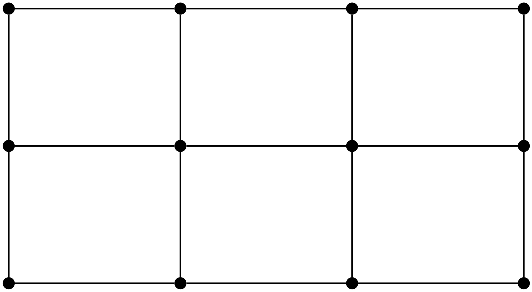
The Relation δ

Two edges e and $f \in E(G)$ are in relation $\delta \iff$

(i) $e = f$

(ii) e and f are opposite edges of a square

(iii) e and f are incident and there is no cordless square containing them



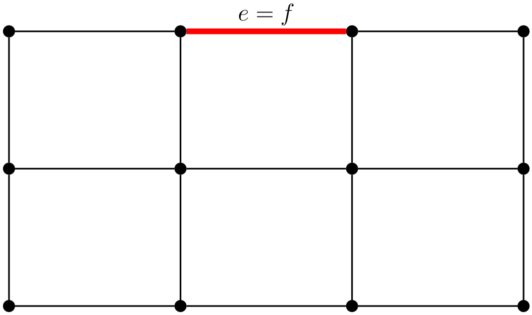
The Relation δ

Two edges e and $f \in E(G)$ are in relation $\delta \iff$

(i) $e = f$

(ii) e and f are opposite edges of a square

(iii) e and f are incident and there is no cordless square containing them



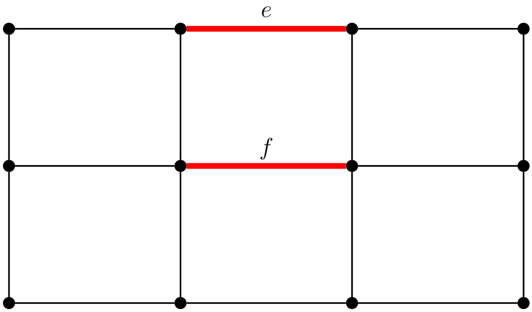
The Relation δ

Two edges e and $f \in E(G)$ are in relation $\delta \iff$

(i) $e = f$

(ii) e and f are opposite edges of a square

(iii) e and f are incident and there is no cordless square containing them



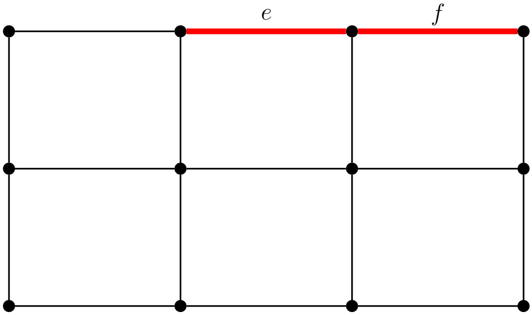
The Relation δ

Two edges e and $f \in E(G)$ are in relation $\delta \iff$

(i) $e = f$

(ii) e and f are opposite edges of a square

(iii) e and f are incident and there is no cordless square containing them

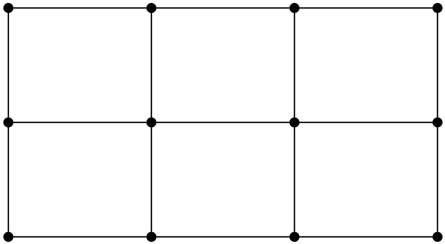


Theorem (W.Imrich, J.Žerovnik[1])

The relation corresponding to the unique prime factorization of a connected graph G is the convex hull of δ.

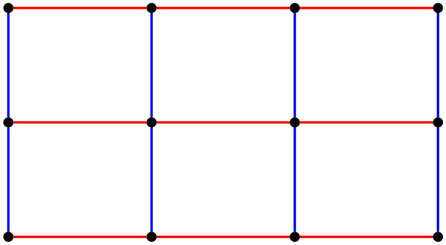
[1] Imrich, W. J. Žerovnik: *Factoring Cartesian-product graphs*. J. Graph Theory, 18(6):557–567, 1994.

if δ^* is convex $\Rightarrow$ δ^* is a product relation[2]



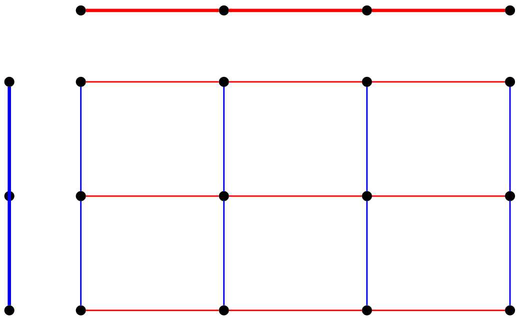
[2] δ^*: transitive closure of δ

if δ^* is convex $\Rightarrow$ δ^* is a product relation[2]



[2] δ^*: transitive closure of δ

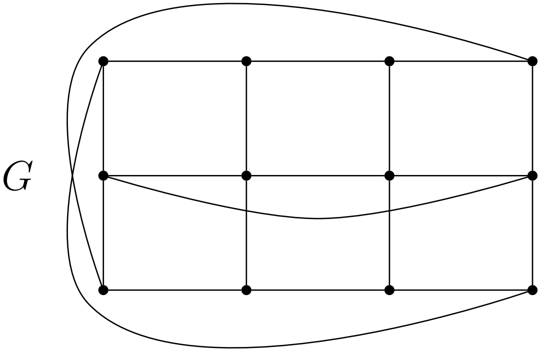
if δ^* is convex $\Rightarrow$ δ^* is a product relation[2]



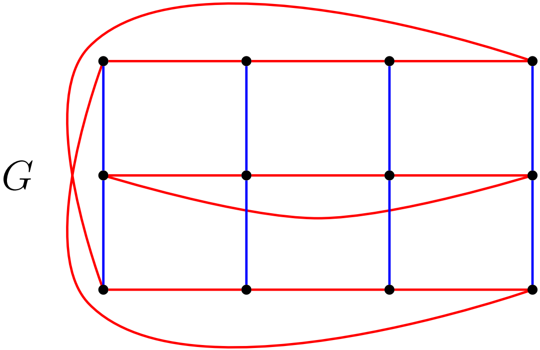
[2] δ^*: transitive closure of δ

What if δ^* is not convex?

A prime graph



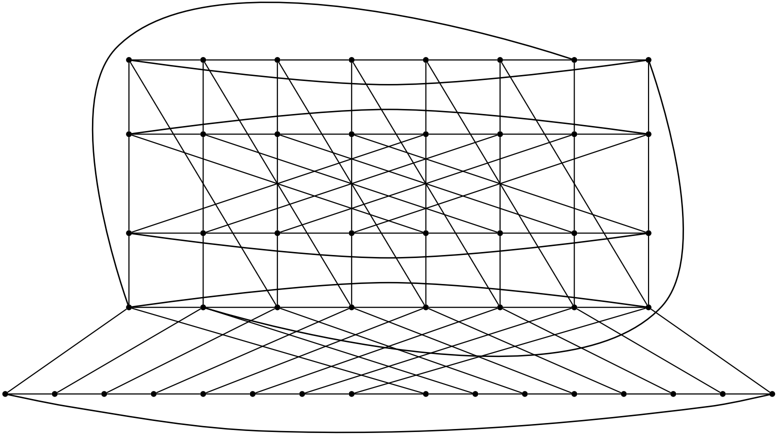
A prime graph with 2 different δ^*-equivalence classes



An Idea

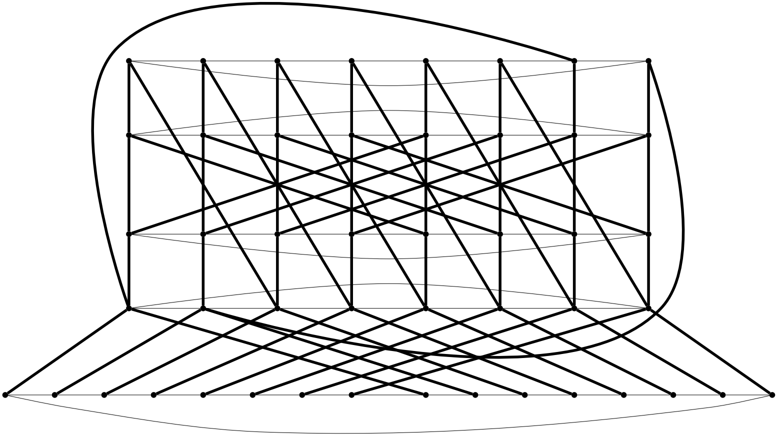
Given: a graph G with nontrivial δ^*:

- how to find the underlying Cartesian product graph?
- how to reconstruct "approximate" factors?



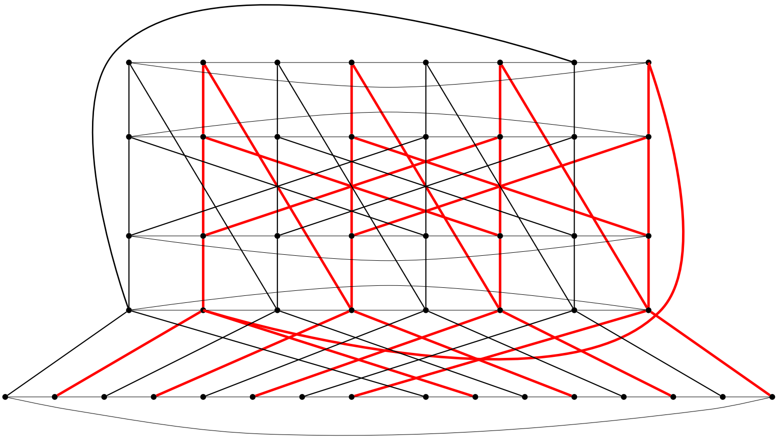
Given: a graph G with nontrivial δ^*:

- how to find the underlying Cartesian product graph?
- how to reconstruct "approximate" factors?



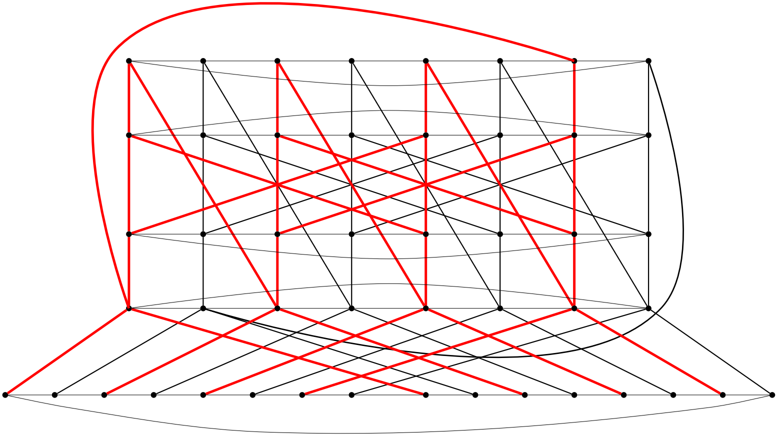
Given: a graph G with nontrivial δ^*:

- how to find the underlying Cartesian product graph?
- how to reconstruct "approximate" factors?



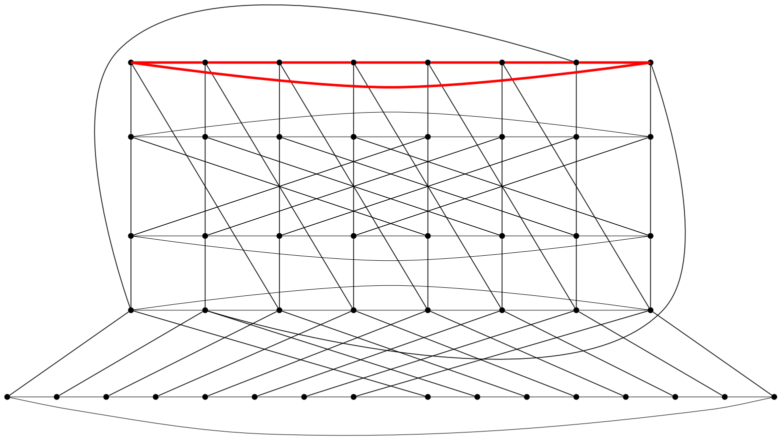
Given: a graph G with nontrivial δ^*:

- how to find the underlying Cartesian product graph?
- how to reconstruct "approximate" factors?



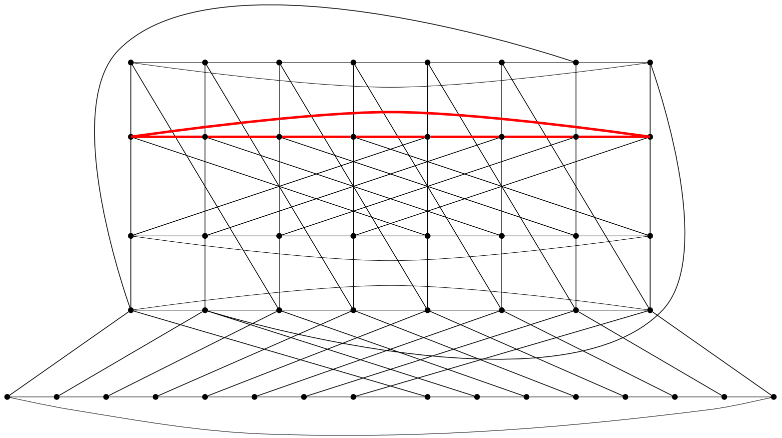
Given: a graph G with nontrivial δ^*:

- how to find the underlying Cartesian product graph?
- how to reconstruct "approximate" factors?



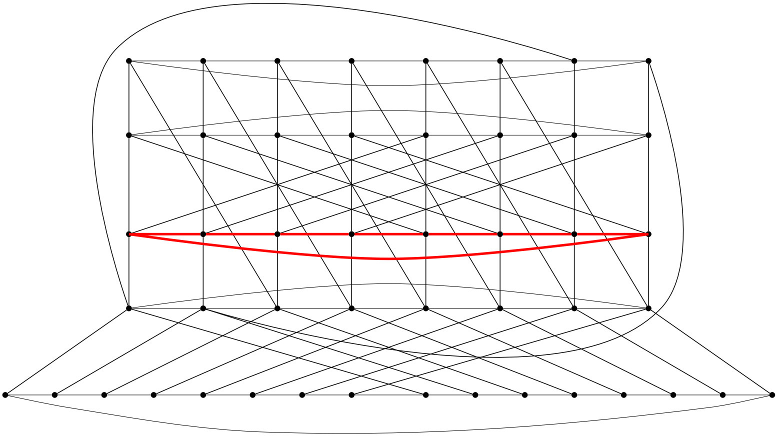
Given: a graph G with nontrivial δ^*:

- how to find the underlying Cartesian product graph?
- how to reconstruct "approximate" factors?



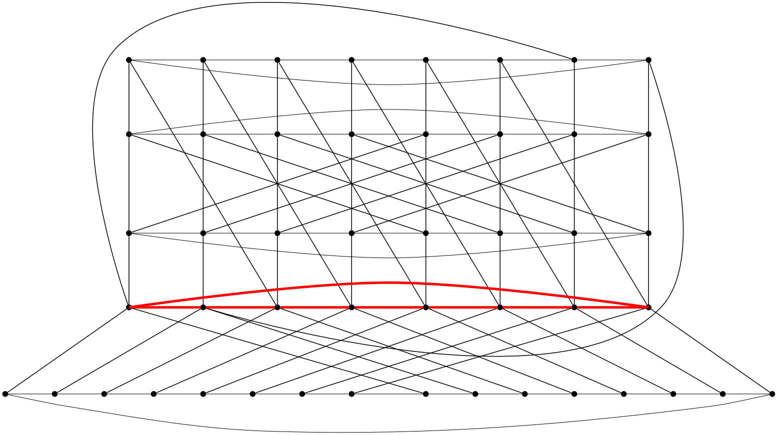
Given: a graph G with nontrivial δ^*:

- how to find the underlying Cartesian product graph?
- how to reconstruct "approximate" factors?



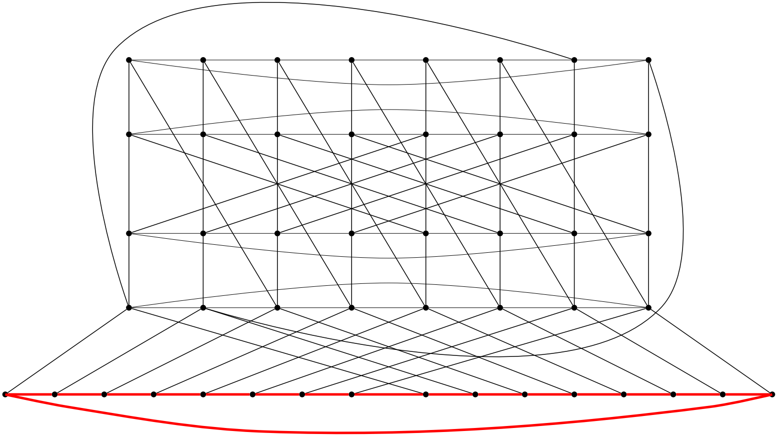
Given: a graph G with nontrivial δ^*:

- how to find the underlying Cartesian product graph?
- how to reconstruct "approximate" factors?



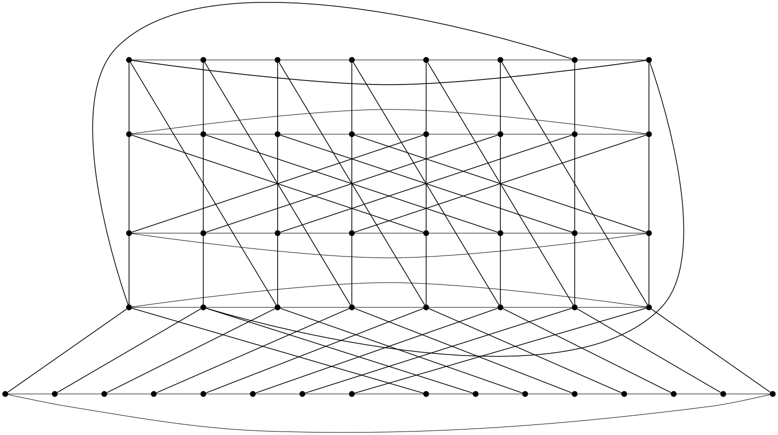
Given: a graph G with nontrivial δ^*:

- how to find the underlying Cartesian product graph?
- how to reconstruct "approximate" factors?



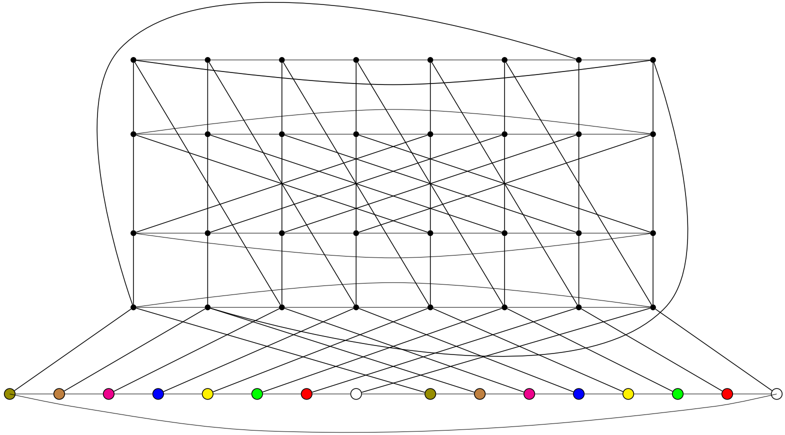
Construct A New Graph

1. merge certain vertices
2. switch some edges



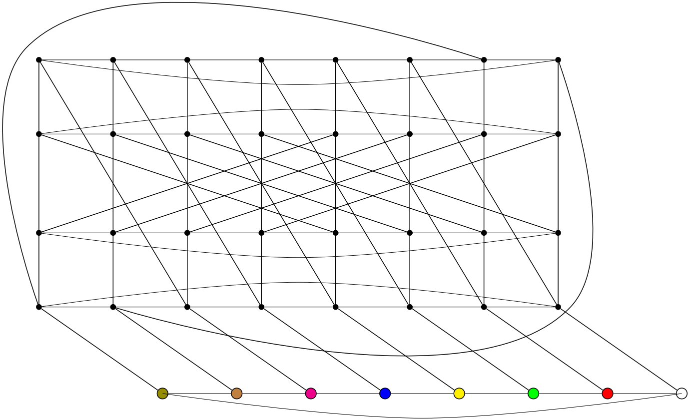
Construct A New Graph

1. merge certain vertices
2. switch some edges



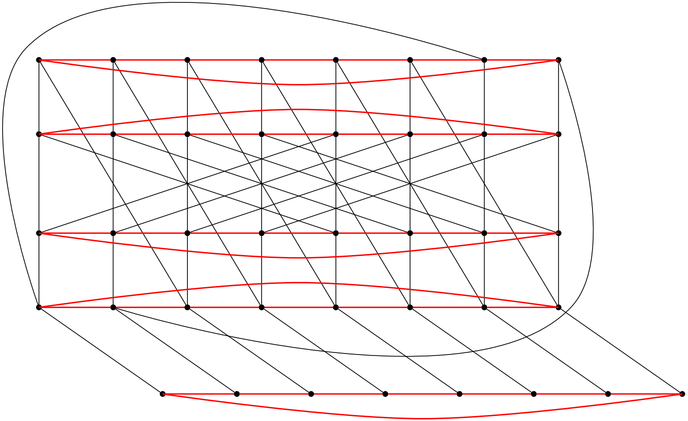
Construct A New Graph

1. merge certain vertices
2. switch some edges



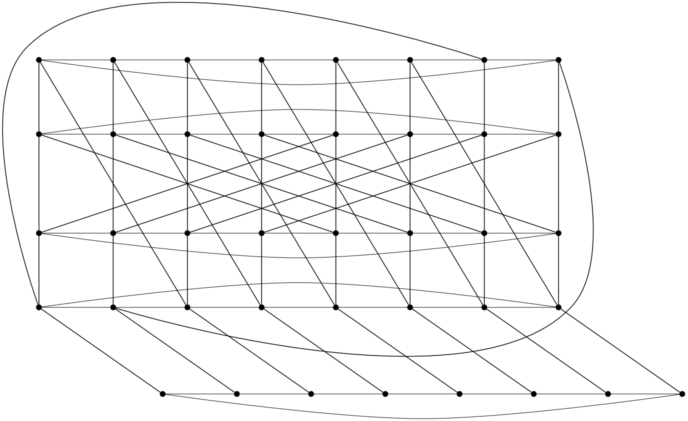
Construct A New Graph

1. merge certain vertices
2. switch some edges



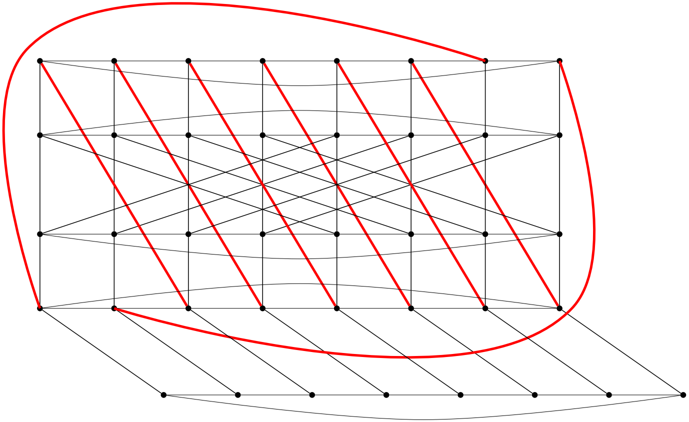
Construct A New Graph

1. merge certain vertices
2. switch some edges



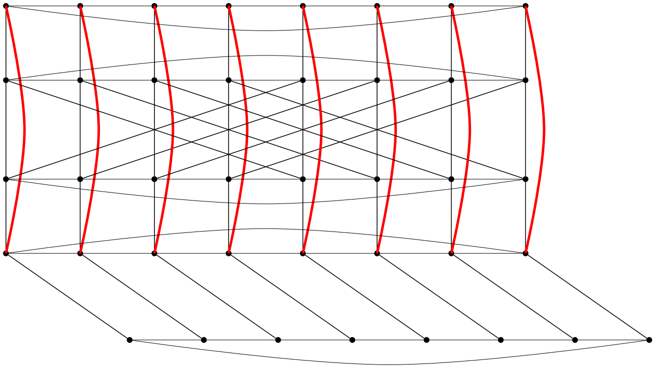
Construct A New Graph

1. merge certain vertices
2. switch some edges



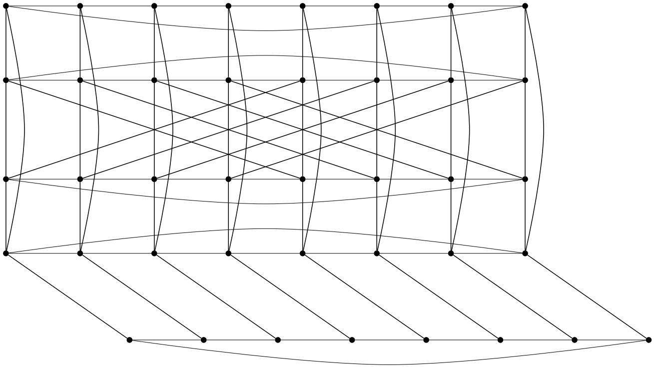
Construct A New Graph

1. merge certain vertices
2. switch some edges



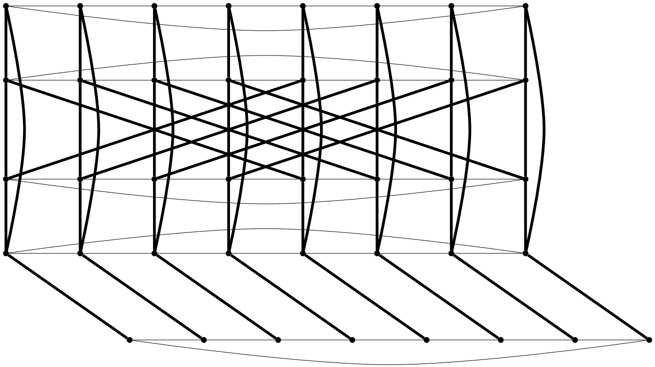
Construct A New Graph

1. merge certain vertices
2. switch some edges



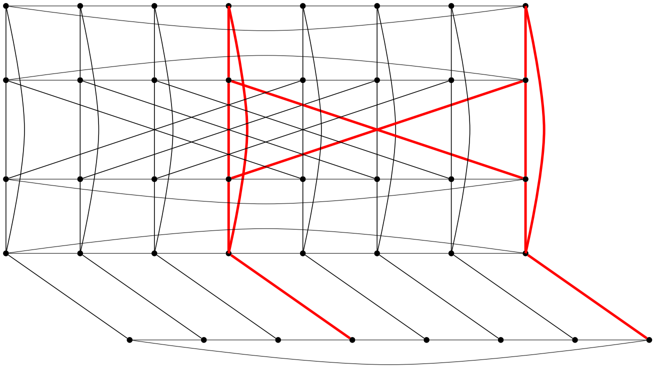
Properties Of The New Graph

1. "preserves" δ^*-equivalence classes
2. isomorphic connected components



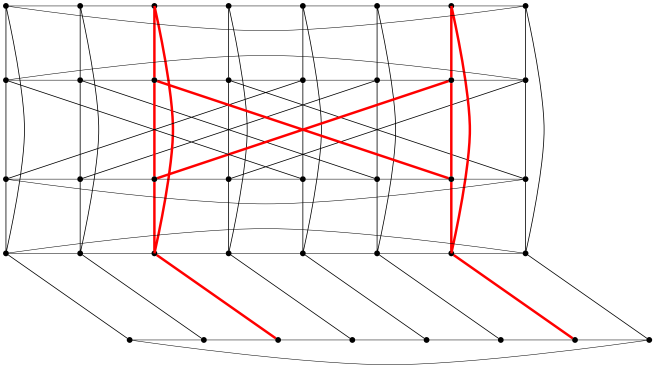
Properties Of The New Graph

1. "preserves" δ^*-equivalence classes
2. isomorphic connected components



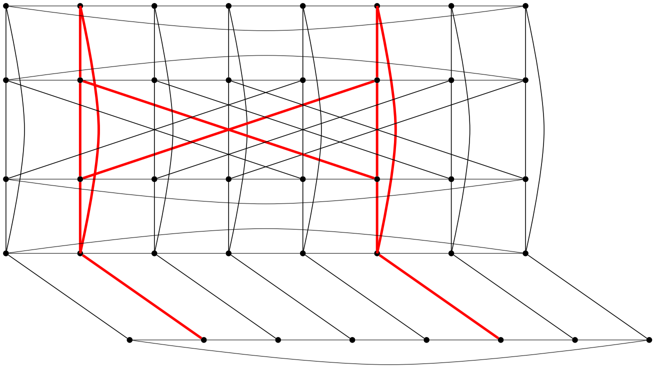
Properties Of The New Graph

1. "preserves" δ^*-equivalence classes
2. isomorphic connected components



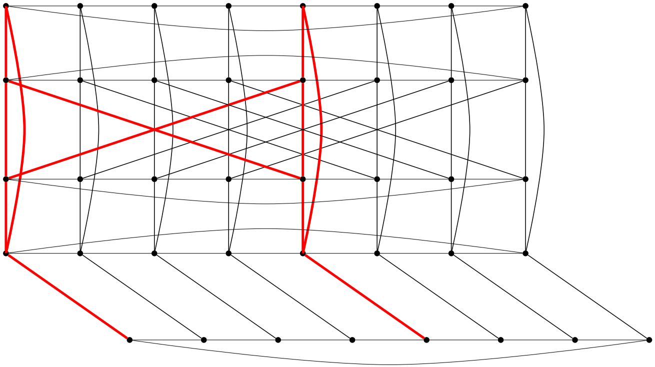
Properties Of The New Graph

1. "preserves" δ^*-equivalence classes
2. isomorphic connected components

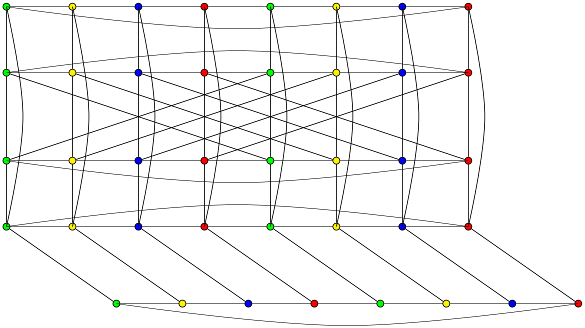


Properties Of The New Graph

1. "preserves" δ^*-equivalence classes
2. isomorphic connected components

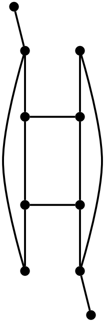


Observation

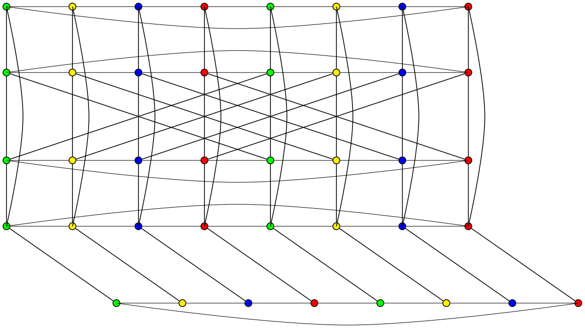


Observation

G' is a graph bundle over basegraph B with fiber F.



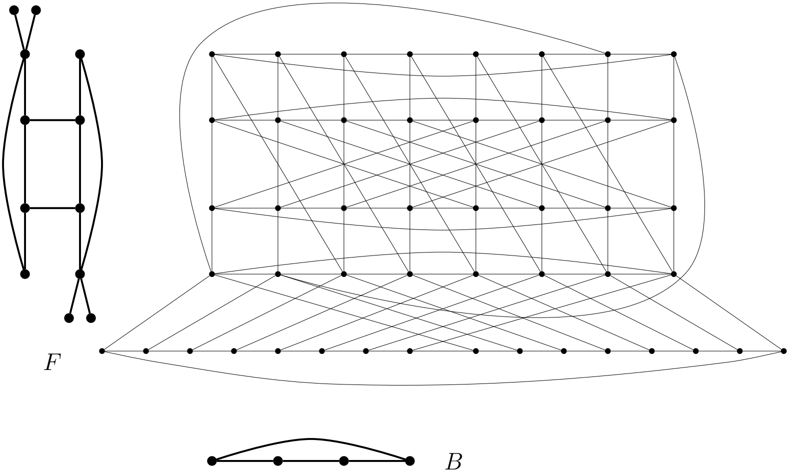
F



B

Observation

pumping up F



Some Facts...
...and Conjectures

Assumptions

- Consider 2 adjacent connected components of an δ^*-e.c. of a graph, i.e.
- consider graph G with nontrivial δ^* such that $\exists\ \varphi$ e.c. of δ^* with exactly 2 induced connected components: $G_\varphi(u) \neq G_\varphi(v)$
- $\overline{\varphi} := \bigcup_{\varphi' \neq \varphi} \varphi'$

Facts 1

- If two vertices $x, y \in V\left(G_{\varphi}(v)\right)$ belong to the same connected component of $G_{\overline{\varphi}}$, then there exists an automorphism $\pi : G_{\varphi}(v) \rightarrow G_{\varphi}(v)$ with $\pi(x) = y$.
- If each automorphism of a connected component $G_{\varphi}(v)$ that maps vertices into the same connected components of $G_{\overline{\varphi}}$ has no fixed points, all connected components of $G_{\overline{\varphi}}$ are isomorphic.

Facts 1

- If two vertices $x, y \in V\left(G_{\varphi}(v)\right)$ belong to the same connected component of $G_{\overline{\varphi}}$, then there exists an automorphism $\pi : G_{\varphi}(v) \rightarrow G_{\varphi}(v)$ with $\pi(x) = y$.
- If each automorphism of a connected component $G_{\varphi}(v)$ that maps vertices into the same connected components of $G_{\overline{\varphi}}$ has no fixed points, all connected components of $G_{\overline{\varphi}}$ are isomorphic.

Conjectures 1

- define relation A on the vertex set $V(G)$: $xAy \Leftrightarrow$
  1. $y \in V(G_\varphi(x)) \cap V(G_{\overline{\varphi}}(x))$ and
  2. there exists an automorphism π on $G_\varphi(x)$ with $\pi(x) = y$ that has fixed points, i.e., there is a vertex $v \in V(G_\varphi(x))$ with $\pi(v) = v$.
- compute quotient graph G/A:
  1. $V(G/A) = \{A_i \mid A_i$ is an equivalence class of A$\}$
  2. $(A_i, A_j) \in E(G/A)$, whenever there is an edge $(x,y) \in E(G)$ with $x \in A_i$ and $y \in S_j$.

Conjectures 1

- define relation A on the vertex set $V(G)$: $xAy \Leftrightarrow$
  1. $y \in V(G_\varphi(x)) \cap V(G_{\overline{\varphi}}(x))$ and
  2. there exists an automorphism π on $G_\varphi(x)$ with $\pi(x) = y$ that has fixed points, i.e., there is a vertex $v \in V(G_\varphi(x))$ with $\pi(v) = v$.
- compute quotient graph G/A:
  1. $V(G/A) = \{A_i \mid A_i \text{ is an equivalence class of A}\}$
  2. $(A_i, A_j) \in E(G/A)$, whenever there is an edge $(x, y) \in E(G)$ with $x \in A_i$ and $y \in S_j$.

Conjectures 1

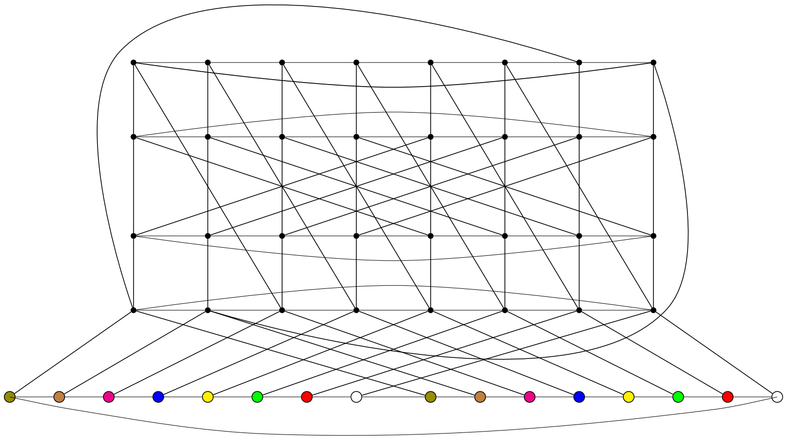
- define relation A on the vertex set $V(G)$: $xAy \Leftrightarrow$
  1. $y \in V(G_\varphi(x)) \cap V(G_{\overline{\varphi}}(x))$ and
  2. there exists an automorphism π on $G_\varphi(x)$ with $\pi(x) = y$ that has fixed points, i.e., there is a vertex $v \in V(G_\varphi(x))$ with $\pi(v) = v$.
- compute quotient graph G/A:
  1. $V(G/A) = \{A_i \mid A_i$ is an equivalence class of A$\}$
  2. $(A_i, A_j) \in E(G/A)$, whenever there is an edge $(x, y) \in E(G)$ with $x \in A_i$ and $y \in S_j$.

Conjecture

- G has nontrivial $\delta^* \Rightarrow G/A$ has nontrivial δ^*,
- G/A has the "same" δ^*-equivalence classes as G.

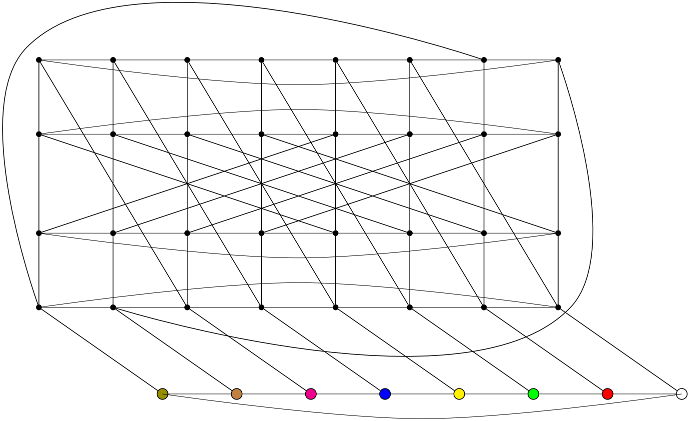
Construct A New Graph

1. merge certain vertices
2. switch some edges



Construct A New Graph

1. merge certain vertices
2. switch some edges



Facts 2

- There exists a connected subgraph H of $G_\varphi(v)$, that contains each connected component of $G_{\overline{\varphi}}$ exactly once.
- Let T be a spanning tree of H. Then there exists a subgraph T' of $G_\varphi(u)$, that is isomorphic to T, with the same properties.
- If each automorphism of a connected component $G_\varphi(v)$ that maps vertices into the same connected components of $G_{\overline{\varphi}}$ has no fixed points, then

$$V(G_\varphi(v)) = \bigcup_{w \in V(T)} \{\pi_i(w) \mid \pi_i|_{G_{\overline{\varphi}}(x)} : G_{\overline{\varphi}}(x) \to G_{\overline{\varphi}}(x)\}.$$

- If $|\{j \mid (v, \pi_j(w)) \in E(G)\}| \leq 1 \; \forall \, v, w \in V(T), v \neq w \Rightarrow$ $(G, p_{\overline{\varphi}}, B_{\overline{\varphi}})$ is a graph bundle.

Facts 2

- There exists a connected subgraph H of $G_\varphi(v)$, that contains each connected component of $G_{\overline{\varphi}}$ exactly once.
- Let T be a spanning tree of H. Then there exists a subgraph T' of $G_\varphi(u)$, that is isomorphic to T, with the same properties.
- If each automorphism of a connected component $G_\varphi(v)$ that maps vertices into the same connected components of $G_{\overline{\varphi}}$ has no fixed points, then

$$V\left(G_\varphi(v)\right) = \bigcup_{w \in V(T)} \{\pi_i(w) \mid \pi_i|_{G_{\overline{\varphi}}(x)} : G_{\overline{\varphi}}(x) \to G_{\overline{\varphi}}(x)\}.$$

- If $|\{j \mid (v, \pi_j(w)) \in E(G)\}| \leq 1 \; \forall \, v, w \in V(T), v \neq w \Rightarrow$ $\left(G, p_{\overline{\varphi}}, B_{\overline{\varphi}}\right)$ is a graph bundle.

Facts 2

- There exists a connected subgraph H of $G_\varphi(v)$, that contains each connected component of $G_{\overline{\varphi}}$ exactly once.
- Let T be a spanning tree of H. Then there exists a subgraph T' of $G_\varphi(u)$, that is isomorphic to T, with the same properties.
- If each automorphism of a connected component $G_\varphi(v)$ that maps vertices into the same connected components of $G_{\overline{\varphi}}$ has no fixed points, then

$$V\left(G_\varphi(v)\right) = \bigcup_{w \in V(T)} \{\pi_i(w) \mid \pi_i|_{G_{\overline{\varphi}}(x)} : G_{\overline{\varphi}}(x) \to G_{\overline{\varphi}}(x)\}.$$

- If $|\{j \mid (v, \pi_j(w)) \in E(G)\}| \leq 1 \ \forall\, v, w \in V(T), v \neq w \Rightarrow$ $(G, p_{\overline{\varphi}}, B_{\overline{\varphi}})$ is a graph bundle.

Facts 2

- There exists a connected subgraph H of $G_\varphi(v)$, that contains each connected component of $G_{\overline{\varphi}}$ exactly once.
- Let T be a spanning tree of H. Then there exists a subgraph T' of $G_\varphi(u)$, that is isomorphic to T, with the same properties.
- If each automorphism of a connected component $G_\varphi(v)$ that maps vertices into the same connected components of $G_{\overline{\varphi}}$ has no fixed points, then

$$V\left(G_\varphi(v)\right) = \bigcup_{w \in V(T)} \{\pi_i(w) \mid \pi_i|_{G_{\overline{\varphi}}(x)} : G_{\overline{\varphi}}(x) \to G_{\overline{\varphi}}(x)\}.$$

- If $|\{j \mid (v, \pi_j(w)) \in E(G)\}| \leq 1 \; \forall \, v, w \in V(T), v \neq w \Rightarrow$ $\left(G, p_{\overline{\varphi}}, B_{\overline{\varphi}}\right)$ is a graph bundle.

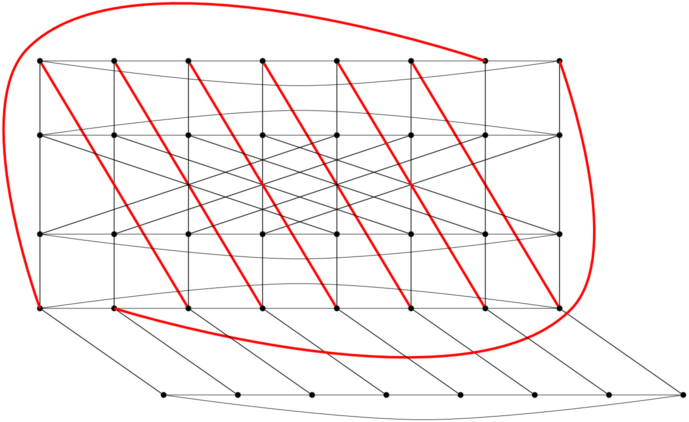
Conjecture 2

Conjecture

By some special "edge switching" one can construct a graph G' from G/A that is a Cartesian product (or at least a graph bundle).

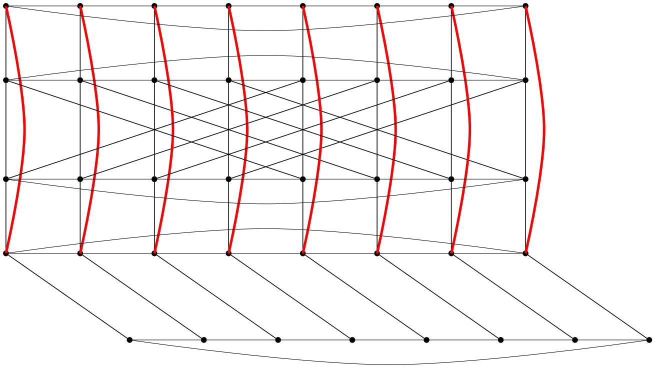
Construct A New Graph

1. merge certain vertices
2. switch some edges



Construct A New Graph

1. merge certain vertices
2. switch some edges



- Proofs?
- How to define/ formalize those "switchings"?
- How to extend this to the entire graph?
- .....

Thanks to Peter Stadler and Marc Hellmuth!

Thanks to Peter Stadler and Marc Hellmuth!

Thank you for your attention!