Computational Complexity of Graph Relations

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Winerseminar, Bled

Outline

How to relate two graphs/networks?

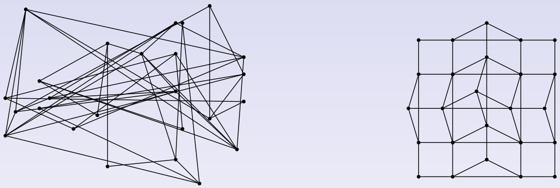
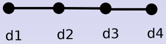


Figure: What are the relations between networks?

Model in biology

- **Domain interaction graph**.



- **Relation**: containing relation=

$$\{d_2, d_3 \in p_1; d_4 \in p_2; d_1, d_3 \in p_3; d_1 \in p_4\}$$

Model in biology

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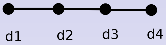

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$$\{d_2, d_3 \in p_1; d_4 \in p_2; d_1, d_3 \in p_3; d_1 \in p_4\}$$

- We define a **protein interaction graph**, the vertex set are proteins, two proteins have interaction if there are two interacted domains which belong to them respectively.

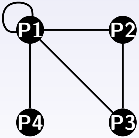

Figure: Protein Interaction Network

From Biology to Mathematics

Firstly, we generalize the containing relation to a binary relation.

Definition

The ***(binary) relation*** *R between two sets A and B is a subset of $A \times B$.*

Definition

Let $G = (V_G, E_G)$ be a simple graph with loops allowed, $R \subset V_G \times B$ be a binary relation, $G * R$ is defined as a graph with vertex set B (B is given by R). For $u, v \in B$, $u \sim v$ if and only if there exist $(x, y) \in E_G$, $(x, u), (y, v) \in R$.

Our fundamental problem

Question

Given any two graphs G and H, how to find R satisfies $G*R = H$.

Definition

If R satisfies $G * R = H$, we say R is a relation from G to H.

Then this problem becomes a *Graph Theory* problem.

Compare to Graph Homomorphisms

- A graph *homomorphism* f from a graph $G \to H$ is a **mapping** $f : V_G \to V_H$ such that whenever (u, v) is an edge of G, we have $(f(u), f(v))$ is an edge of H.

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- A graph *Multihomomorphism* $G \to_m H$ is a mapping $\varphi : V_G \to 2^{V_H}/\{\emptyset\}$ (i.e., associating a nonempty subset of vertices of H with every vertex of G) such that whenever u_1, u_2 is an edge of G, we have (v_1, v_2) is an edge of H for every $v_1 \in \varphi(u_1)$ and every $v_2 \in \varphi(u_2)$.

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- If we consider $G * R = H$ with **full domain**, i.e., every vertex in G has at least an image in H, then R can be seen as a *surjective multihomomorphism*.

- **homomorphism $\Longleftrightarrow$ multihomomorphism**
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- Multihomomorphism is a generalization of homomorphism. *Graph relation* is a generalization of *surjective graph homomorphism*.
- If $G * R = H$, then there exist a homomorphism from G to H, but NOT vice versa.

Composition and Decomposition

Proposition

$(G * R) * S = G * (R \circ S)$.

Lemma

For any R *satisfing* $G * R = H$, *there exists* R_D, R_C *such that* $R = R_D \circ R_C$, $(G * R_D) * R_C = H$.

$$\begin{tikzcd} G \arrow[rr, "R"] \arrow[dr, "R_D"'] & & H \\ & G' \arrow[ur, "R_C"'] & \end{tikzcd}$$

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Question

Given any two graphs G and H, how to find R satisfies $G * R = H$.

The existence of the relations is non-trivial:

- it could have a relation

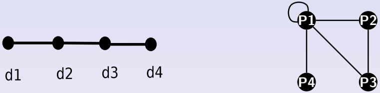

Figure: there exists a relation of G to H

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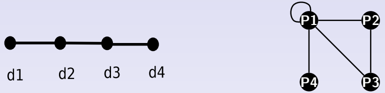

Figure: there exists a relation of G to H

- it could have no relations

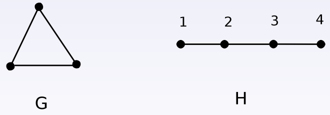

Figure: no relations of G to H

Diameter decrease

The *distance* between two vertices in a graph is the number of edges in a shortest path connecting them. The *diameter* of a graph is the greatest distance between any pair of vertices.

Proposition

If $G * R = H$, then $max\{diam(G), 2\} \geq diam(H)$.

Corollary

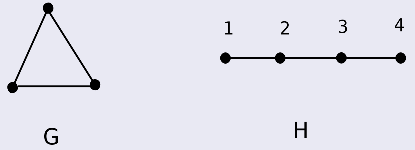

Figure: $2 < diam(H) = 3$, so no relations of G to H

.

Chromatic number increase

A *coloring* of a graph is a labelling of the graph's vertices with colors such that no two vertices sharing the same edge have the same color. The smallest number of colors needed to color a graph G is called its *chromatic number*, $\chi(G)$.

Proposition

Suppose G and H are simple graphs. If $G * R = H$, then $\chi(G) \leq \chi(H)$.

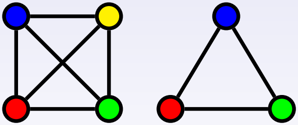

Figure: $\chi(K_4) > \chi(K_3)$, so no relation from K_4 to K_3.

Definition

*Two graphs G and H are **strongly relationally equivalent**, $G \sim H$, if $G * R = H$ and $H * R^+ = G$, where R^+ is the transpose of R, i.e., $(u, x) \in R^+$ if and only if $(x, u) \in R$.*

Lemma

Strongly relational equivalence is an equivalence relation on graphs.

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Lemma

Strongly relational equivalence is an equivalence relation on graphs.

Question

How to find the simpliest(smallest) represented elments?

Definition

Denote thin graph of graph G by G_{thin}, which is obtained from G by contracting the vertices with the same neighborhoods together.

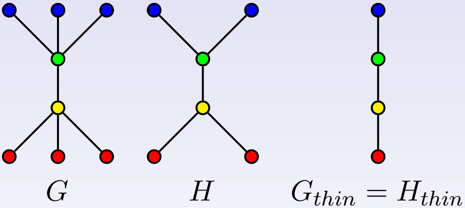

Figure: Non-isomorphic graphs G and H with isomorphic thin graphs.

Theorem

Two graph are strongly relationally equivalent iff their thin graphs are the same.

In other words, *thin graphs* ares the represented elements of strongly relational equivalent classes.

Observation

We could simplify the equation $G * R = H$ *to* $G_{thin} * R = H_{thin}$.

Let $\mathcal{H}$ be a fixed graph.

Definition

The homomorphism problem $\textsc{Hom}(\mathcal{H})$ *takes as input some finite* $\mathcal{G}$ *and asks whether there is a homomorphism from* $\mathcal{G}$.

Definition

The surjective homomorphism problem $\textsc{Sur-Hom}(\mathcal{H})$ *asks whether or not an input graph* $\mathcal{G}$ *admits a surjective homomorphism to* $\mathcal{H}$.

Definition

The relation problem $\textsc{Rel}(\mathcal{H})$ *asks whether or not an input graph* $\mathcal{G}$ *admits a relation to* $\mathcal{H}$.

Clearly two relationally equivalent graphs result in the same relation problem.

Polynomial Equivalence to SUR-HOM($\mathcal{H}$)

This is an observation of Prof. Jarik Nešetřil.
From decomposition lemma, we know $G * R = H$ iff there is a graph $G' = G * R_D$ which has a (full) homomorphism to G and has a surjective homomorphism to H. For a fixed H, given a graph G, we duplicate every vertex of G at most $|H|$ times, there are polynomially many G'. Thus, the H relation problem is polynomial-time equivalent to SUR-HOM($\mathcal{H}$).

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$$\begin{tikzcd}
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SUR-HOM($\mathcal{H}$) is still *OPEN*.

- If H is K_2, there exists relation R such that $G * R = H$ iff G is a bipartite graph. So in this case the problem is polynomial.
- If H is K_3, the problem is polynomial equivalent with 3-colourable problem, which is NP-complete.

Sur-Hom($\mathcal{H}$) VS Hom($\mathcal{H}$)

This is an observation of Jan Hubička.

Proposition

Existence of homomorphism from G to H is the same as existence of surjective homomorphisms from $G + H$ to H.

So Sur-Hom($\mathcal{H}$) is obviously hard for all graphs where homomorphisms are *hard*.

Definition

Two graphs G and H are ***(weakly) relationally equivalent****, $G \sim H$, if there are relations R and S such that $G * R = H$ and $H * S = G$.*

Strong relational equivalence implies weak relational equivalence. To see this, simply note the the definition of the weak from is obtained from the strong one by setting $S = R^T$.

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Observation

Weak and strong relational equivalence are not the same.

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Example

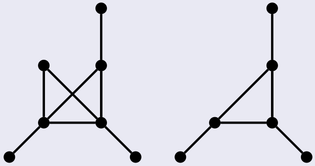

Figure: relations exist in both directions with different thin graphs.

Future works

Represented elements of weakly relational equivalence classes are not thin graphs, and more precise than thin graphs.

Question

How to find the represented element of weakly relational equivalence classes?

Special thanks to

Christoph
Peter
Jürgen
Honza
Frank and Tina

Thanks for your attention!!!

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