How to find Cartesian Product Graphs in “Graphs with Square Property"

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BLED

Introduction

- We consider graphs that have nontrivial equivalence relation on their edge set that satisfy the *square property*
- The square property is closely related to the factorization of a *Cartesian product*
- Any product relation on the edge set of a connected Cartesian product satisfies the square property

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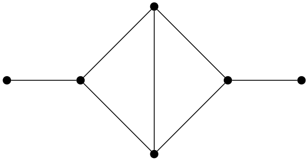
Q: Conversely, does a graph with equivalence relation having the square property yield a product like structure?

Graphs

Graph $G = (V, E)$



- **here:** finite, undirected, simple graphs

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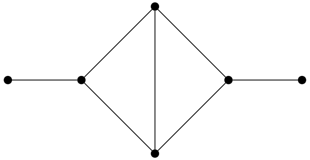

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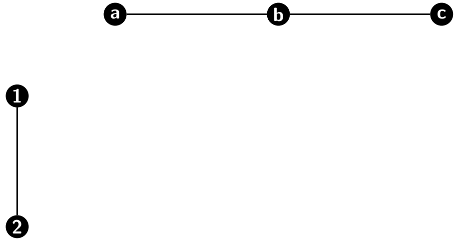
The Cartesian Product

$G_1 = (V_1, E_1)$ $G_2 = (V_2, E_2)$



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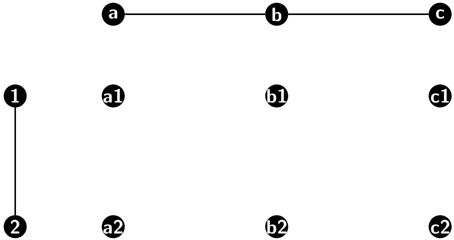
$$V(G_1 \square G_2) = V_1 \times V_2$$



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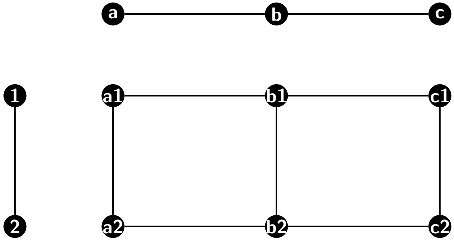
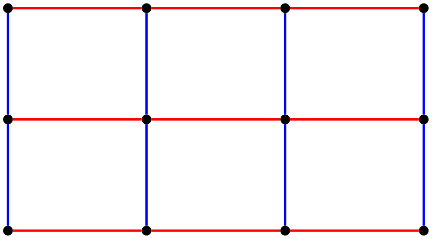
$$V(G_1 \square G_2) = V_1 \times V_2$$
$$E(G_1 \square G_2) = \{[(u_1, u_2), (v_1, v_2)] \mid [u_1, v_1] \in E_1, u_2 = v_2, \text{ or } u_1 = v_1, [u_2, v_2] \in E_2\}.$$



The Square Property

equiv. relation R on $E(G)$ has *square property* if two adjacent edges of different equiv. classes span exactly one square.



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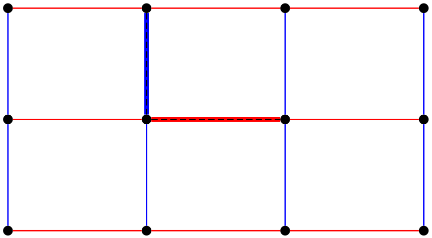

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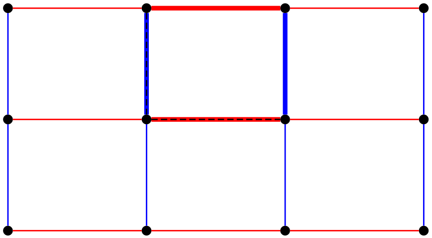

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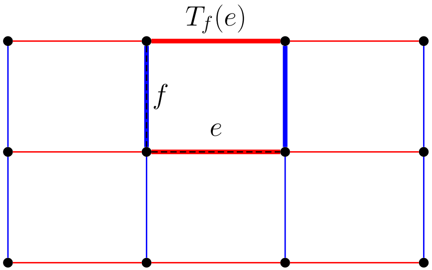
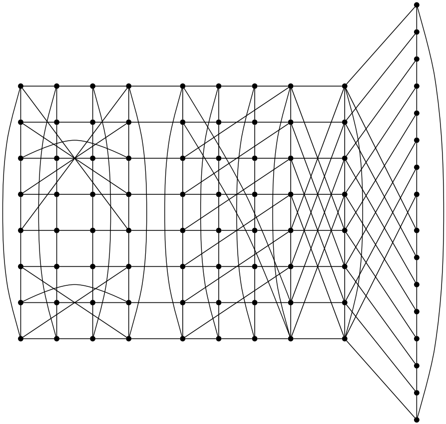


- e, f adjacent edges of different equiv. classes
  Translation of e along f, $T_f(e)$:= opposite edge of e in the (unique) square spanned by e and f.

Some Notation



- Graph G with nontrivial equiv. rel. R on $E(G)$ with square property
- equiv. classes A and B
- G_A: spanning subgraph of G generated by A
- $G_A(v)$: conn. comp. of G_A containing v
- G_B: spanning subgraph of G generated by B
- $G_B(u)$, $G_B(v)$: conn. comp. of G_B containing u resp. v

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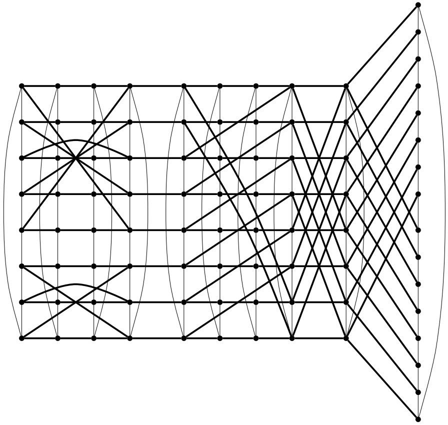

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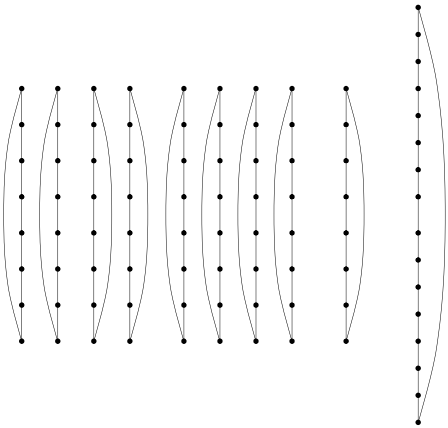

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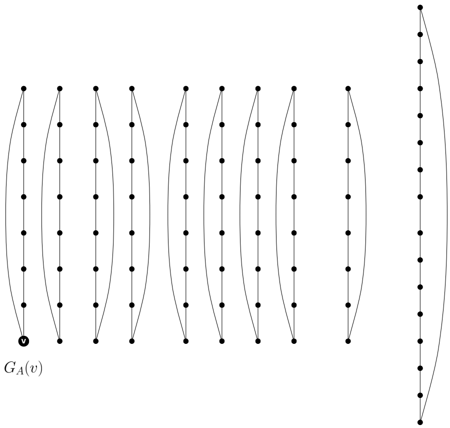

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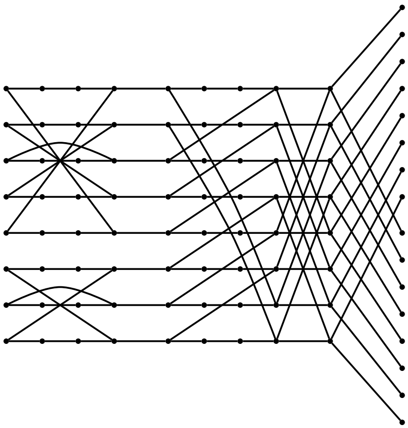

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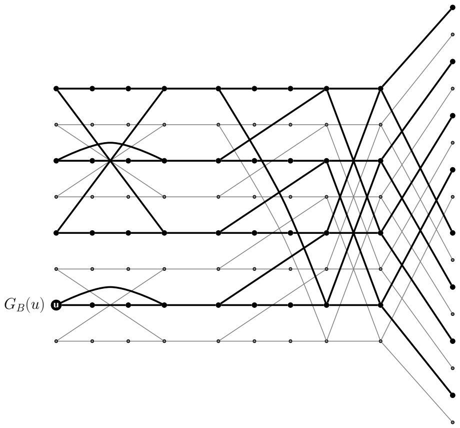

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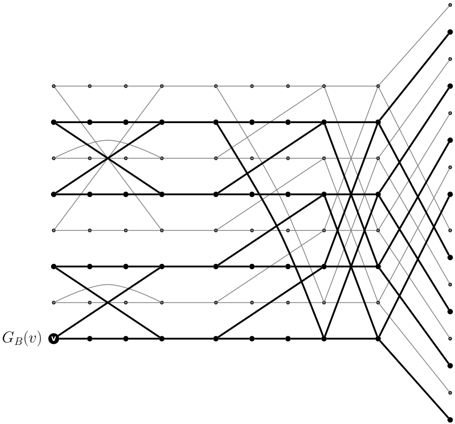

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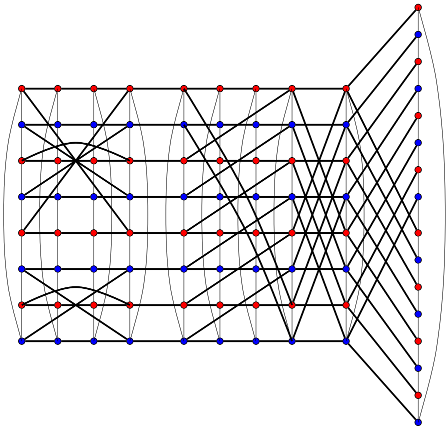

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Conjecture:

- Let G be a graph and R a nontrivial equiv. rel. on $E(G)$ satisfying the square property with equivalence classes A and B. Then there exists connected subgraphs $T \subseteq G_A$, $U \subseteq G_B$ such that there is a spanning subgraph $H \subseteq G$ with $H \cong T \Box U$.

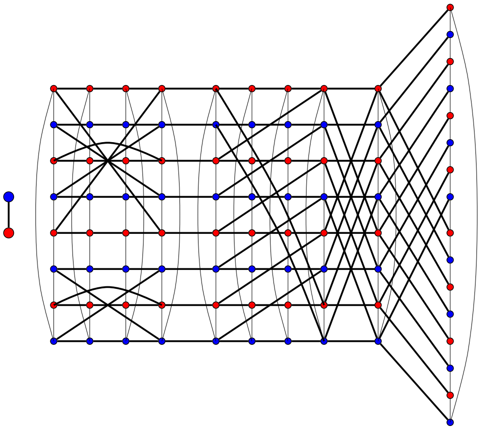
Problem:

- How to find H, T, U?

Idea:

- “peel“ H out of G

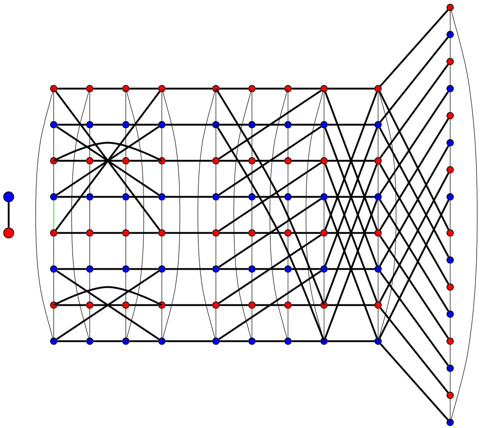
Some More Notation



quotient graph Q_B w.r.t. B:

- $V(Q_B) = \{G_B(x) \mid x \in V(G)\}$
- $[G_B(u), G_B(v)] \in E(Q_B) \Leftrightarrow \exists x \in V(G_B(u))\, y \in V(G_B(v)) : [x, y] \in A$

An Idea



Choose a spanning tree $T_B \subseteq Q_B$

for each edge of T_B choose a representative edge $e \in V(G)$

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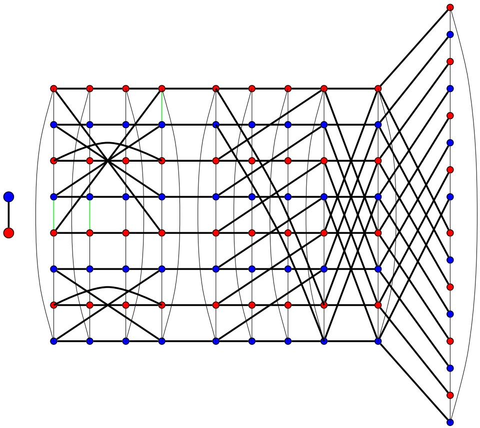

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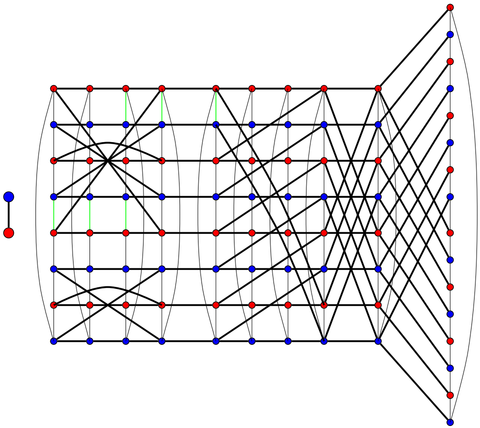

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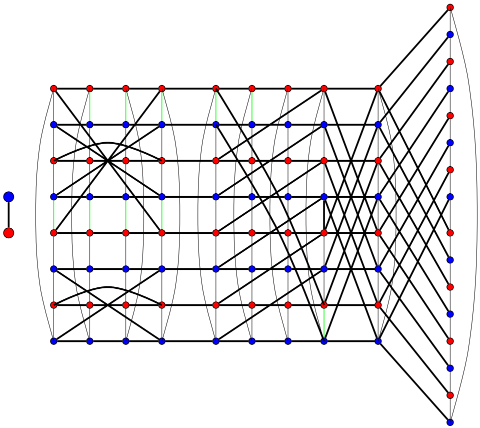

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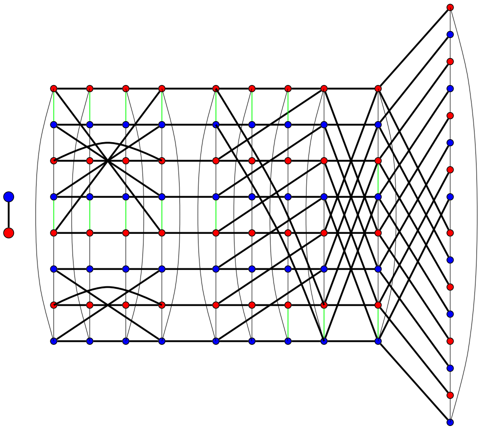

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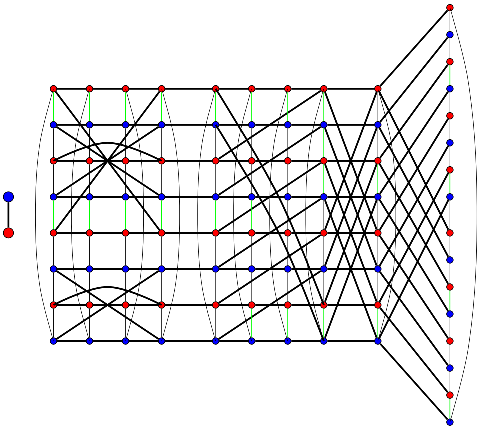

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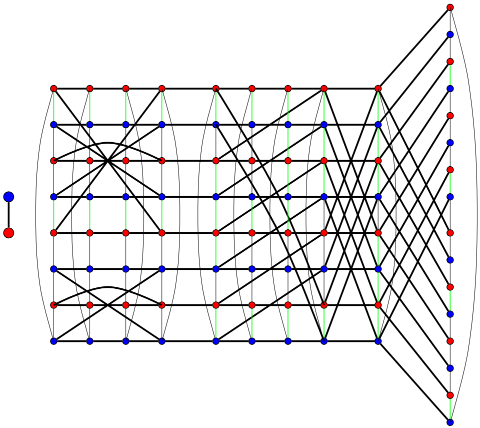

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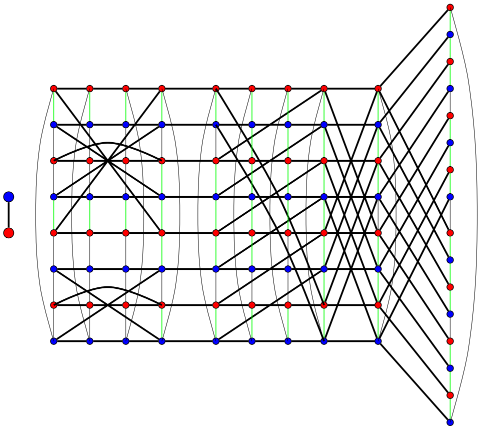

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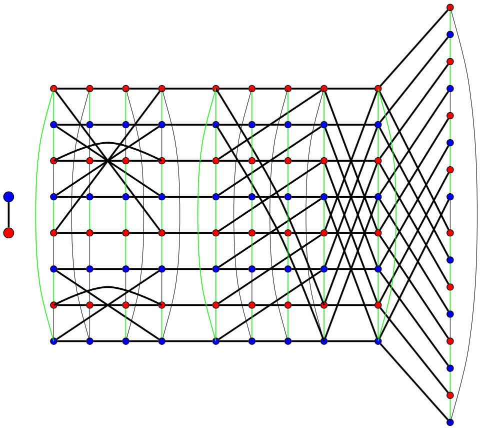

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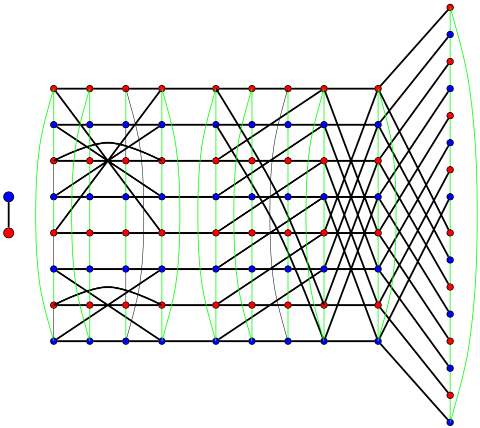

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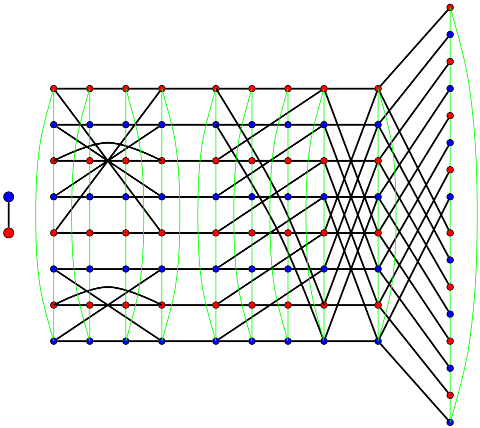

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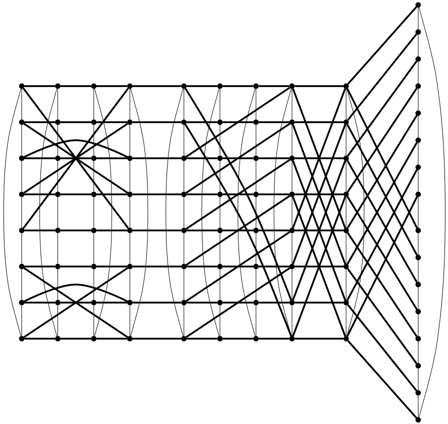

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$\hookrightarrow$ new graph G_1:

- $V(G_1) = V(G)$
- $E(G_1) = B \cup \bigcup_e E_1(e)$

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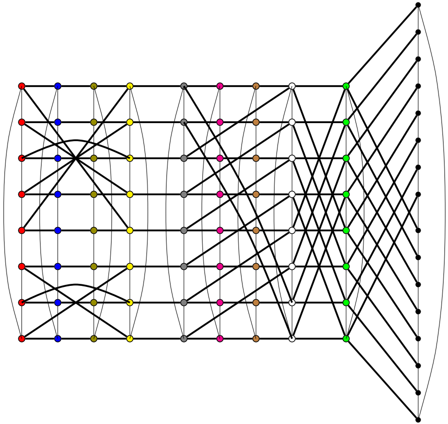

repeat this procedure on G_1 with equiv. class $A_1 \subseteq A$:

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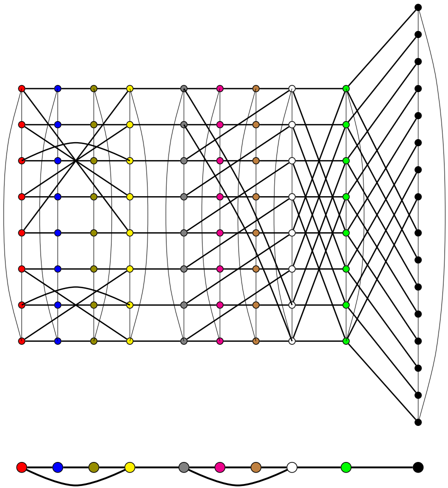

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- Choose a spanning tree $T_B \subseteq Q_B$
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- define the set $E_2(e)$

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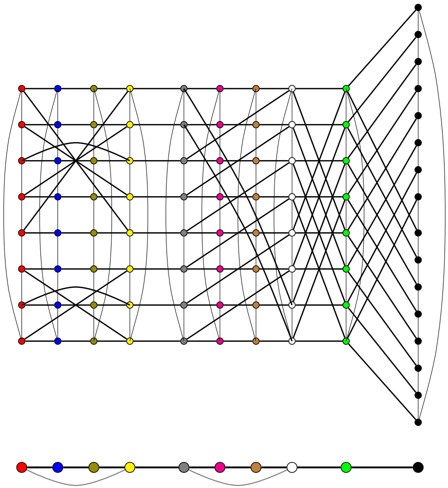

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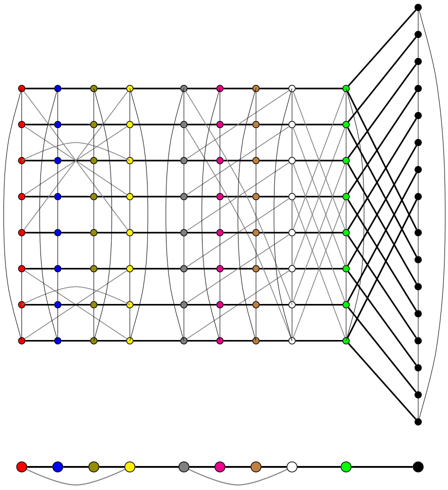

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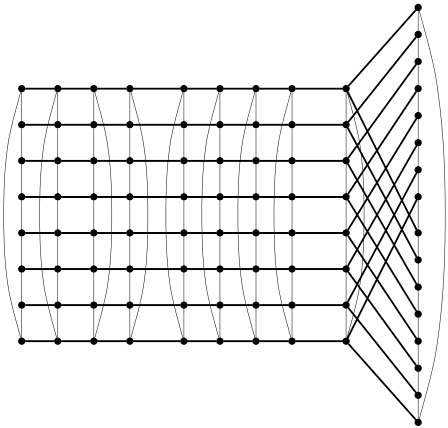

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- Choose a spanning tree $T_B \subseteq Q_B$
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- define the set $E_2(e)$

$\hookrightarrow$ new graph G_2:

- $V(G_2) = V(G)$
- $E(G_2) = A_1 \cup \bigcup_e E_2(e)$

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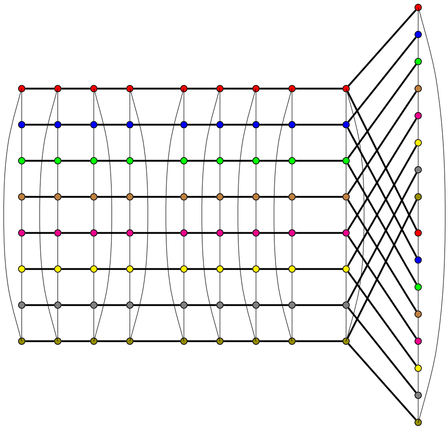

- repeat this procedure on G_2 with equiv. class $B_2 \subseteq B$...

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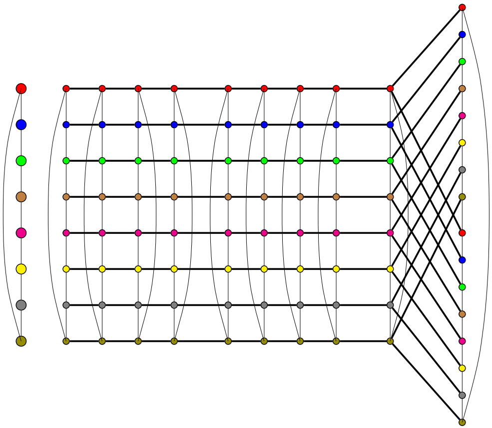

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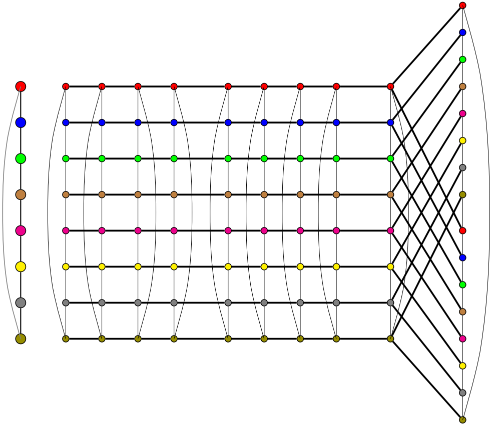

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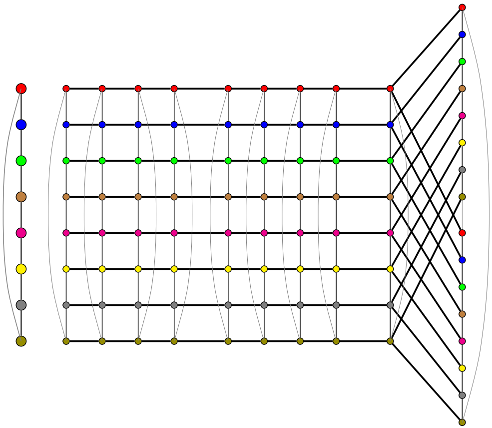
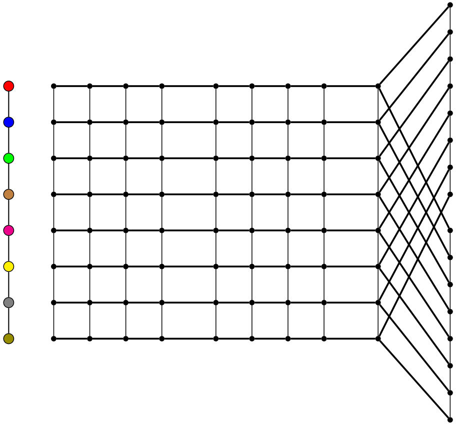


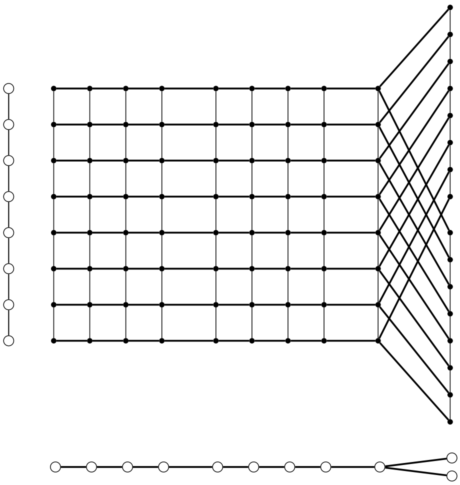
- repeat this procedure on G_2 with equiv. class $B_2 \subseteq B$...

An Idea



- a Cartesian product graph

An Idea



- a Cartesian product graph

What do we have

For the G_i holds

- $G_{i+1} \subseteq G_i \subseteq G$, $V(G_i) = V(G)$
- G_i connected
- $\exists$ equivalence relation R_i on $E(G_i)$ having square property
- R_i has equivalence classes A_i, B_i with $A_{i+1} \subseteq A_i \subseteq A$, $B_{i+1} \subseteq B_i \subseteq B$
- $G_i = G_{i+1} \Rightarrow G_j = G_i \forall j > i \hookrightarrow H := G_i$

???

- under which conditions is H a *nontrivial* product?
  e.g. a neccessary condition is that
  $gcd\{G_A(x) \mid x \in V(G)\}, gcd\{G_B(x) \mid x \in V(G)\} > 1$
- is there an estimate for the number of steps needed?
- ...

Thank you for your attention!