Polynomial Solutions to Hard Problems for undirected 2-quasi best match graphs

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joint work with Peter F. Stadler, Marc Hellmuth, and Federico Romaniello

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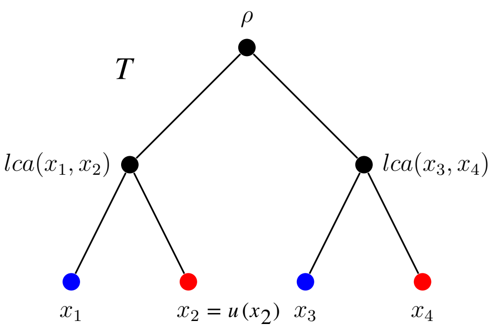
UNIVERSITÄT LEIPZIG

BIO INFORMATICS Leipzig

2QBMG: explaining tree

2QBMG: EXPLAINING TREE

- Tree **T** is rooted at ρ, phylogenetic, $\sigma : L(T) \longrightarrow \{\bullet, \bullet\}$.
- u: $L(T) \longrightarrow V(T)$ is a **truncation map** for T if $u(x) = t$ s.t. t in the path from x to ρ.



$$u(x) = \rho \ \forall x \neq x_2, x \in L(T)$$

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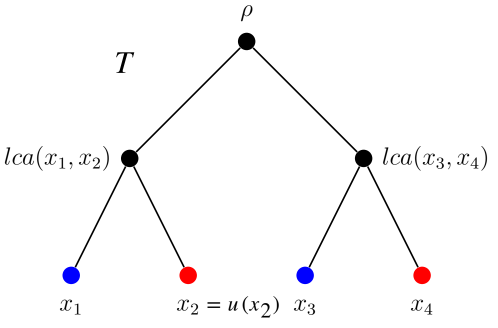

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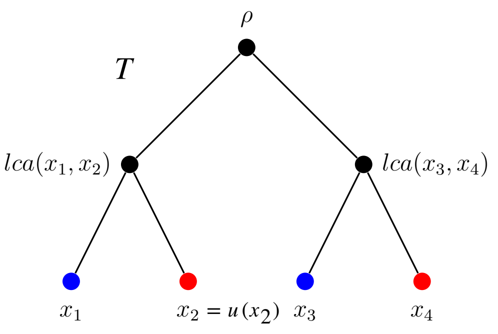

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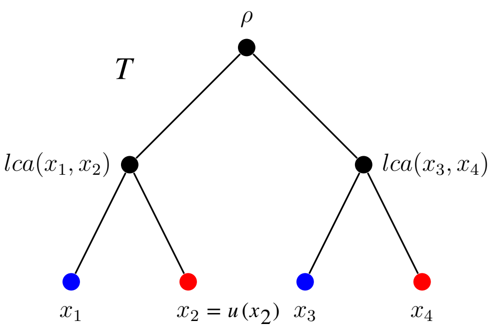

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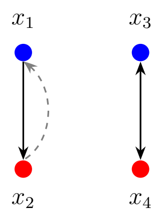

2qBMG(T, σ, u)

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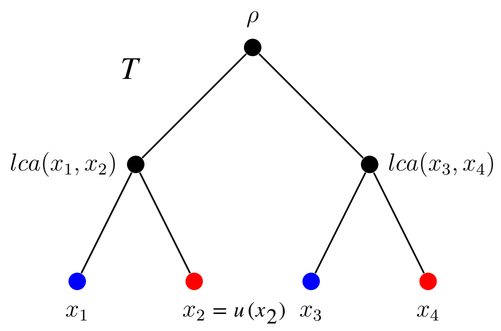

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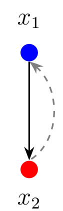

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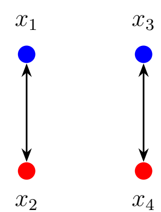

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2BMG = sink-free 2qBMG

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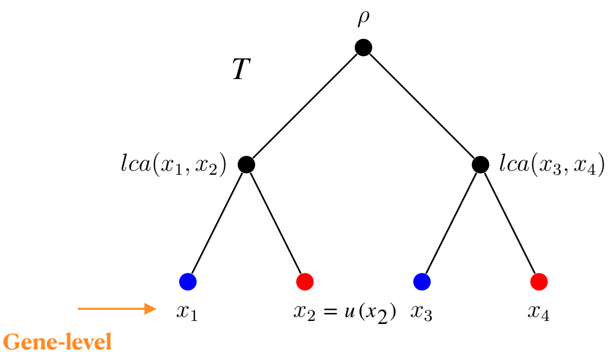

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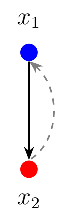

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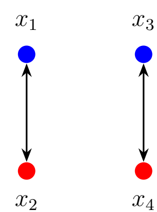
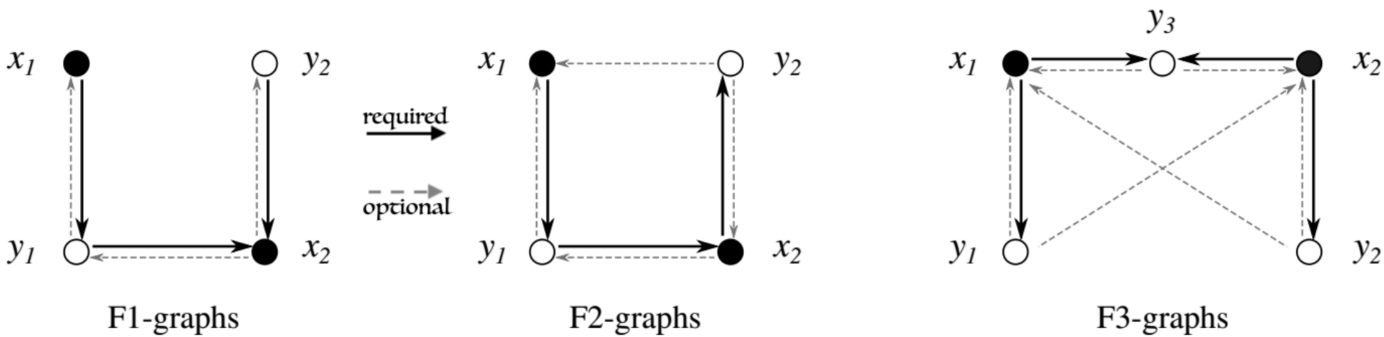


2BMG$(T, \sigma,$ ***u'***$)$

2BMG = sink-free 2qBMG

2QBMG: FORBIDDEN INDUCED-SUBGRAPHS

- A bipartite digraph is a 2-qBMG iff it contains no induced F1-, F2-, and F3-graph[1].



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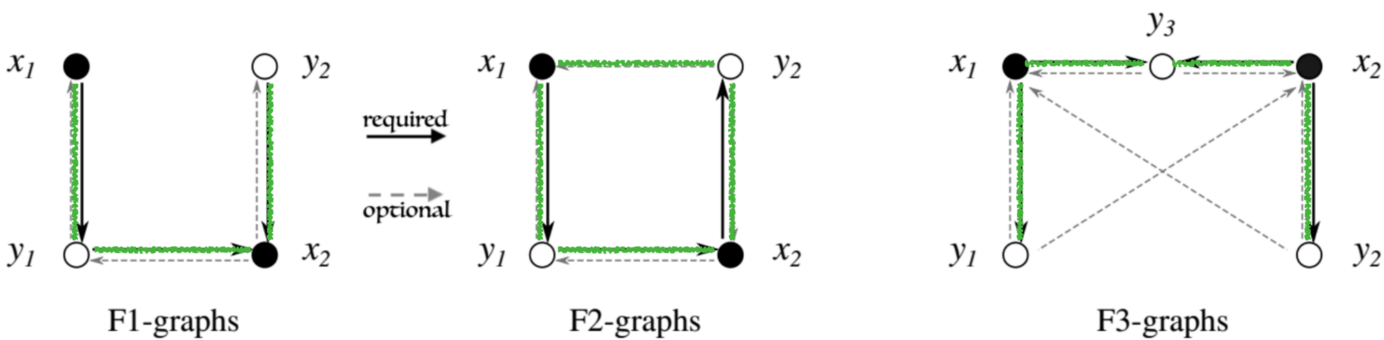

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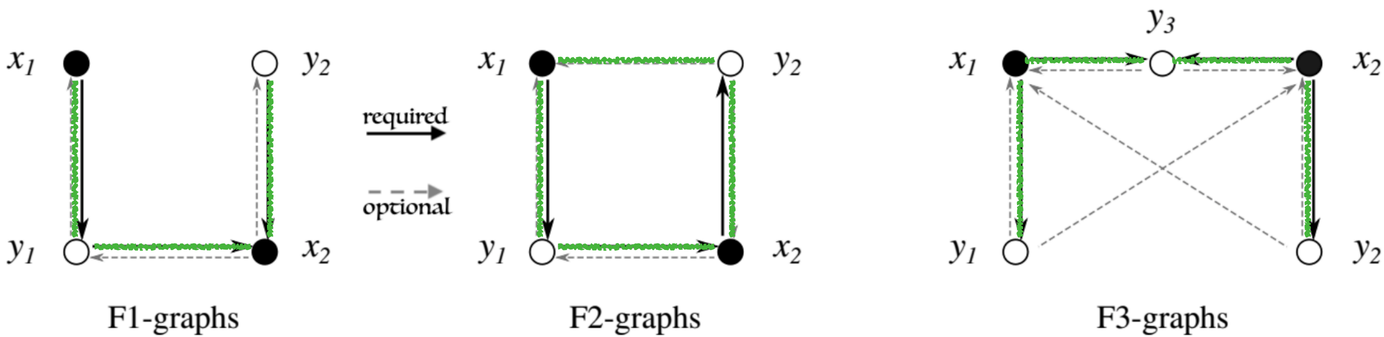


- **un2qBMG** is the undirected underlying graph G of a 2qBMG $\overrightarrow{G}$.

  Question 1: Is a un2qBMG P_4-, C_4-, or P_5-free?[2]

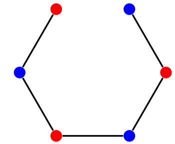
[1] David Schaller, Peter F Stadler, and Marc Hellmuth (2021). **"Complexity of modification**

[2] Annachiara Korchmaros (2024). **"Forbidden Paths and Cycles in the Undirected Underlying Graph of a 2-quasi Best Match Graph".** In: *24th Conference Information Technologies – Applications and Theory (ITAT 2024)*.

un2qBMG: chordal bipartite

Proposition

P_6 is the minimum forbidden induced subgraph for un2qBMGs.



P_6 path-graph

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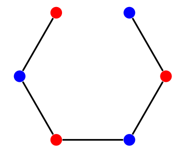
Every un2qBMG is P_6- and C_6-free.



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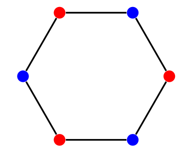

C_6 cycle-graph

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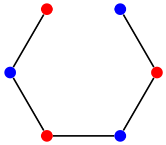
*Every un2qBMG is P_6-free and **chordal bipartite** (ie C_l-free for $l \geq 6$).*

- Every C_l contains P_6 as induced subgraph for $l > 6$.



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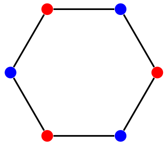

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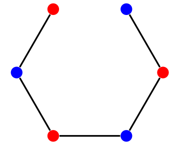

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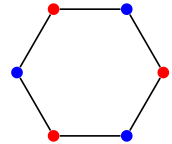

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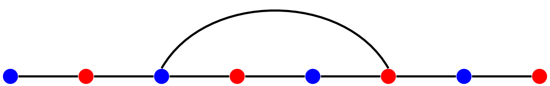


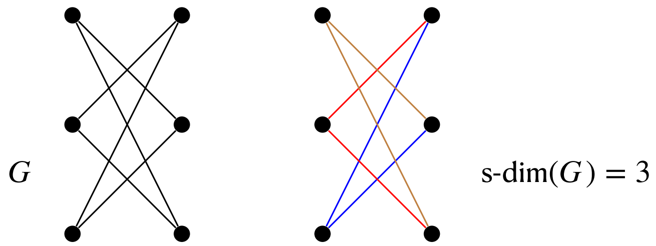
sunset-graph

Proposition 2

Every un2qBMG is sunset-free.

Biclique cover problem

- A collection of (induced) bicliques covering $E(G)$ is a **biclique cover**.



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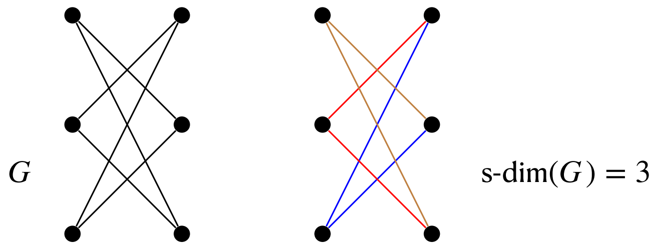

- The minimum cardinality of biclique cover of G is the bipartite dimension and referred as **s-dim**(G) for bipartite graphs[1] .

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MOTIVATION

Known results:

1. Computing s-dim (**biclique cover problem**) is NP-complete for bipartite and chordal biparite graphs[1].

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Question 2: Is un2qBMG domino-free? convex? distance hereditary?

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DOMINO-FREE

- **Domino-free** graphs do not contain D as induced subgraph.

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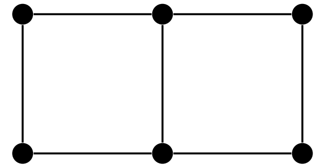

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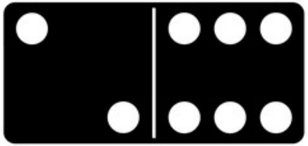

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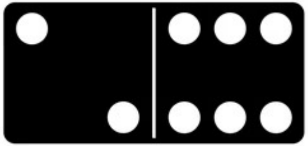
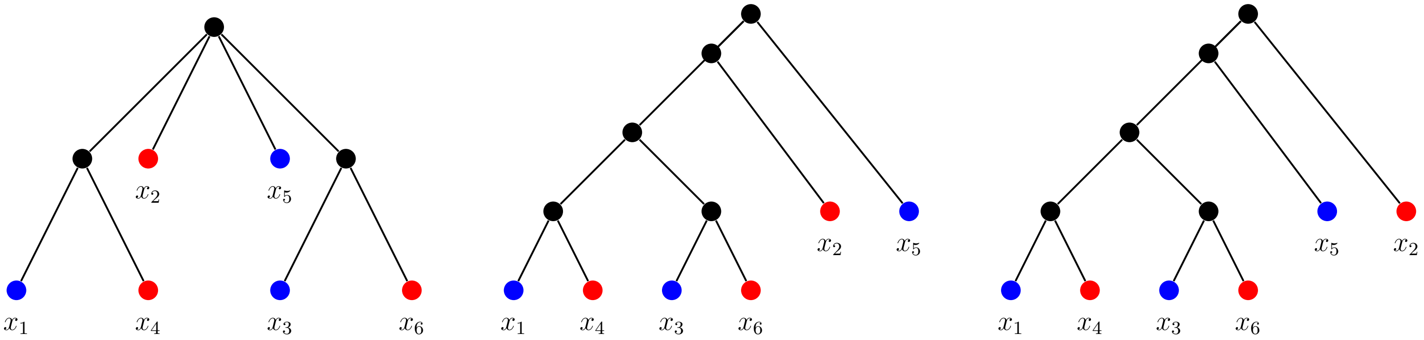


Theorem 2

*There are exactly **15** non-color-preserving isomorphic 2qBMGs on 6 vertices s.t. un2qBMG=D, explained by three possible tree topologies.*



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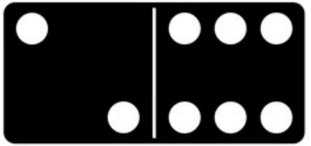

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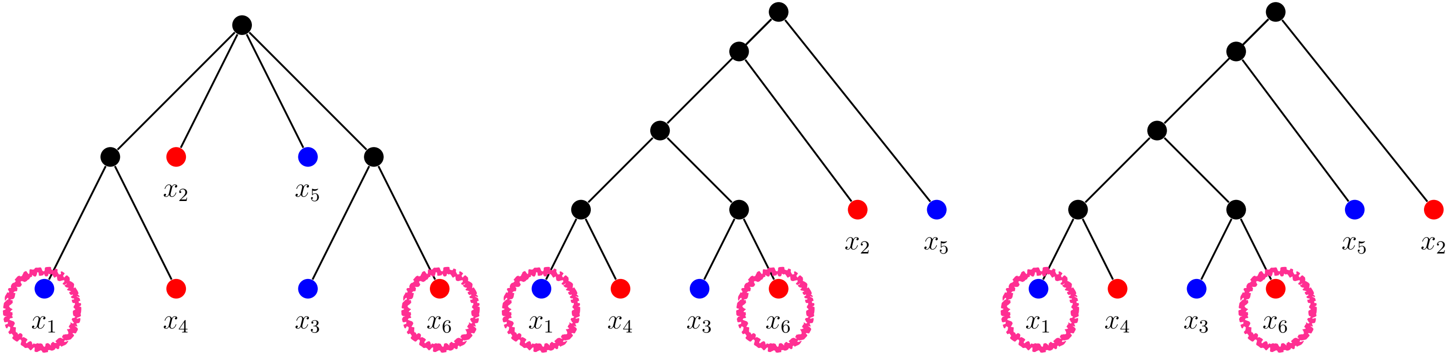

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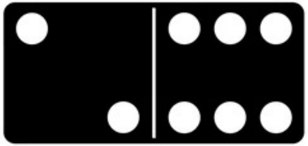

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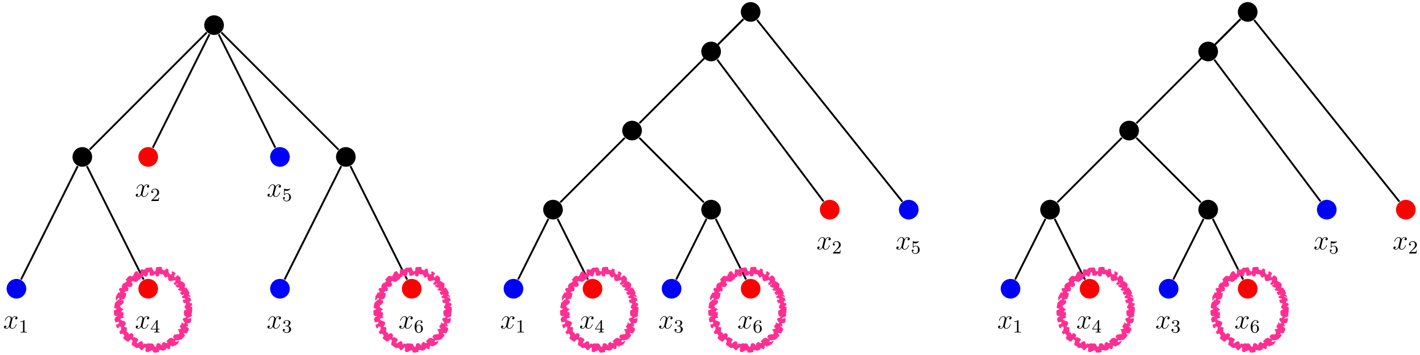

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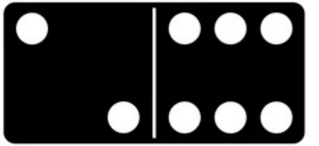

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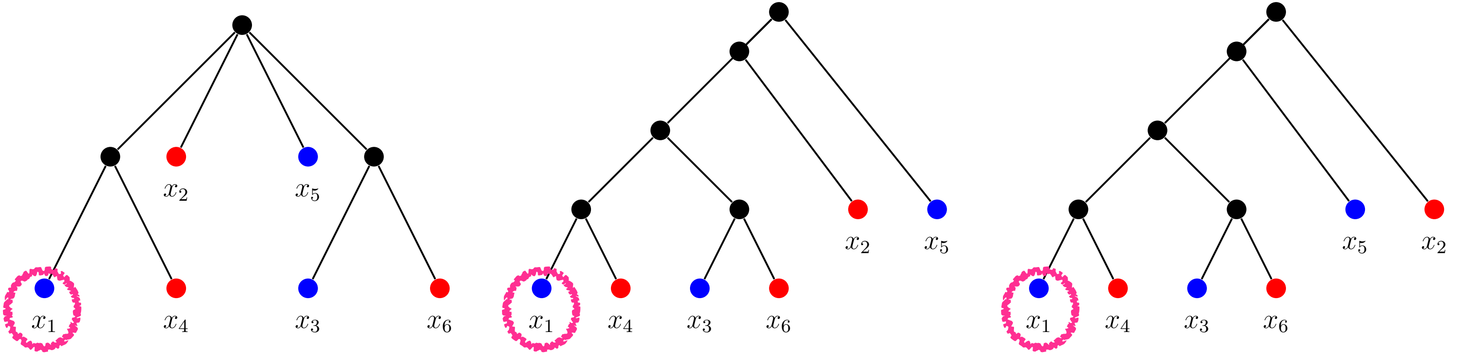

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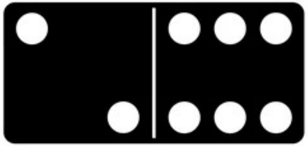

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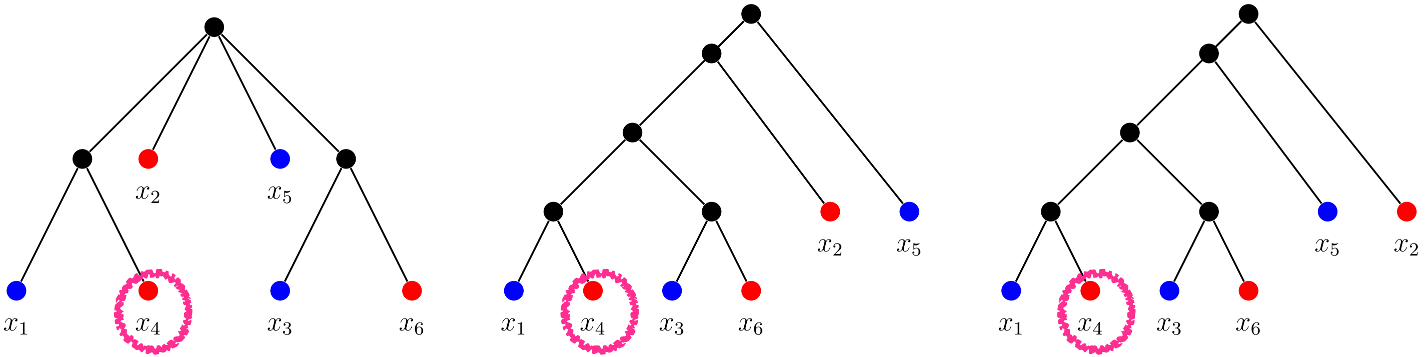

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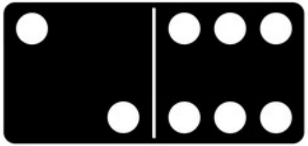

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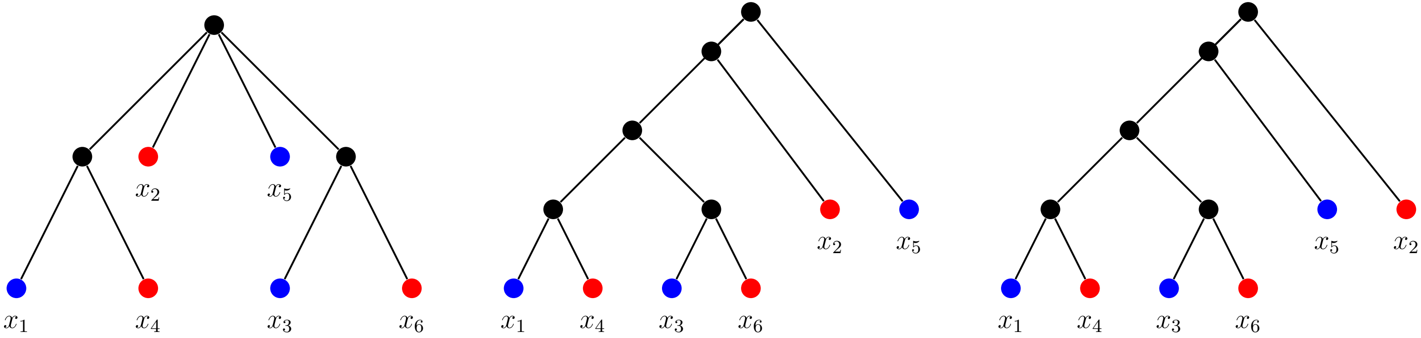

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DISTANCE HEREDITARY & CONVEX GRAPHS

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$M_c = \begin{pmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix}$ *and* $M_p = \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \end{pmatrix}$ *do not satisfy C1P.*

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Conjecture

un2qBMGs are not convex graphs.

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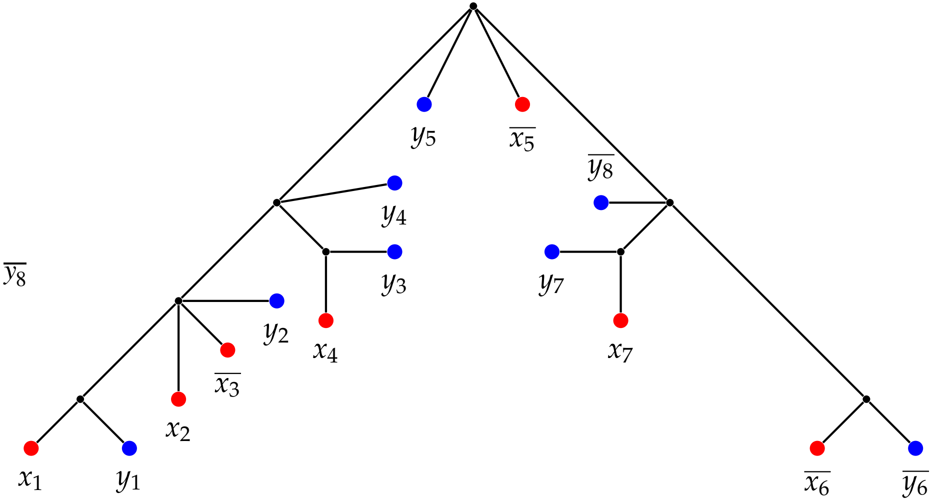
Linear biclique cover

- $T := (T, \sigma, \mu)$ with $\mu(L(T)) = L(T) \cup \{\rho\}$.
- T **explains** G if T explains $\overrightarrow{G}$ and G=un2qBMG($\overrightarrow{G}$). $\mu(x) = x$ iff x is a **sink** of $\overrightarrow{G}$.

sinks: $\overline{x_3}, \overline{x_5}, \overline{x_6}, \overline{y_6}, \overline{y_8}$



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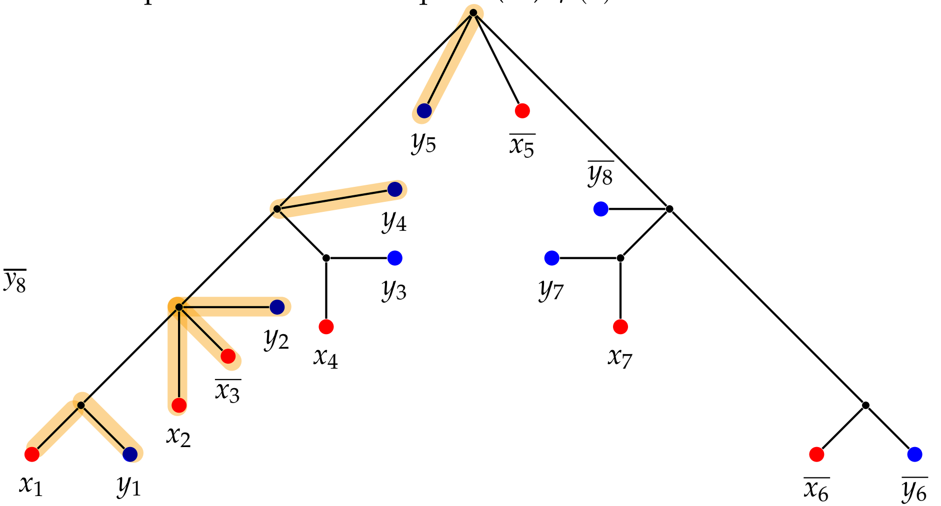

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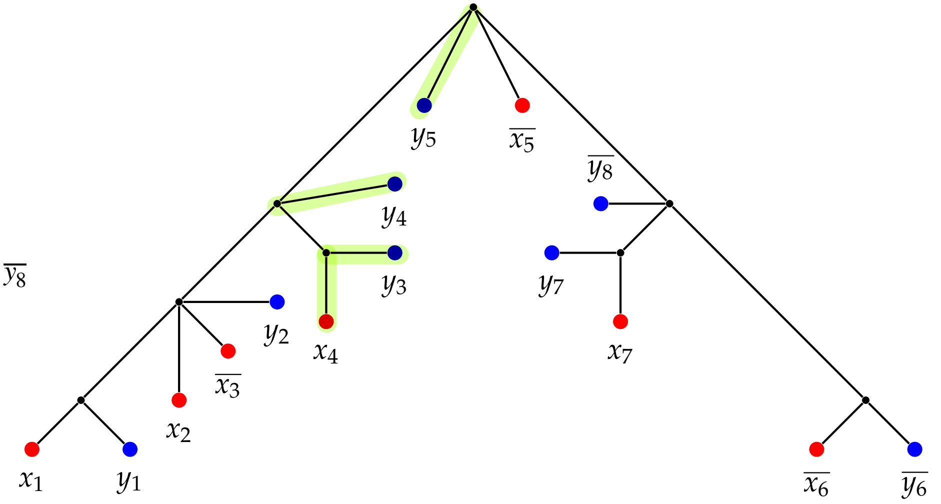

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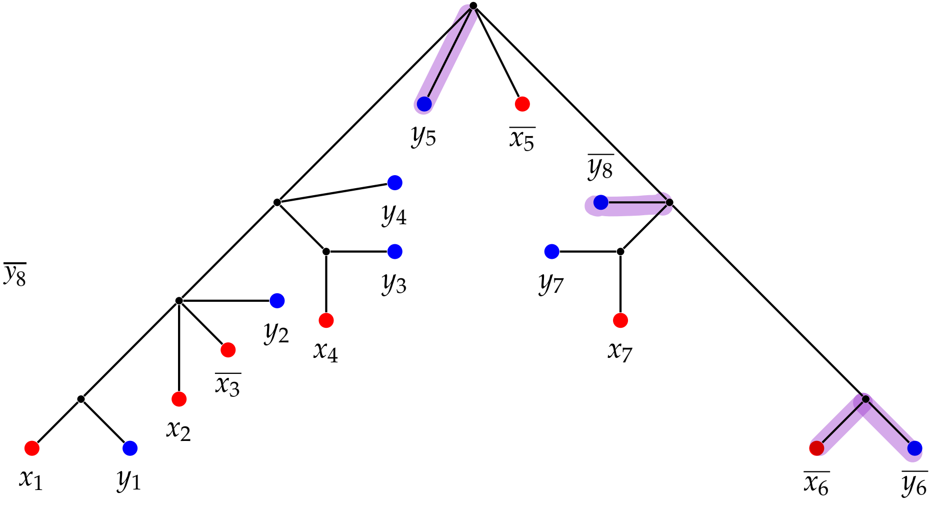

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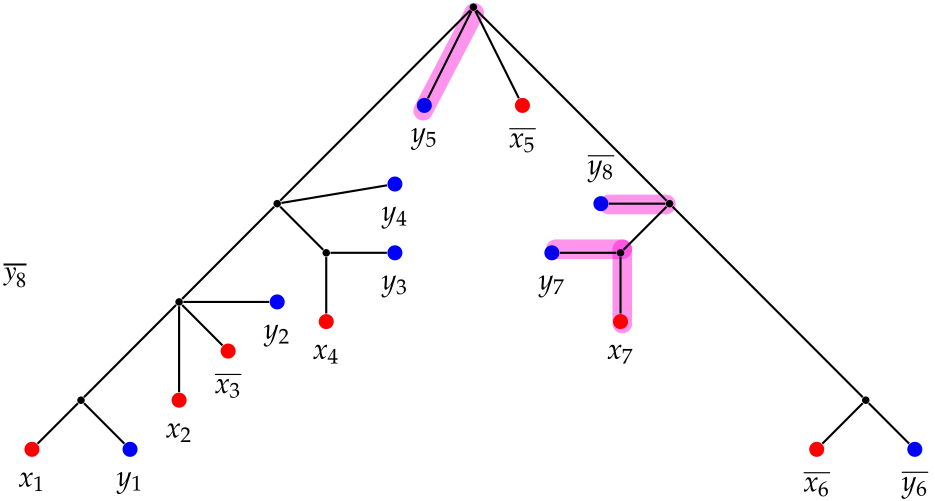

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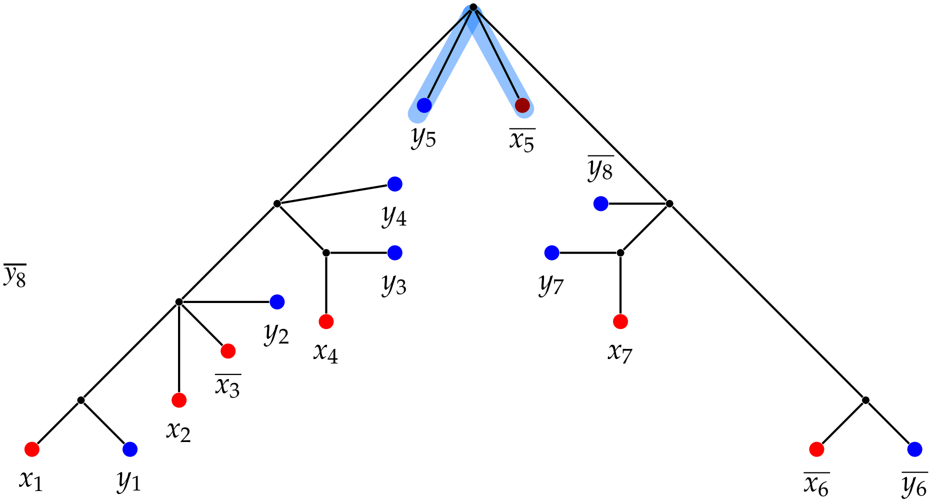

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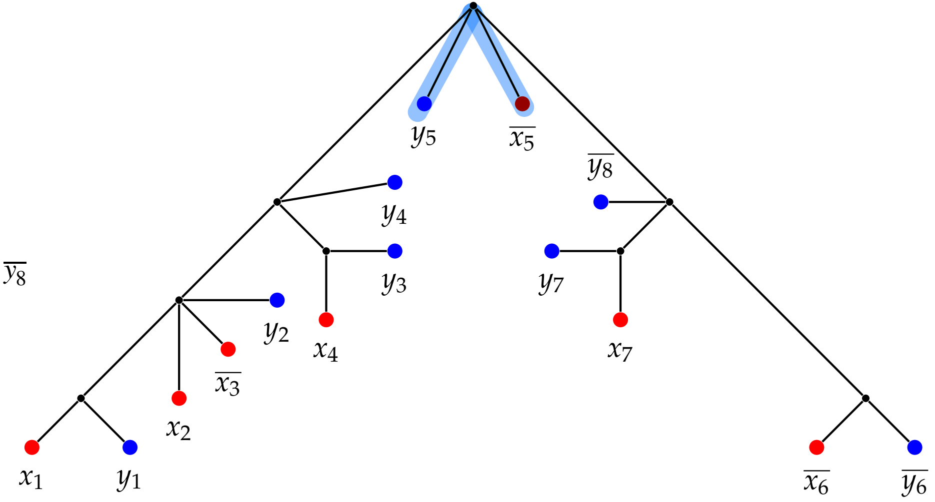

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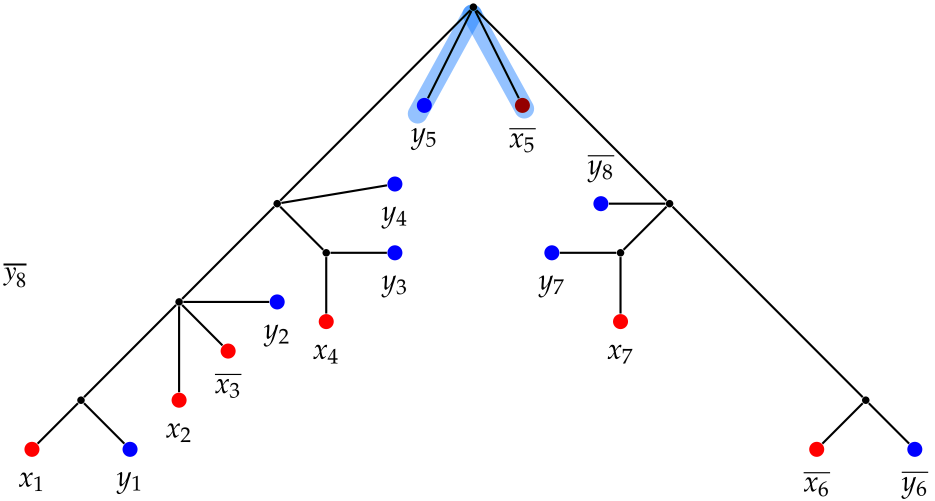

Theorem 3

If G is explained by T, s-dim(un2qBG)$\leq$ 1+ numb. of starts of T. The bound is tight when the root of T has two children of different colors, one of which is a sink.

Maximum 2-transitivity problem

- $A, B \subseteq V(G)$; A **2-dominates** B if $\forall\, b \in B, \exists\, a_1, a_2 \in A$ st $a_1 b, a_2 b \in E(G)$.

Subhabrata Paul and Kamal Santra (2024). **“Algorithmic study on 2-transitivity of graphs”.** In: *Discrete Applied Mathematics* 358, pp. 57–75.

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Known results :

1. Computing $Tr_2(G)$ (maximum 2-transitivity problem) is NP-complete for bipartite graphs.
2. Computing $Tr_2(G)$ is **polynomial** for bipartite chain graphs.

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BIPARTITE CHAIN GRAPHS

- A bipartite graph is **bipartite chain** if there exists an ordering of vertices st $N(x_n) \subseteq N(\bullet) \subseteq \cdots \subseteq N(x_1)$ and $N(y_m) \subseteq N(\bullet) \subseteq \cdots \subseteq N(y_1)$.

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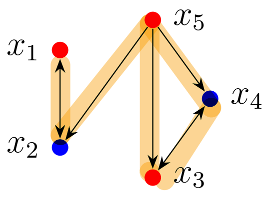
Fact

un2qBMGs are not bipartite chain graphs.



G is not a bipartite chain graph!

$N(x_2) \nsubseteq N(x_4)$ and $N(x_4) \nsubseteq N(x_2)$

LINEAR 2-TRANSITIVITY

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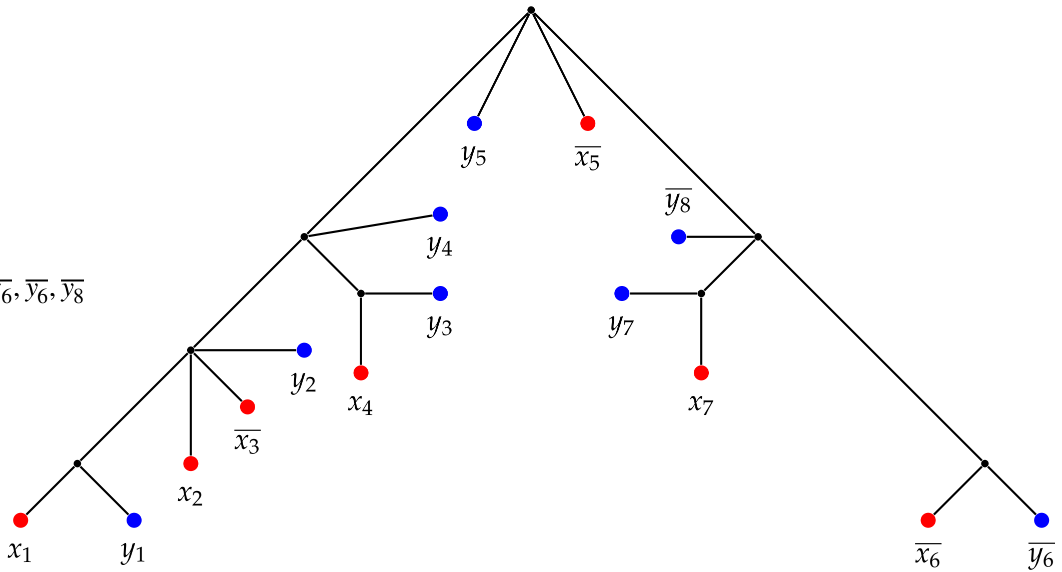

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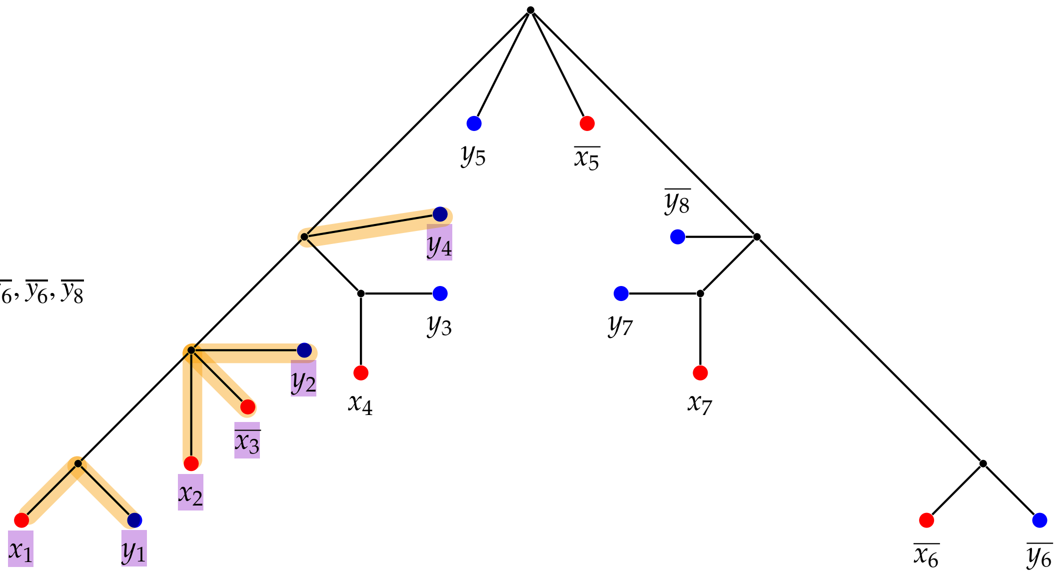

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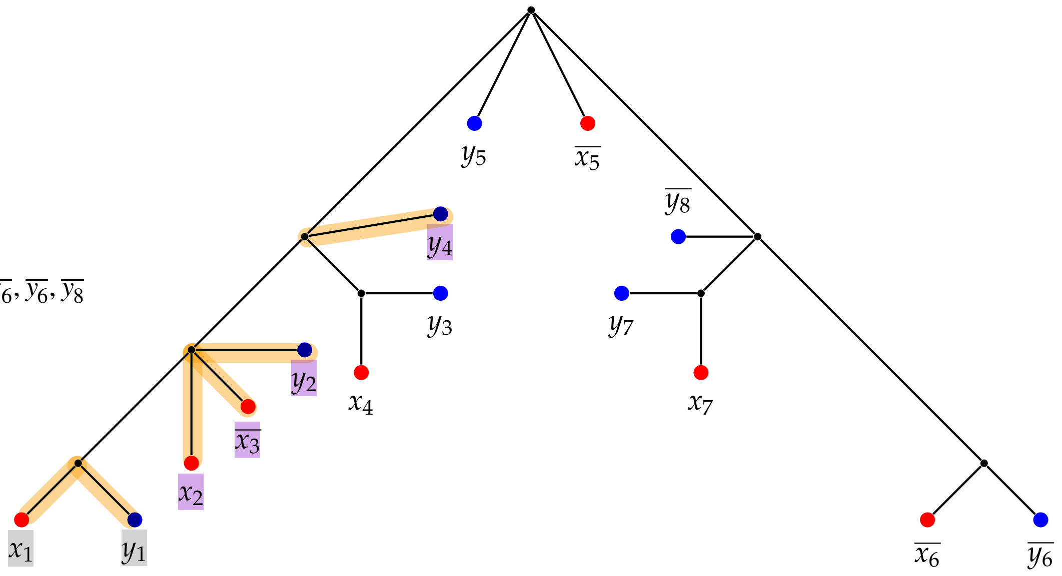

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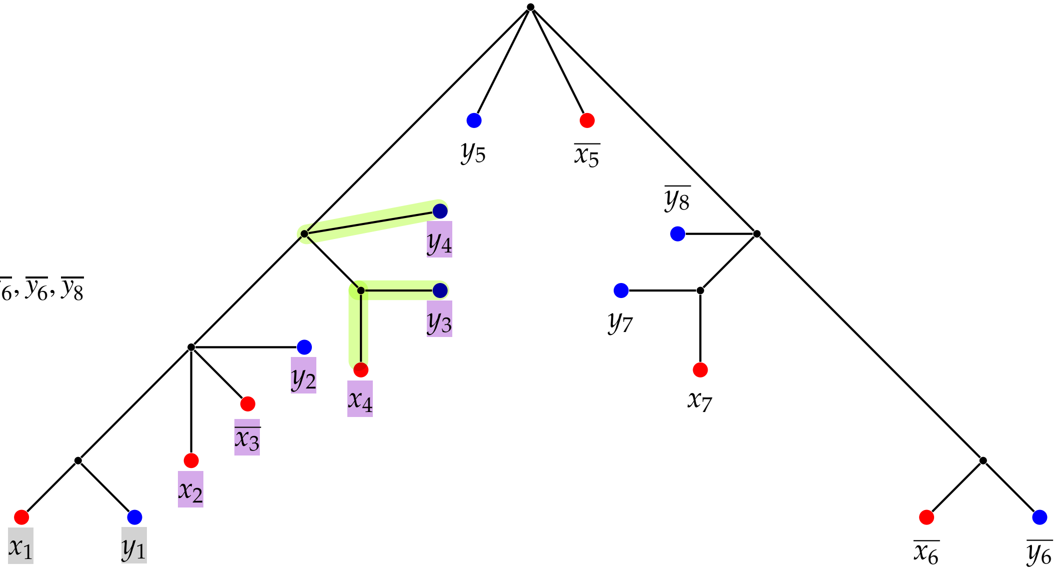

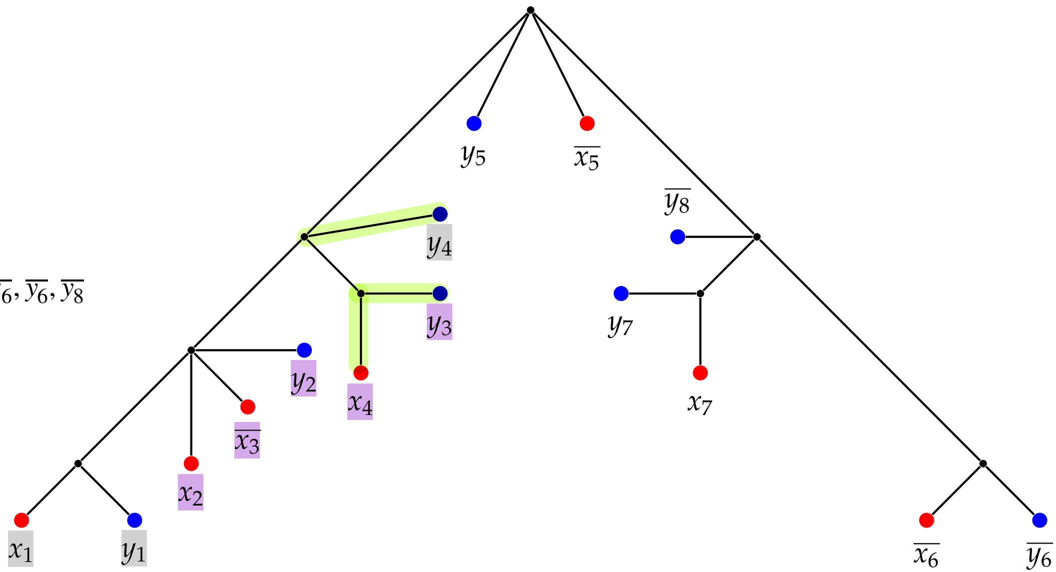

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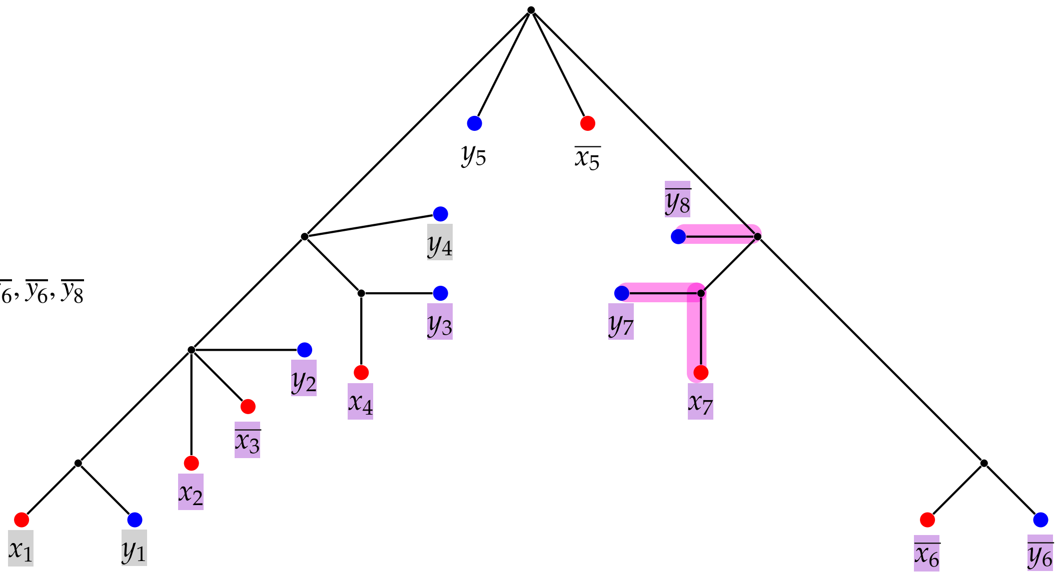

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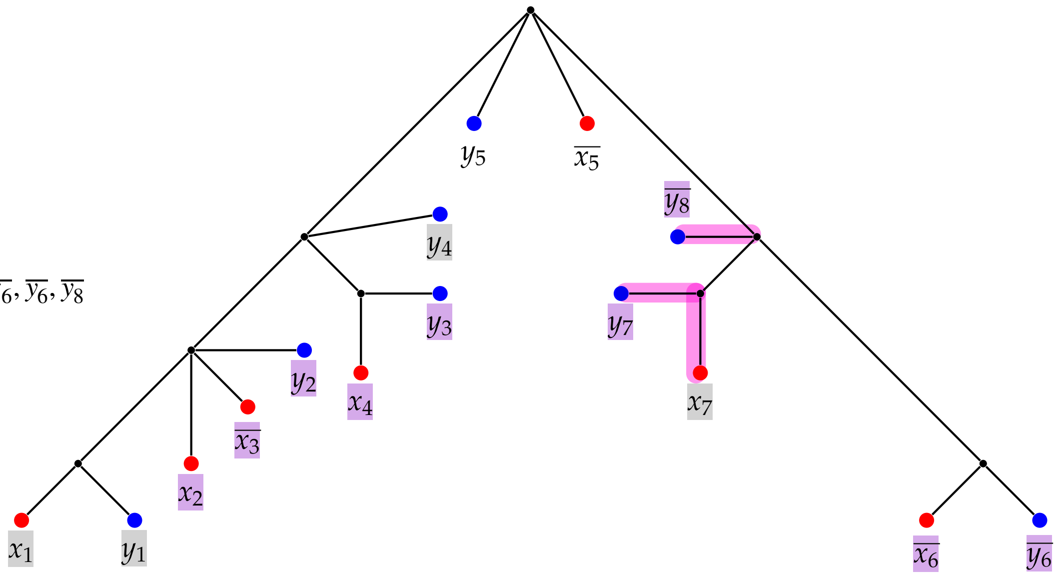

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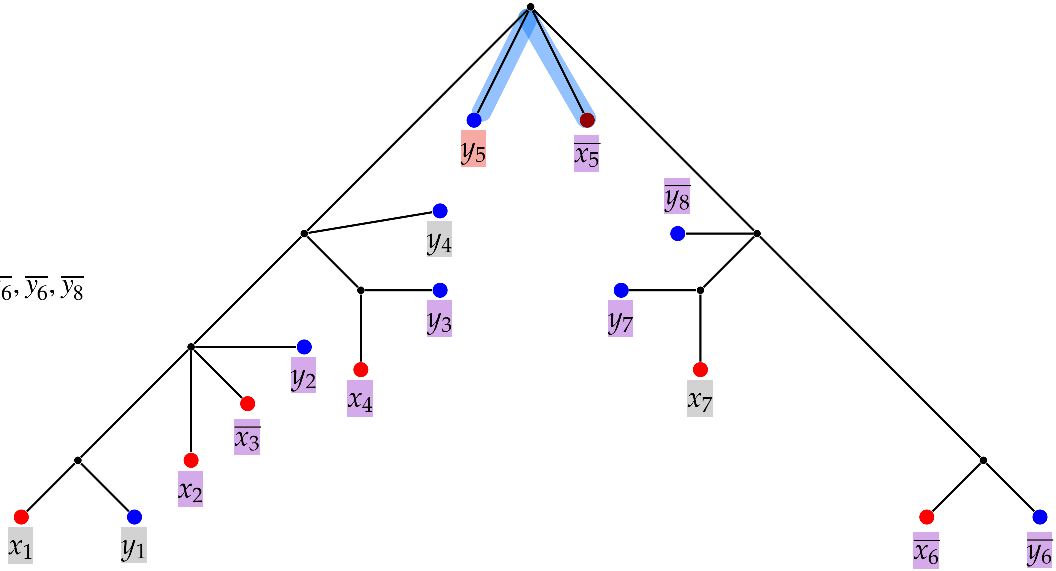

LINEAR 2-TRANSITIVITY

$T_{r2}(G) = 3$

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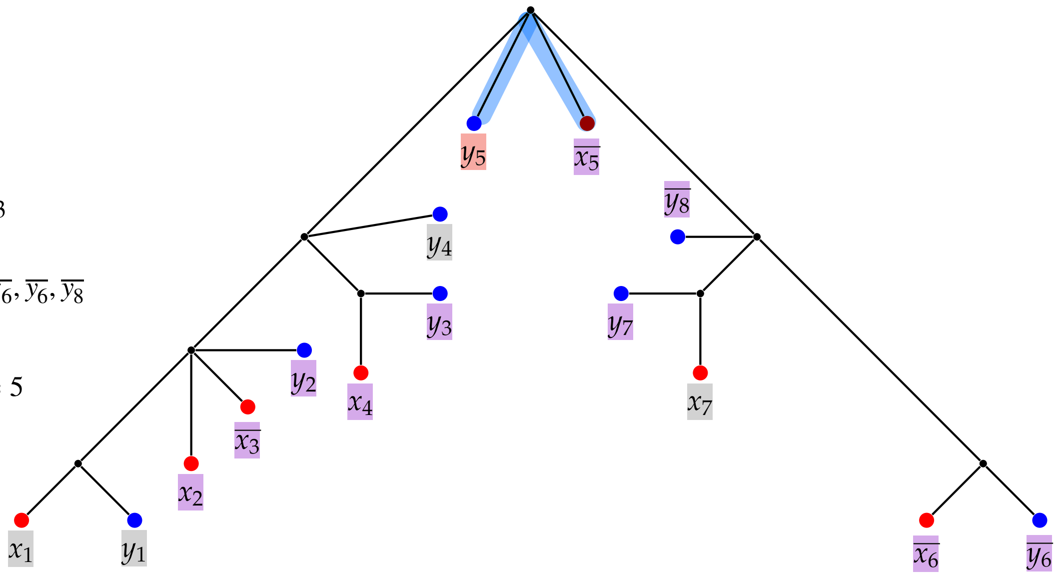

Working in progress..

1. Can we recognize if a graph is un2qBMG in a polynomial time? ((P6, C6)-free in linear time)

2. Can we build a tree that explains an un2qBMG is polynomial time?

3. How difficult is to edit a graph to a un2qBMG?

SCAN ME