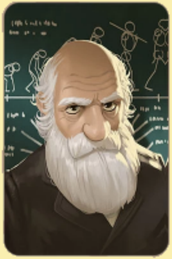
Representing distance-hereditary graphs with trees

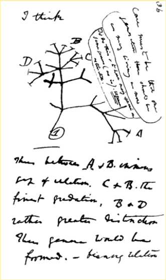
Guillaume Scholz

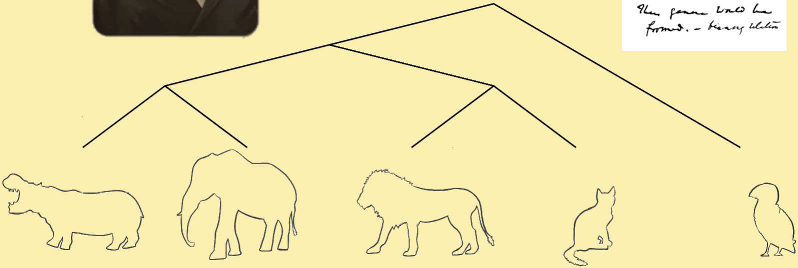
UNIVERSITÄT LEIPZIG

TBI Winterseminar - Bled 2025

Phylogenetic tree (∼1850)







Multi-rooted fusion graph (2015)

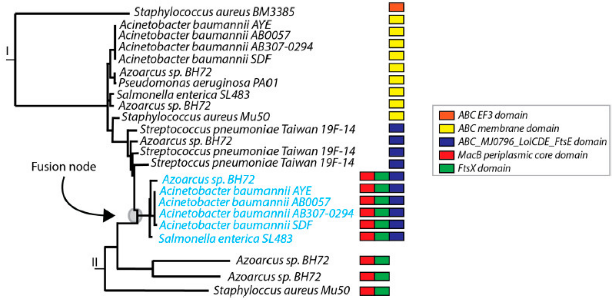


Figure 5. Two-rooted fusion graph. This two-rooted graph was constructed using the two phylogenetic trees from Figure 4. The trees were mid-point rooted and merged using Adobe Illustrator. The two roots are marked I and II. The grey dot, labelled "Fusion node" indicates the approximate location of the fusion event. The coloured squares display the domain architecture of the genes.

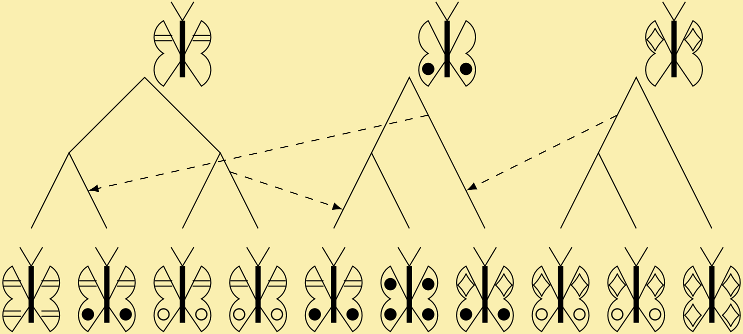
computation
ISSN 2079-3197
www.mdpi.com/journal/computation

Article

Evolution by Pervasive Gene Fusion in Antibiotic Resistance and Antibiotic Synthesizing Genes

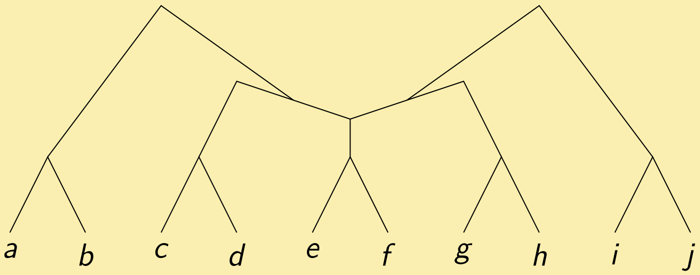
Orla Coleman †, Ruth Hogan †, Nicole McGoldrick †, Niamh Rudden † and James O. McInerney *

Representing introgression [1]



[1] G. E. Scholz, A.-A. Popescu, M. I. Taylor, V. Moulton and K. T. Huber. OSF-Builder: A new tool for reconstructing and representing phylogenetic histories involving introgression, *Systematic Biology* (2019) 68(5):717-729.

Arboreal network



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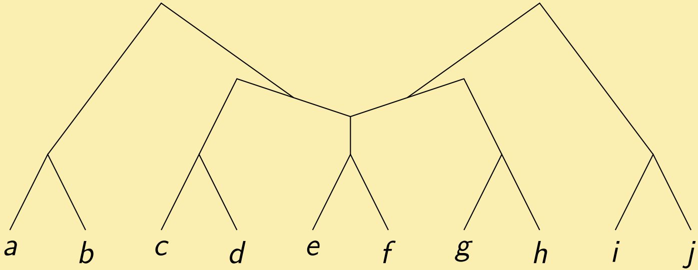
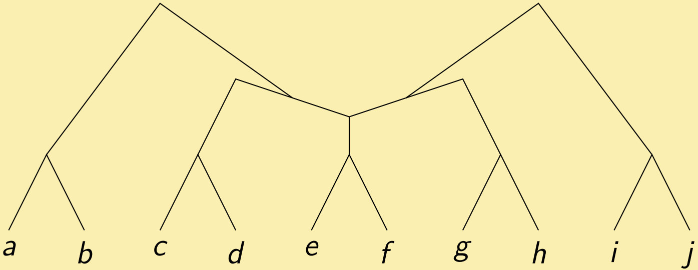


Arboreal network: a directed acyclic graph whose underlying undirected structure is a tree.

Shared ancestry graph



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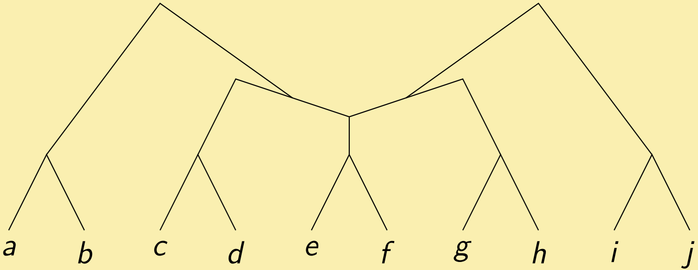

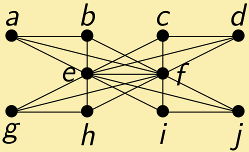

Shared ancestry graph

Shared-ancestry graph: records pairs of leaves sharing an ancestor in a given multirooted network.

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Question: Can we characterize undirected graphs that are the shared ancestry graph of some arboreal network?

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Answer: Yes. And this class of graphs is already well known!

Ptolemaic graphs (early 80s)

G is Ptolemaic if the inequality:

$$d(u,v)d(x,y) + d(u,x)d(v,y) \geq d(u,y)d(v,x)$$

holds for all pairwise-connected vertices x, y, u, v.

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Theorem [2]: A graph G is the shared ancestry graph of some arboreal network N if and only if G is connected and Ptolemaic.

[2] K. T. Huber, V. Moulton and G. E. Scholz. Shared ancestry graphs and symbolic arboreal maps. *SIAM Journal on Discrete Mathematics* (2024) 38(4): 2553-2577.

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In that case, N can be built in polynomial time.

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Distance-hereditary graphs (late 70s)

G is distance hereditary if for all connected vertices u, v and all connected induced subgraph H, $d_G(u, v) = d_H(u, v)$.

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G is distance hereditary if for all connected vertices u, v and all connected induced subgraph H, $d_G(u, v) = d_H(u, v)$.

Ptolemaic graphs are distance-hereditary.

Recursive characterization

G is distance hereditary if it can be recursively constructed from a single vertex by one of:
(R1) Add a pendant vertex.
(R2) Add a true-twin.
(R3) Add a false-twin.

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G is Ptolemaic if it can be recursively constructed from a single vertex by one of:
(R1) Add a pendant vertex.
(R2) Add a true-twin.
(R3*) Add a false-twin to a vertex whose neighbors form a clique.

Recursive construction

Given: A connected Ptolemaic graph G

A vertex x such that G is obtained from $G - x$

via one of (R1), (R2), (R3*)

An arboreal network whose shared ancestry graph is $G - x$

Build an arboreal network whose shared ancestry graph is G

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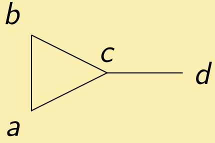

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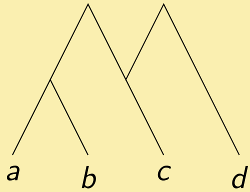

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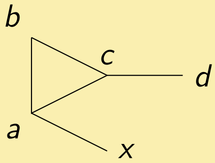
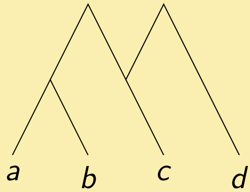


(R1)

N



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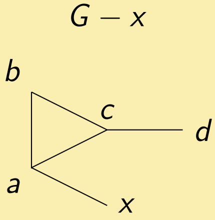

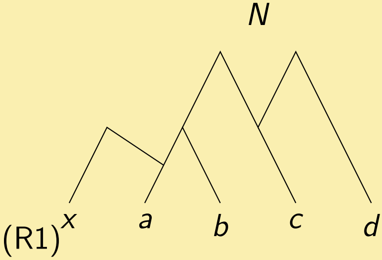

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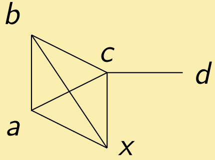
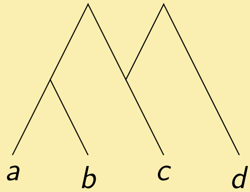


(R2)

N



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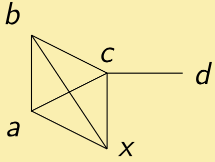

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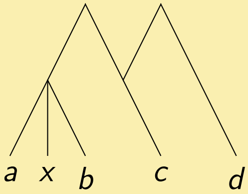

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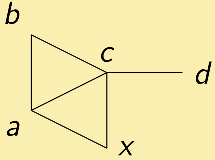
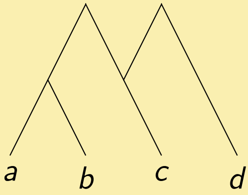


(R3*)

N



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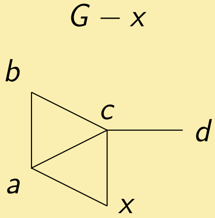
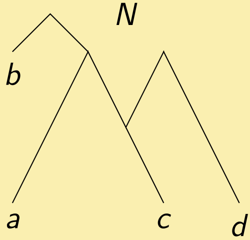


(R3*)



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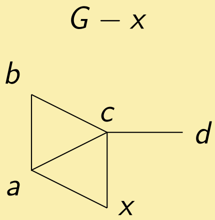
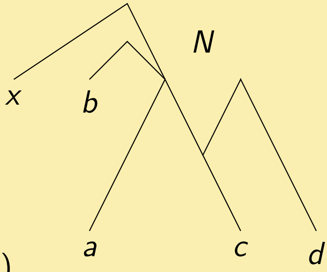
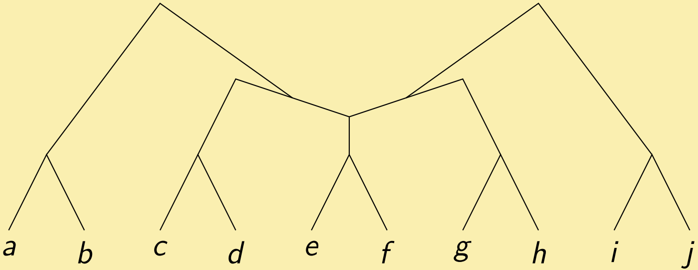




(R3*)

But that's not it!



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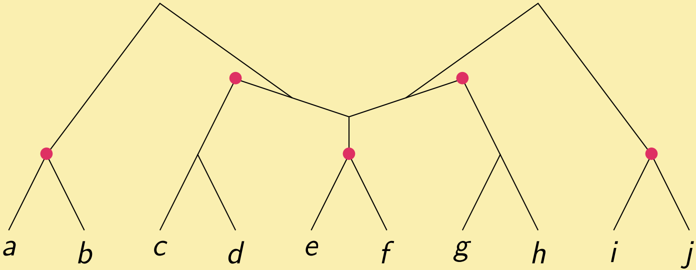

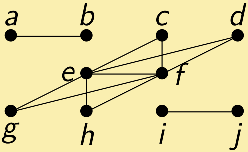

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Theorem: A graph G can be represented by a labelled tree if and only if G is a cograph.

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Theorem [3]: A graph G can be represented by a labelled arboreal network if and only if G is distance-hereditary.

[3] G. E. Scholz. Representing distance-hereditary graphs with multi-rooted trees. (submitted)

Recursive construction (again)

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A vertex x such that G is obtained from $G - x$
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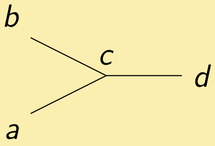

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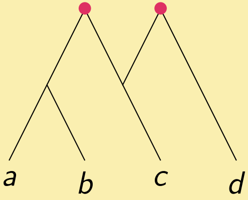

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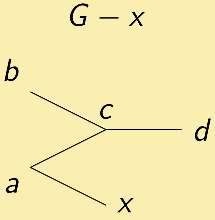
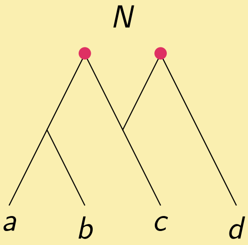


(R1)



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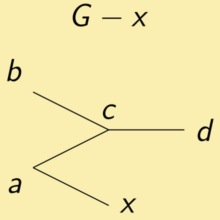

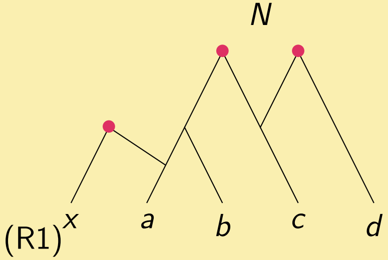

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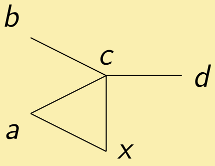
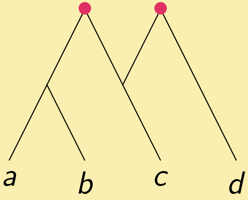


(R2)

N



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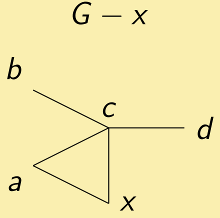

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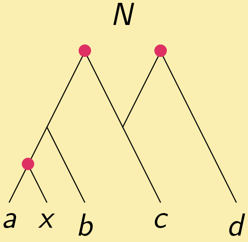

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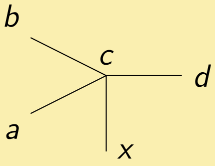

(R3)

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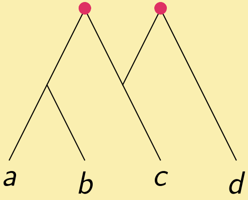

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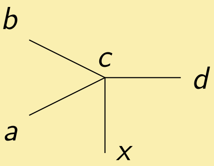
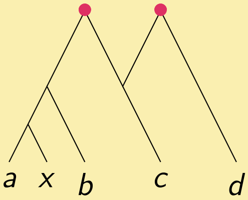


(R3)

N



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Theory

Multirooted networks offer a new and exciting playground for mathematicians, with connections to graph theory, combinatorics, algorithmics, ...