Sparse Semicycle Bases in Graphs

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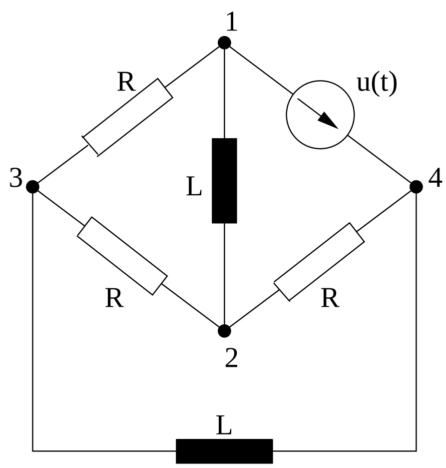
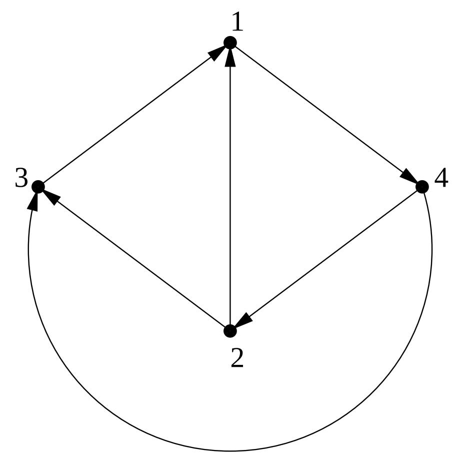
Kirchhoff Voltage Law

The sum of the voltages along any mesh in an electric network is zero.





Model: Directed Graph $G = (V, E)$.

- A semicycle is a simple closed walk C in G, represented by a vector $b(C)$.

$$b_i(C) = \begin{cases} -1, & \text{if } e_i \in C, \text{ and } e_i \text{ backwards}, \\ 0, & \text{if } e_i \notin C \\ +1, & \text{if } e_i \in C \text{ and } e_i \text{ forwards}. \end{cases}$$

Definition: A semicycle basis $\mathcal{B}$ is a maximal linearly independent set of semicycles in G.

The incidence vectors span an $\mathbb{R}$-vector space $\mathcal{C}$, of $\dim \mathcal{C} = m - n + 1$ for G connected.

Problem: Construct sparsest semicycle basis $\mathcal{B}$ of G.

Solvability in time $O(m^3 n)$ with memory requirement $O(n^2)$:

Modification of Horton'87 for undirected weighted graphs.

- Use matroid property of systems of independent semicycles $\rightarrow$ Greedy algorithm can be applied
- Polynomially-sized set of candidate semicycles

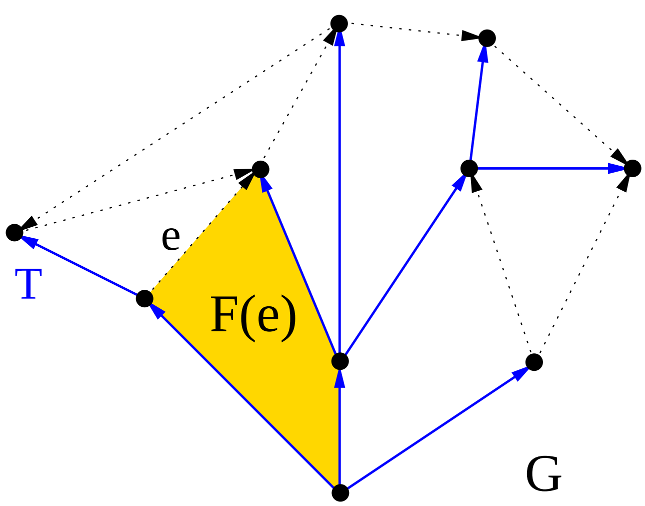
Aim: Find faster approximative methods to construct sparse semicycle bases with low space requirement.

I. Optimize fundamental tree semicycle basis:

* Construct a spanning tree T in G.

* Add non-tree edge e to T and store unique semicycle $F_T(e) = T + e$

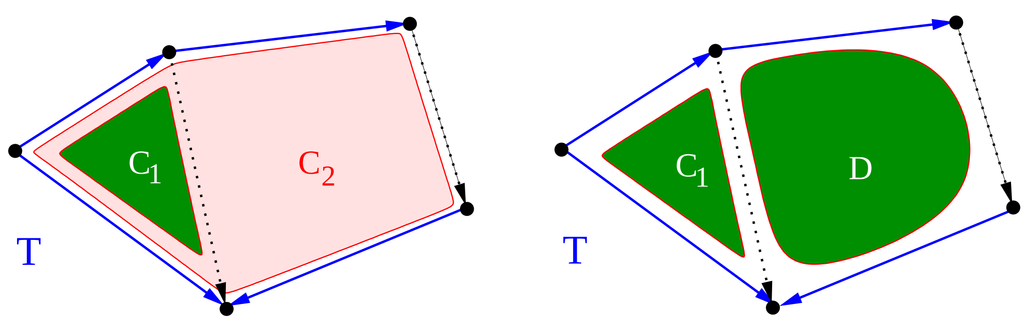
Lemma 1. *The set of fundamental semicycles $\{F_T(e) | e \notin T\}$ is a semicycle basis.*



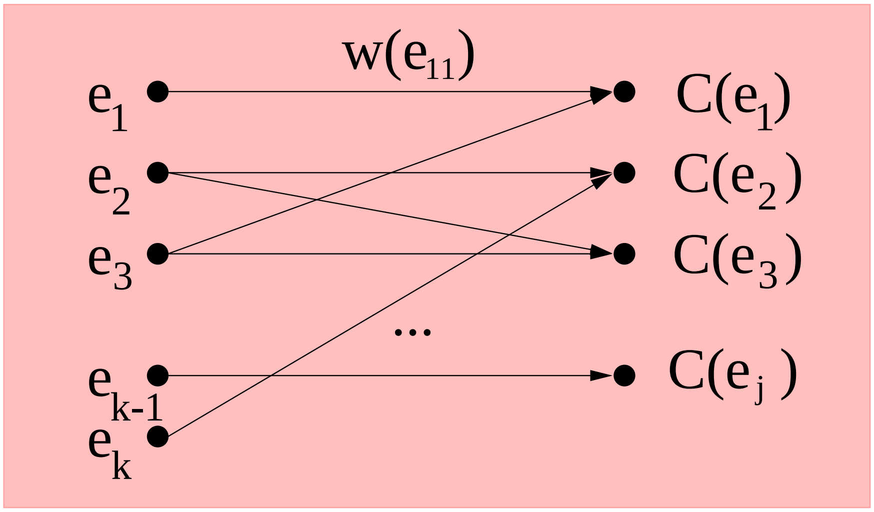
Idea: Exchange semicycles in a fundamental semicycle basis such that as many as possible are shortest semicycles.

Algorithm

1. Create a fundamental semicycle basis with respect to a spanning tree T.
2. For each non-tree edge e construct a shortest semicycle $C(e)$.
3. If $|F_T(e)| > |C(e)|$, this semicycle is a candidate for replacement.
4. Choose an independent set of such $C(e)$.



5. Maximum Bipartite Matching



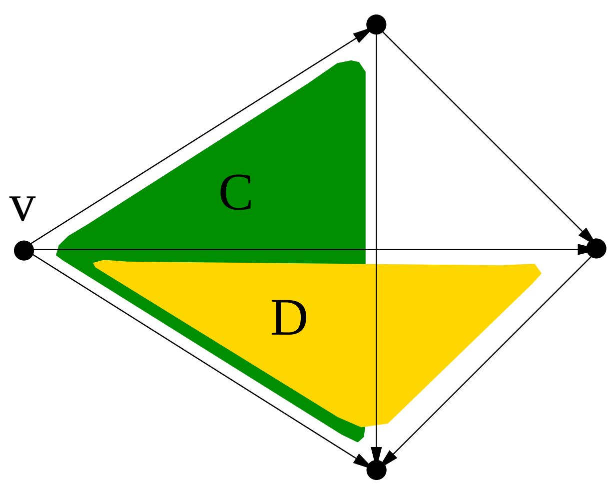
$$w(e_{ij}) = F(e_i) - C(e_j)$$

Proposition 2. *By exchanging fundamental semicycles for semicycles of a maximum bipartite matching of value* M*, the resulting semicycle basis has sparsity*

$$\left(\sum_{i=1}^{m-n+1} |F_T(e_i)| \right) - M.$$

II. Ordering Method

<u>Definition:</u> A *star generator* of a vertex $v \in G$ is a linearly independent set of degree$(v) - 1$ semicycles which contain v and cover all edges adjacent to v.



<u>Idea:</u> For all vertices v, construct a star generator $S(v)$ and remove v from G.

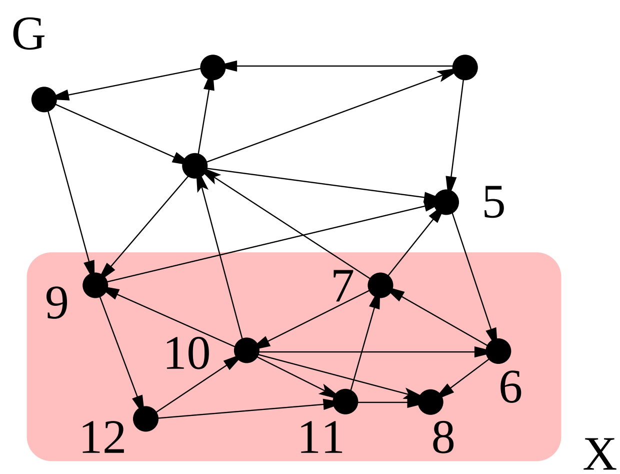
Horton'87 $\Rightarrow$:

Proposition 3. *This method yields a semicycle basis.*

A vertex ordering does not always provide a sparse semicycle basis.

Find a good elimination ordering $V = \{v_1, \ldots, v_n\}$ such that for each $G_i = G \setminus \{v_1, \ldots, v_{i-1}\}$

$$\min |S_{G_i}(v_i)| \subseteq \min |S(v_i)|$$



If such an ordering exists, the resulting semicycle basis is minimal.

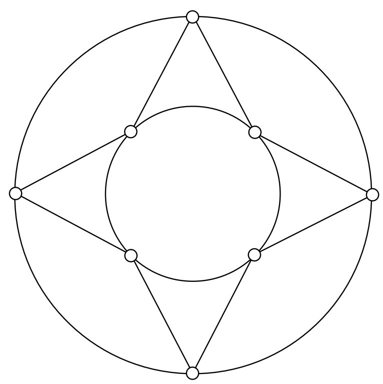
This is possible for special graph classes.

Proposition 4. *If* G *has a unique minimum semicycle basis such that every edge is contained at least twice in one of its member semicycles, then there exists no good elimination ordering.*

Unique bases?

III. Matrix Extension

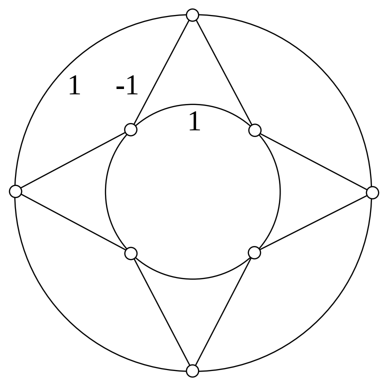
There is a minimum semicycle basis which contains any fixed set of independent shortest semicycles $\{C(e) : e \in E\}$ (Horton'87, Mardon'93).



Let $A \in \mathbb{R}^{r \times m}$ be the matrix with incidence vectors of $\{C(e) : e \in E\}$. If the basis is not complete, then $r = \text{rank}(A) < m - n + 1$.

$$\begin{pmatrix} 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ . & . & . & & & & & & & & & & & & & \end{pmatrix}$$

Therefore there exists a vector $w \in \mathbb{R}^m \backslash \{0\}$ with $w \in \text{kern} A$. Take w as weight vector for the edges of G.



$w = \{-1, 0, 1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1\}$

Find a semicycle in G with odd weight according to w.

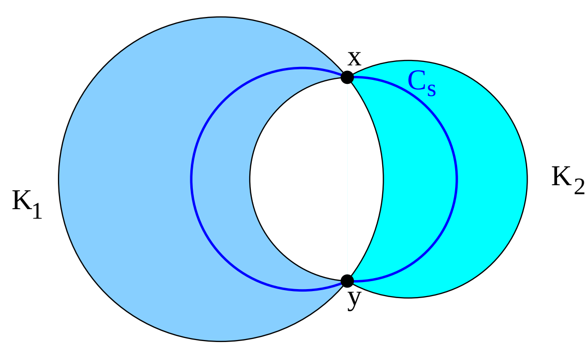
Lemma 5. *If such a semicycle C exists, it is linearly independent from all semicycles in A.*

Add C to A and repeat this step if necessary.

IV. Partitioning

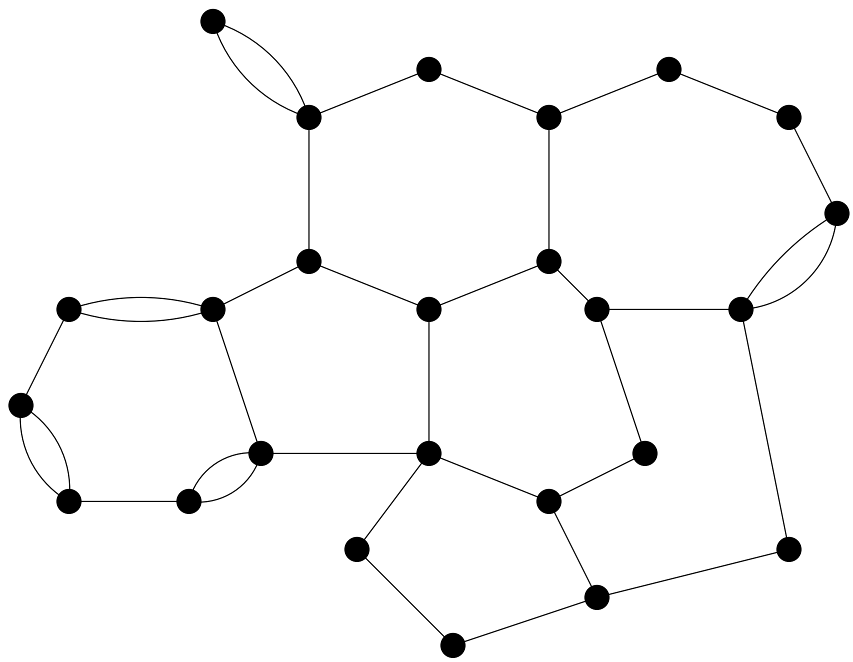
Let G be two-connected, but not three-connected, and $K_1, \ldots, K_r$ its three-connected components.

C is the direct sum of the semicycle spaces of its $K_1, \ldots, K_r$ and $r-1$ 'connecting' semicycles.



Proposition 6. *Let $r = 2$. There exists a minimum semicycle basis of G which consists of the union of two minimum semicycle bases of K_1 and K_2, both containing a shortest connecting cycle C_s.*

V. Comparison



Algorithm	Minbasis	Fundamental
Basis length	56	76
Time (sec.)	0.13	0.0

Match(I.)	Order(II.)	Extension(III.)
56	56	56
0.046	0.04	0.1