Packing ***and Coloring***

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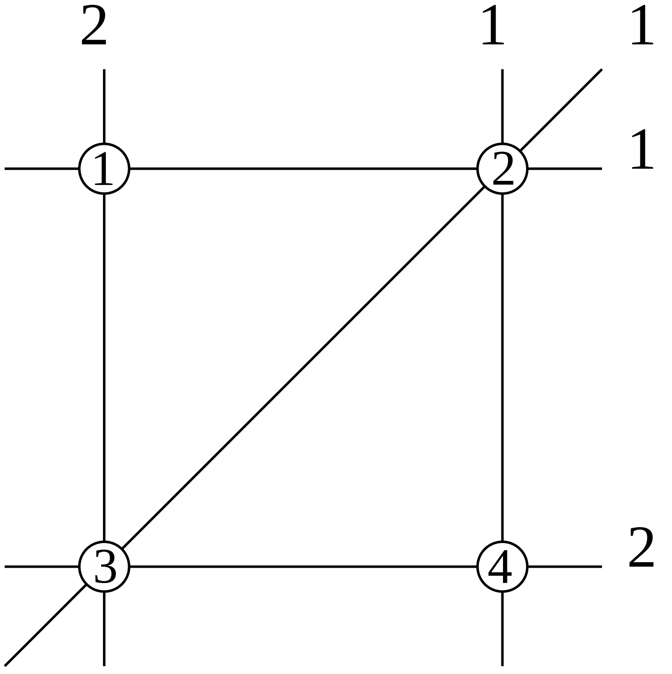
Let $F \subseteq \mathbb{Z}_2$ be a finite set.
Let $1, \ldots, k$ be rational x-rays.
$b_i(F) = |F \cap i| \in \mathbb{N}, i = 1, \ldots, k$



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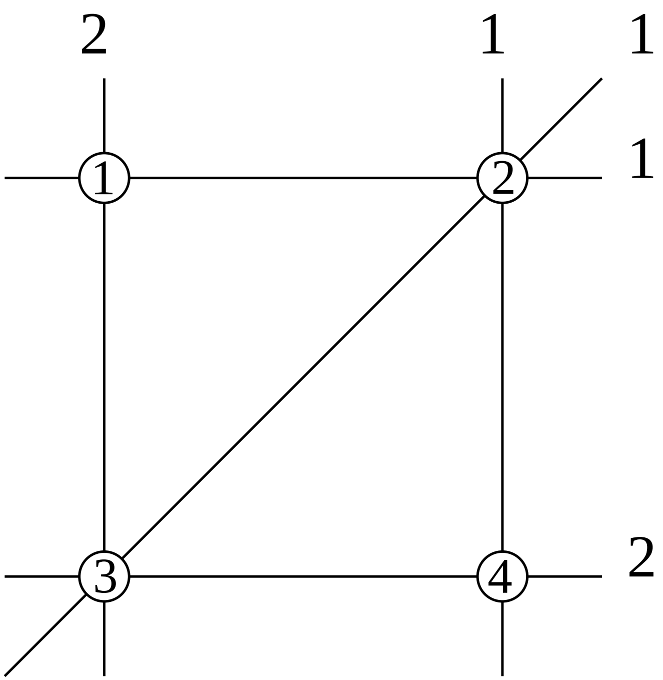
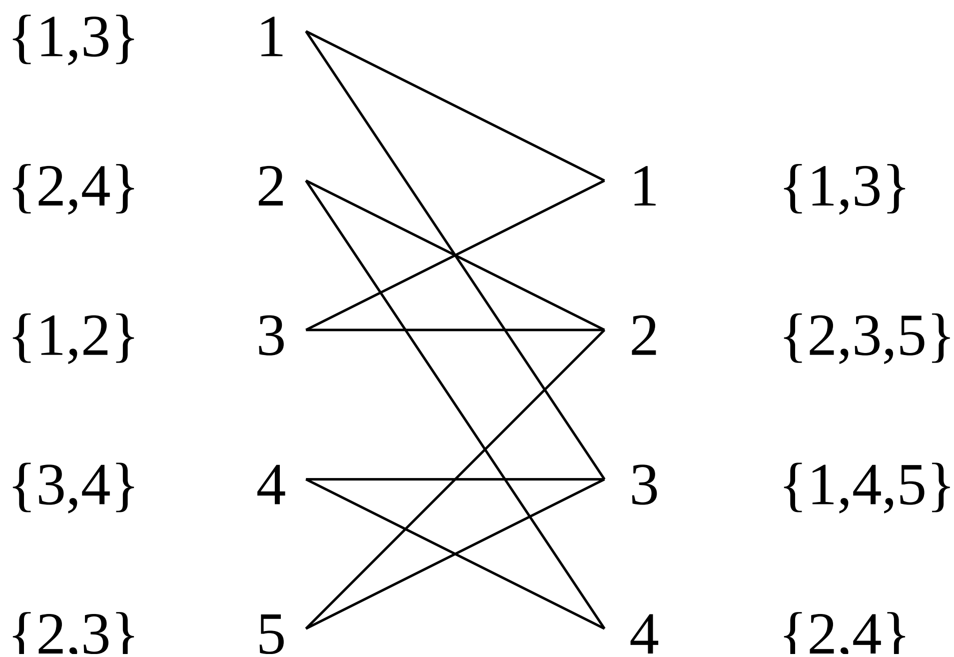


Goal: Reconstruct F from given values b_i!

Reformulation

The following bipartite graph represents the incidence structure of the problem.



Formulation as a packing problem

Let $\mathcal{C} = \{\{1,3\},\{2,3,5\},\{1,4,5\},\{2,4\}\}$ and $b^\top = (2,1,1,2,1)$.

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Then the reconstruction problem is to find a subset of $\mathcal{C}$ of maximum cardinality such that each element j is contained in at most b_j sets.

The LP formulation

The incidence matrix:

$$A = \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 \end{pmatrix}$$

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The generalized packing problem:

$$\max \sum_{i=1}^{N} x_i \text{ s.t.}$$

$$Ax \le b, \text{ and } x \in \{0,1\}^N.$$

The generalized set multipacking problem

$G \subseteq \mathbb{N}$ a finite ground set of p elements.

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$$\max w^\top x \text{ s.t.}$$

$$Ax \le b, \text{ and } x \in \{0,1\}^N,$$

where $N := |\mathcal{C}|$ and $w \in \mathbb{R}_+^N$ positive weights, $b \in \mathbb{N}^p$ capacities.

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Choose from a given collection of weighted sets a subset of minimum weight which covers all elements in the union of the sets at least a prescribed number of times.

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An example

Let $\mathcal{C} = \{\{1,3\},\{2,3,5\},\{1,4,5\},\{2,4\}\}$ with weights $w^\top = (2,3,3,2)$ and capacities $b^\top = (2,1,1,2,1)$.

$$A = \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 \end{pmatrix}$$

Questions

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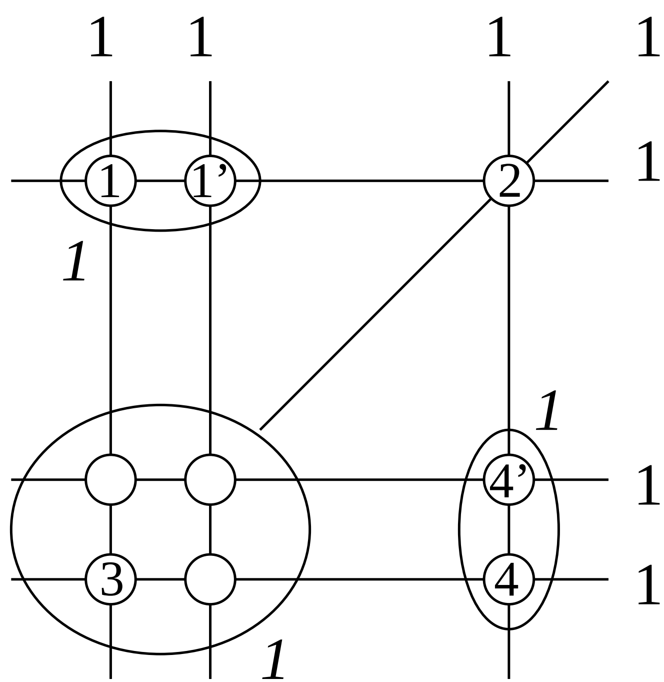
- Idea: Reduce weighted problems to simple problems, for which estimations are known
- Result: For the generalized set multipacking problem, we obtain the same ratio as in the simple case

The transformation idea



$\{1,4,5\}$ is transformed to

$$\{1,4,5,v_2\},\{1',4,5,v_2\},\{1,4',5,v_2\},\{1',4',5,v_2\}.$$

The transformation

We assign to C_j all those sets that can be formed by all combinations of the copies of each element. Finally, to every set, we add the element v_j.

$G = (g_1, \ldots, g_p)$. Let $q := \sum_{i=1}^{p} b_i$

$d :=$ number of sets $C_j \in \mathcal{C}$ that contain an element g_i with capacity $b_i > 1$

$y \mapsto y' \in \mathbb{N}^{q+d} =$
$(y_1, y_1', \ldots, y^{(b_1-1)}, \ldots, y_p, y_p', \ldots, y^{(b_p-1)}, v_1, \ldots, v_d)^\top$.

$b \mapsto b' = 1_{q+d}$.

LP Formulation of the transformed problem

$\max w'^{\top} x$ s.t.

$A'x \leq b',$ and $x \in \{0,1\}^{N'}$.

$\min w'^{\top} x$ s.t.

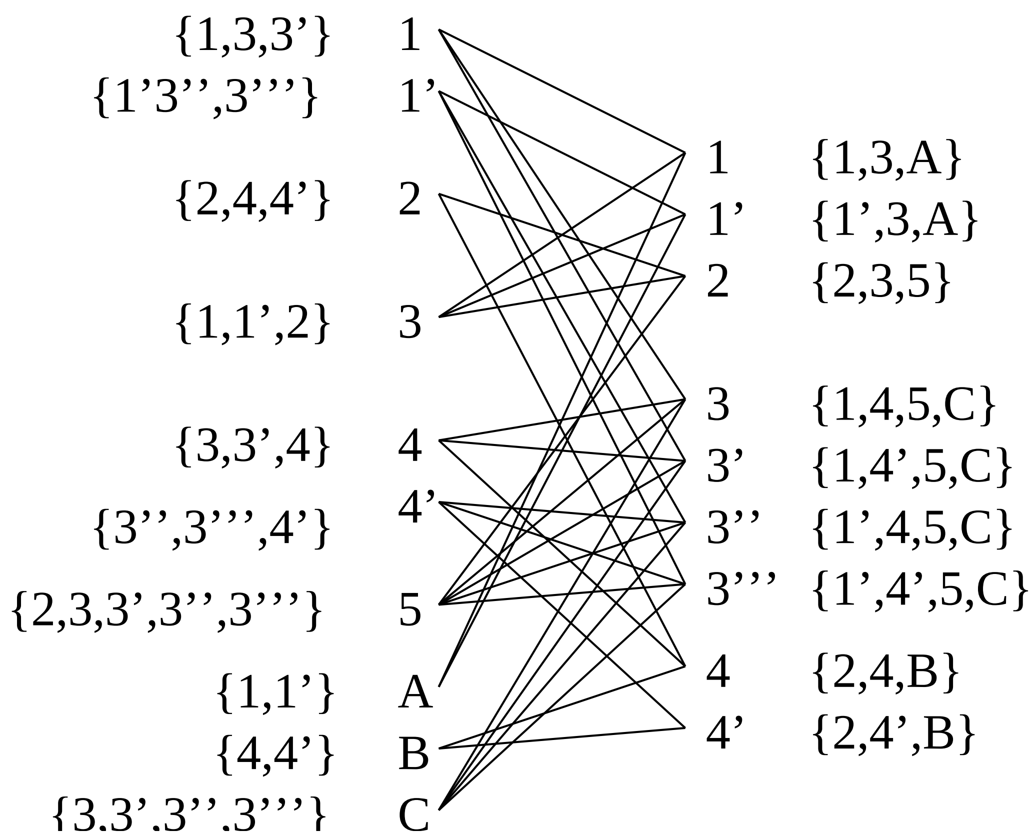
$A'x \geq b',$ and $x \in \{0,1\}^{N'}$.

Example

$q = 10$ and $N' = 9$. $w'^{\top} = (2, 2, 3, 3, 3, 3, 3, 2, 2)$ and

$$A' = \begin{pmatrix} 1 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 & 1 & 1 & 1 & 0 & 0 \\ \hline 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 \end{pmatrix}$$

Example


{1,3,3'} 1
{1'3'',3'''} 1'
{2,4,4'} 2
{1,1',2} 3
{3,3',4} 4
{3'',3''',4'} 4'
{2,3,3',3'',3'''} 5
{1,1'} A
{4,4'} B
{3.3'.3''.3'''} C
1 {1,3,A}
1' {1',3,A}
2 {2,3,5}
3 {1,4,5,C}
3' {1,4',5,C}
3'' {1',4,5,C}
3''' {1',4',5,C}
4 {2,4,B}
4' {2,4',B}

Size and Solution

Lemma: Let $b^* = \max_i b_i$. Then the transformed problem has at most $b^* \cdot p + N$ elements and at most $N \cdot k^{b^*}$ sets.

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The weights of the solutions of the original and the transformed problem coincide because each copied set can belong at most once to a solution.

Let U be an optimal solution of the generalized set multipacking problem. A solution V is t-optimal for $t > 0$ if no subset of $r \leq t$ sets in U can replace sets in V such that the solution is feasible and has strictly greater weight.

- V t-optimal for the original problem. Then all solutions V' of the transformed problem which correspond to V are t-optimal.

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- V t-optimal for the original problem. Then all solutions V' of the transformed problem which correspond to V are t-optimal.
- Every set in V' may have $k + 1$ elements.

First estimations

[Gritzmann-de Vries-Wiegelmann-99], [Arkin-Hassin-98], [Hurkens-Schrijver-89]:
Corollary: The ratio of a weighted t-optimal solution V and an optimal solution U of the weighted k-set multipacking problem is at most

$$\frac{w(U)}{w(V)} \leq k + \frac{1}{t}$$

Corollary: In the unweighted case, the ratio of a t-optimal solution V and an optimal solution U are

$$\frac{|U|}{|V|} \leq \begin{cases} \frac{(k+1)k^s - k - 1}{2k^s - k - 1} & : t + 1 = 2s - 1 \\ \frac{(k+1)k^s - 2}{2k^s - 2} & : t + 1 = 2s \end{cases}$$

Improvement

We find a solution V^* corresponding to V in which every set has at most k neighbors.

Lemma: Let C be a set belonging to both solutions U' and V'. Then the ratio

$$\frac{w(U')}{w(V')} \leq \frac{w(U') - w(C)}{w(V') - w(C)},$$

if $w(U') \geq w(V')$.

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- Call $N_{C_j}(V')$ the solution obtained by replacing a set C'_j by C_j and replacing the elements of C_j if present in V' by the elements of C'_j
- $N_{C_j}(V')$ is feasible and has the same weight as V'; both solutions correspond to V.

V^* ***found!***

- The procedure is repeated. It terminates after at most $\min\{|V'|, |U'|\}$ steps with no element having more than k neighbors, in a solution V^*.

V^* found!

- The procedure is repeated. It terminates after at most $\min\{|V'|, |U'|\}$ steps with no element having more than k neighbors, in a solution V^*.
- Since V is assumed to be t-optimal, V^* is t-optimal. Using the Lemma, we can remove equal sets.

Results

Corollary: The ratio of a weighted t-optimal solution V and an optimal solution U of the generalized set multipacking problem is at most

$$\frac{w(U)}{w(V)} \leq k - 1 + \frac{1}{t}$$

Corollary: In the unweighted case, the ratio of a t-optimal solution V and an optimal solution U are

$$\frac{|U|}{|V|} \leq \begin{cases} \frac{k(k-1)^s - k}{2(k-1)^s - k} & : t + 1 = 2s - 1 \\ \frac{k(k-1)^s - 2}{2(k-1)^s - 2} & : t + 1 = 2s \end{cases}$$

The problem with covering

It is open whether the performance ratio for the covering problem is equal to that of the simple case.

- The transformed problem is a mixed problem
- In many respects, this problem is more difficult to handle
- Good algorithms are known for special cases

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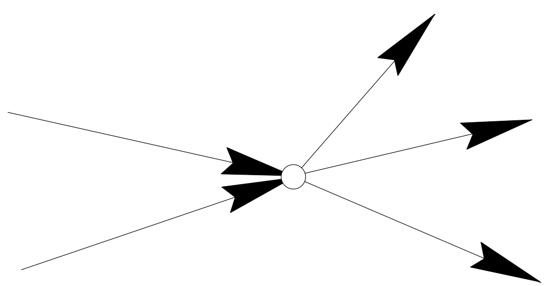
Basic property
Let $G(D)$ be the underlying undirected graph of D.
Then

$$\chi(G(D)) \geq \chi(D).$$

Proposition: All simple planar digraphs are 3-colorable.



There are simple planar graphs with arboricity 3.

Planar Digraphs

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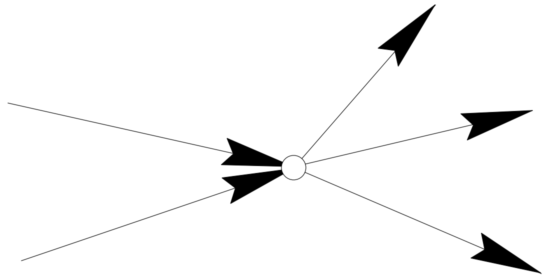

There are simple planar graphs with arboricity 3.

Conjecture (Škrekovski): All simple planar digraphs are 2-colorable.

Properties of a Minimal Counterexample

G planar graph, D orientation of G, $\chi(D) = 3$, minimal number of vertices, maximal number of edges.

D can be chosen to have the following properties:

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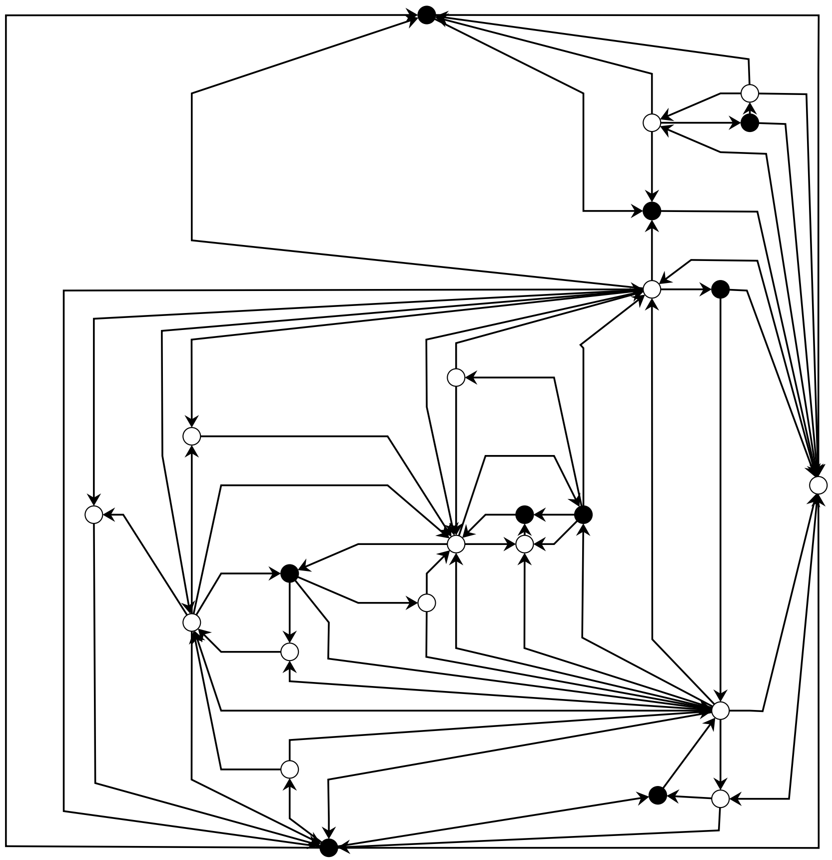
- Dual of G has no hamiltonian cycle
- G is not perfect.

These properties are not sufficient!



The gadget

- An equivalence relation: $u \sim v$ iff for every coloring $c : V(D) \to [n]$ of D the value $|c(u) - c(v)| =: k_{uv}$ is independent of c.

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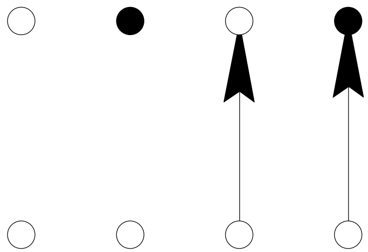
- Digraph D is a gadget for the surface Σ, if there exists an embedding of D in Σ such that one of D's $\sim$-equivalence classes contains two vertices of the same face.
- Four equivalent types of a planar gadget:



Existence of a gadget

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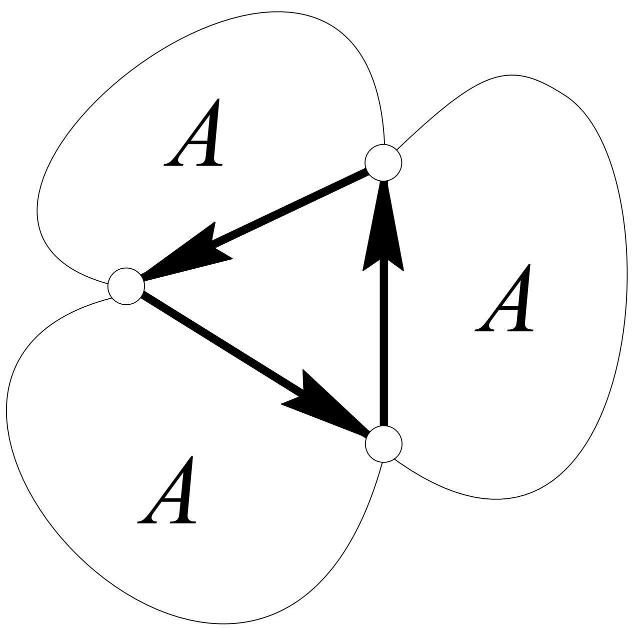
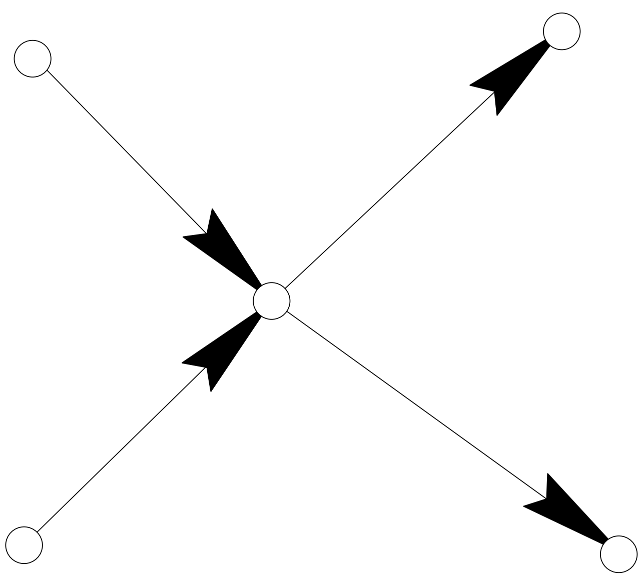
Proof:

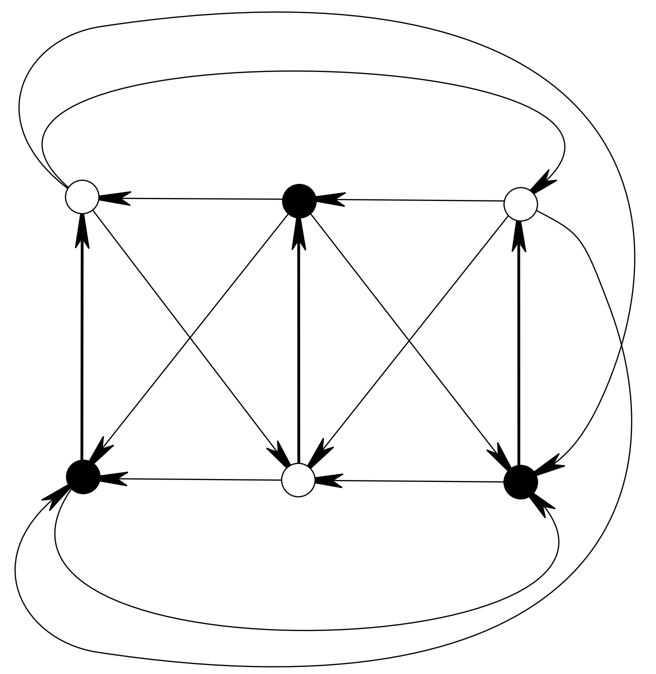




1. equivalent conjecture

Conjecture: A simple planar gadget does not exist.

Nonplanar gadget:



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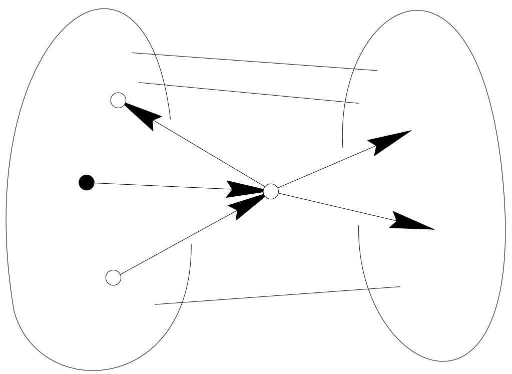

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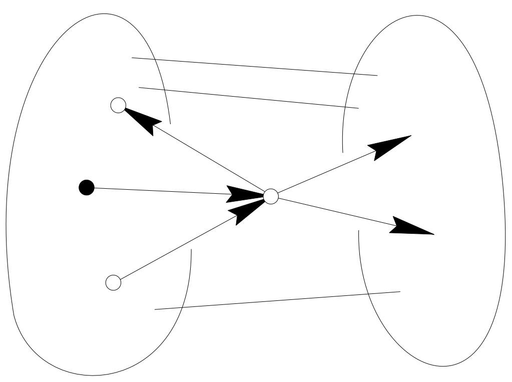

Proposition: A digraph is k-colorable if and only if it is greedily k-colorable.

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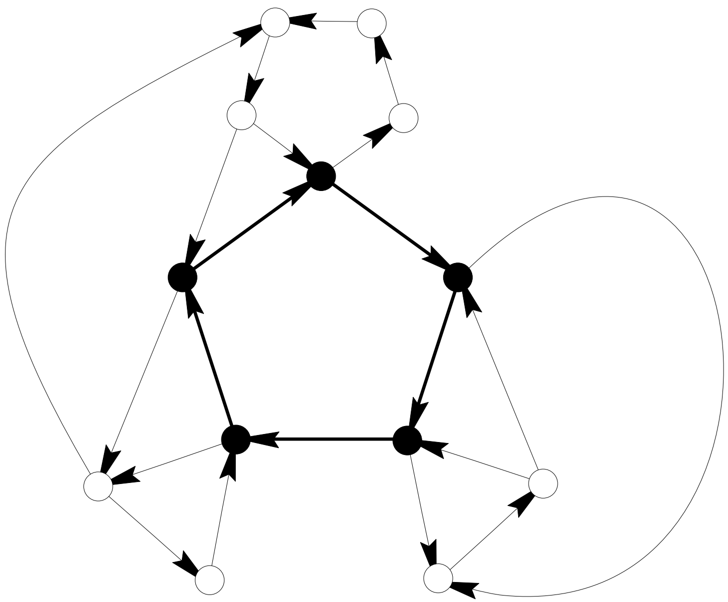
An obstruction



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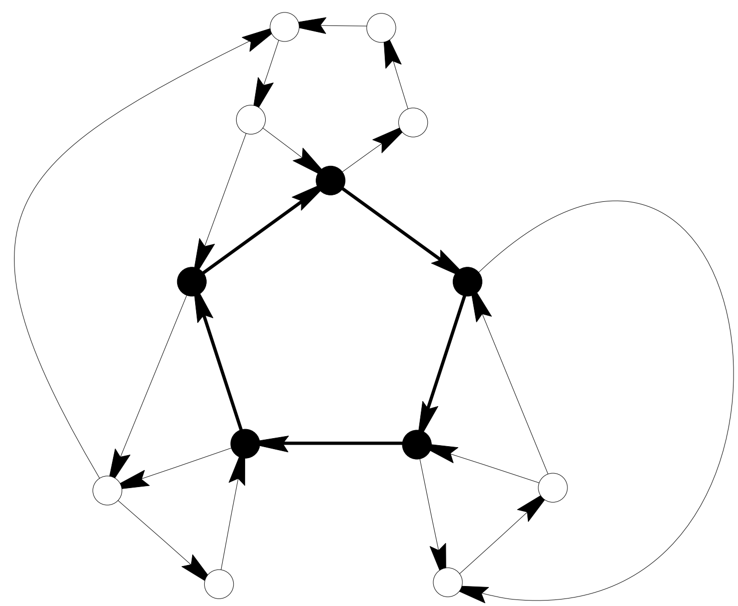

Proposition: Simple D can be colored greedily with ≥ 3 colors if and only if there exists an obstruction O in D and $f(v) < f(w)$ for all outer vertices v and inner vertices w of O.

Outlook

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- Generalized set multicover problem???
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