Prime Factor Theorem for a Generalized Direct Product

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Definition *An* $\mathfrak{N}$-system *consists of a non-empty finite set* X *and a system* $\mathfrak{N}$ *of collections of subsets of* X *that associates to each* $x \in X$ *a collection*

$$\mathfrak{N}(x) = \{N^1(x), N^2(x), \ldots, N^{d(x)}\}$$

of subsets of X *with the following properties:*

(N0) $\mathfrak{N}(x) \neq \emptyset$*.*

(N1) $N^i(x) \subseteq N^j(x)$ *implies* $i = j$*.*

(N2) $x \in N^i(x)$ *for* $1 \leq i \leq d(x)$*.*

Remark. If $d(x) = 1$ for all $x \in X$ then $(X, \mathfrak{N})$ describes a directed (or undirected) graph with loops.

Then $\mathfrak{N} = N^1(x)$ for every x, and $N^1(x)$ is the neighborhood of x.

Definition *Let $(X_1, \mathfrak{N}_1)$ and $(X_2, \mathfrak{N}_2)$ be two $\mathfrak{N}$-systems. We define their direct product $(X_1, \mathfrak{N}_1) \times (X_2, \mathfrak{N}_2)$ as follows:*

(1) The vertex set is $X = X_1 \times X_2$.

(2) The $\mathfrak{N}(x_1, x_2)$ are the sets $\{N' \times N'' | N' \in \mathfrak{N}_1(x_1),\ N'' \in \mathfrak{N}_2(x_2)\}$

Lemma *The direct product of two* $\mathfrak{N}$-systems *is an* $\mathfrak{N}$-system*.*

If $\mathfrak{N}_1(x_1) = \{N(x_1)\}$ and $\mathfrak{N}_2(x_2) = \{N(x_2)\}$ then

$$\mathfrak{N}(x_1, x_2) = \{N(x_1) \times N(x_2)\}.$$

If $(X_1, \mathfrak{N}_1)$ and $(X_2, \mathfrak{N}_2)$ both represent graphs, then their product again represents a graph.

In this case the product is the direct product of graphs.

Let $(X, \mathfrak{N}) = (X_1, \mathfrak{N}_1) \times (X_2, \mathfrak{N}_2)$

We say the decomposition is nontrivial, if both X_1 and X_2 have at least two elements.

Then X_1 and X_2 are called proper factors.

An $\mathfrak{N}$-system is prime if it has no nontrivial decomposition.

The proper factors of $(X, \mathfrak{N})$ are smaller than X.

Thus every finite $\mathfrak{N}$-system has a prime factorization.

Is the prime factorization unique?

What can that mean?

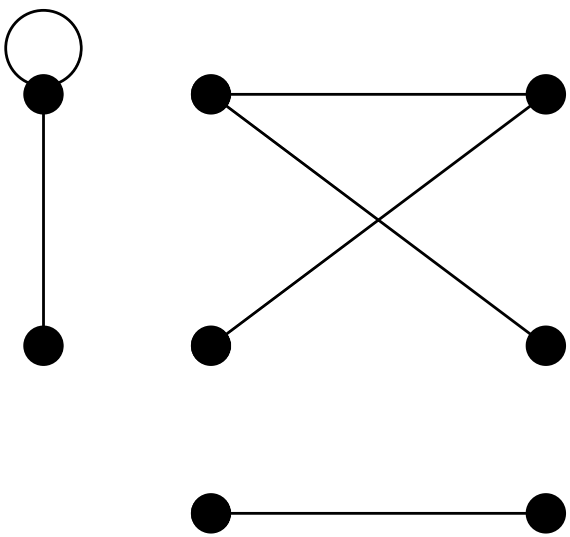
Unique up to the order and isomorphisms of the factors.

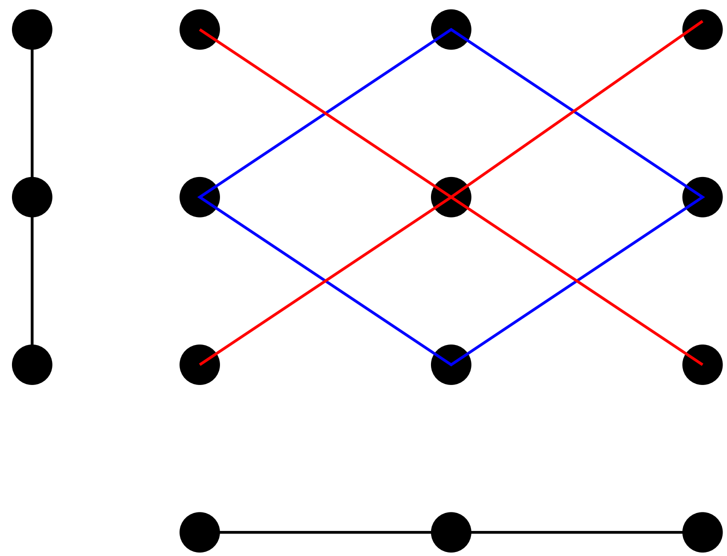
Does that make sense?

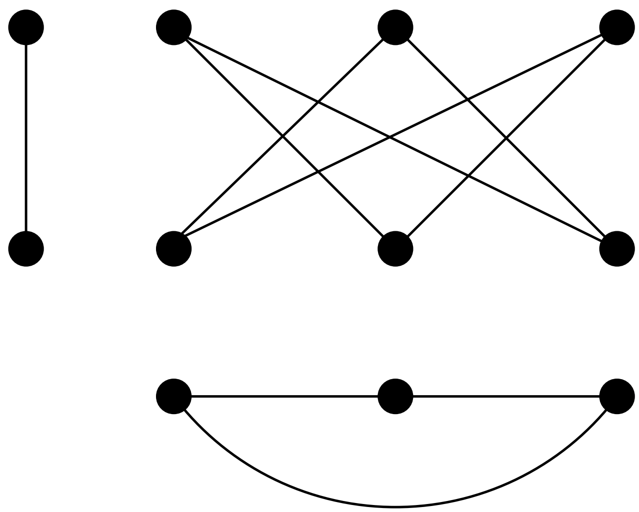
Yes, if the product is commutative and associative.

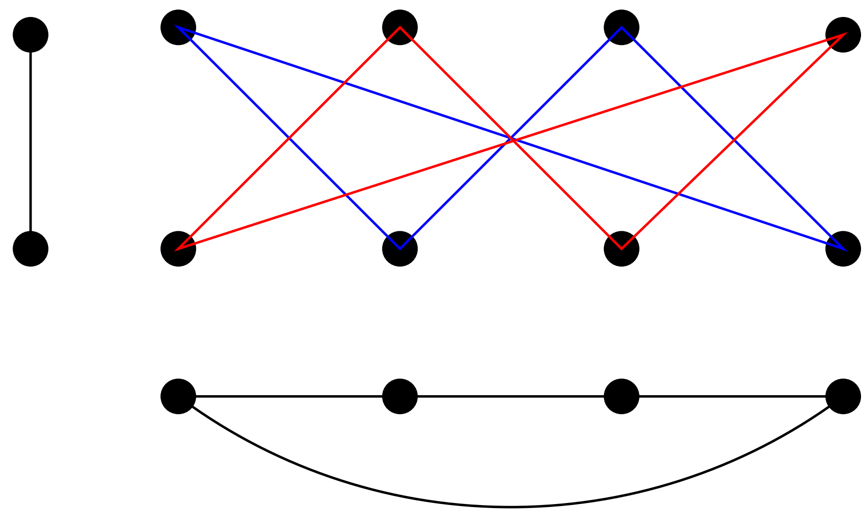
Is it true for the direct product of graphs?

Examples of direct products of undirected graphs









Let $|G|, |H| \geq 2$. Then $G \times H$ is connected if and only if both G and H are connected and at least one of them is non-bipartite.

If both G and H are connected and bipartite, then $G \times H$ has exactly two connected components. (Weichsel)

The existence of unique prime factor decompositions depends very much on the class of graphs considered.

1. It is not unique for disconnected graphs.

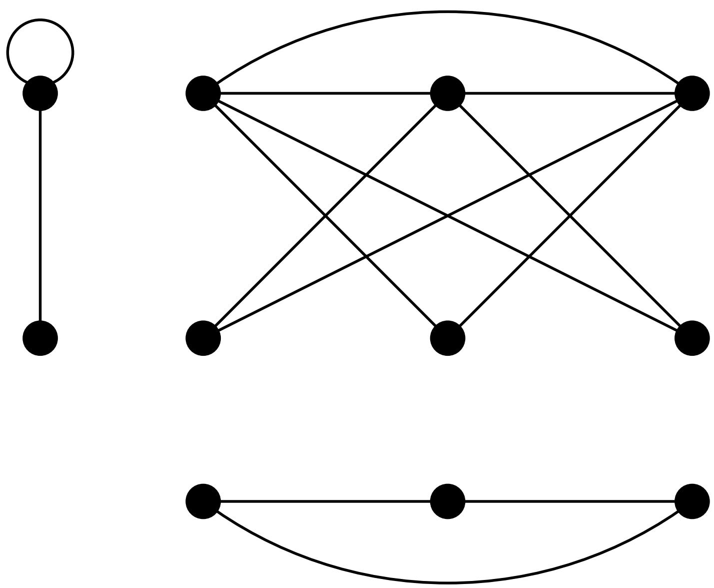
2. It is not unique for undirected graphs without loops. Example:

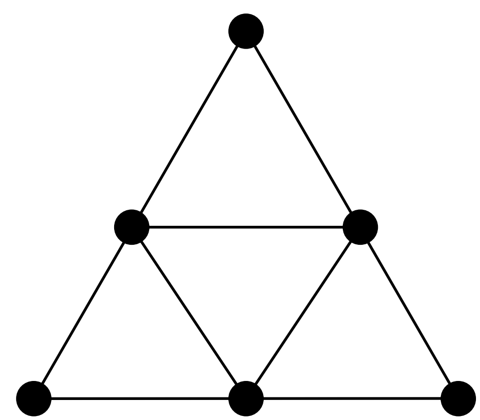
Let L be a triangle in that the midpoints of the sides are connected by edges and P_4 a path on 4 vertices. Then

$$K_3 \times P_4 = L \times K_2$$

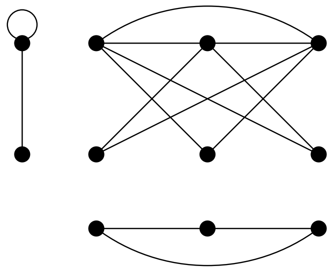
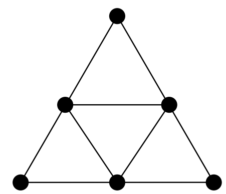
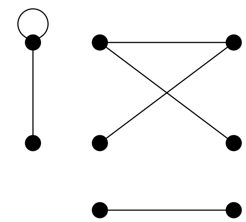
are two distinct prime factorizations of one and the same graph in the class of undirected graphs without loops.

This is easily understood if we consider graphs with loops too.





$K_3 \times K \qquad = \qquad L$

Note that $L = K_3 \times K$, and $P_4 = K \times K_2$, where K is an edge with a loop added to one vertex. Thus

$$K_3 \times K \times K_2$$

is a refinement of $L \times K_2$ and $K_3 \times P_4$.

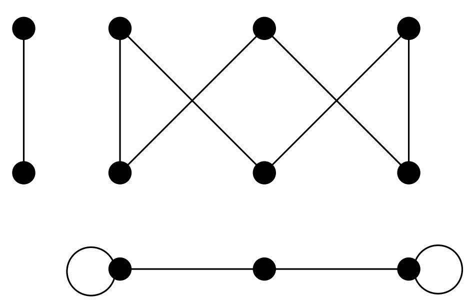
We speak of a common refinement of the factorizations.

3. It is not unique for bipartite connected graphs with loops.

Let P be a path of length 2 with loops at the endpoints. Then

$$P \times K_2 = K_3 \times K_2$$

are two distinct prime factorizations of the cycle C_6.



Ralph McKenzie* showed that the prime factorization of (finite) connected directed graphs with loops is unique.

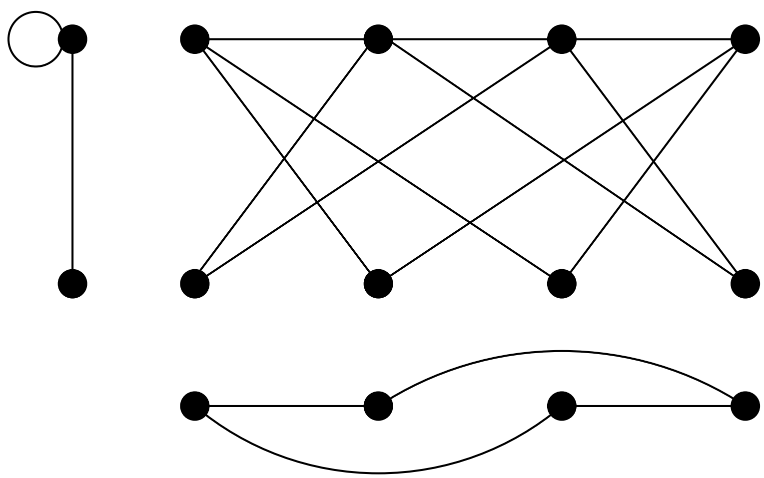
He showed more: Any two factorizations of a (finite or infinite) connected (directed) graph with loops have a common refinement.

For finite graphs this implies unique prime factorization.

Does that mean unique coordinates for every vertex in the product? Or does it just mean that the factors are uniquely determined?

*Cardinal multiplication of structures with a reflexive relation, Fund. Math. 70 (1971) 59-101

Example for nonunique coordinatization:



This happens when two vertices have the same neighborhood. We say two vertices are in the relation R if they have the same neighborhood.

In a graph without loops no two adjacent vertices x, y are in the relation R; the neighborhood of x contains y but not x and the neighborhood of y contains x but not y.

Thus no two vertices in K_n have the same neighborhood! But any two in K_n^s, that is, in K_n with loops added to every vertex.

It is clear what we mean by G/R and that there is a natural homomorphism from G onto G/R. We say a graph is thin if R is trivial.

Note that $(G \times H)/R = G/R \times H/R$.

Definition *Let* $\Gamma(X, \mathfrak{N})$ *be the directed graph (with loops) with vertex set* X *and edge set*

$$E = \left\{ (x, y) \middle| y \in \bigcup_{i=1}^{d(x)} N^i(x) \right\} \tag{1}$$

We say that $(X, \mathfrak{N})$ *is connected if* $\Gamma(X, \mathfrak{N})$ *is connected.*

Lemma $\Gamma((X_1, \mathfrak{N}_1) \times (X_1, \mathfrak{N}_1)) = \Gamma(X_1, \mathfrak{N}_1) \times \Gamma(X_2, \mathfrak{N}_2)$

Every factorization of $(X, \mathfrak{N})$ yields on of $\Gamma(X, \mathfrak{N})$.

$\Gamma(X, \mathfrak{N})$ may have more factorizations than $(X, \mathfrak{N})$.

How do we find the factorizations of $(X, \mathfrak{N})$?

We find the factorizations of $\Gamma(X, \mathfrak{N})$ and throw away those that do not factorize $(X, \mathfrak{N})$.

If $\Gamma(X, \mathfrak{N})$ is thin this is easy because of the unique coordinatization.

Do we get non-unique prime factorization then as in the example with directed graphs without loops?

One can show that for thin $\Gamma(X, \mathfrak{N})$ any two factorizations of $(X, \mathfrak{N})$ have a common refinement.

Theorem *Suppose the $\mathfrak{N}$-system $(X, \mathfrak{N})$ system has a connected digraph $\Gamma(X, \mathfrak{N})$. If $\Gamma(X, \mathfrak{N})$ is thin, then $(X, \mathfrak{N})$ has a unique prime factor decomposition.*

Actually, it suffices to assume that $(X, \mathfrak{N})$ is thin, that is, that no two elements x and y have the same neighborhoodsystems $\mathfrak{N}(x) = \mathfrak{N}(y)$.

Does not look like a severe restriction.

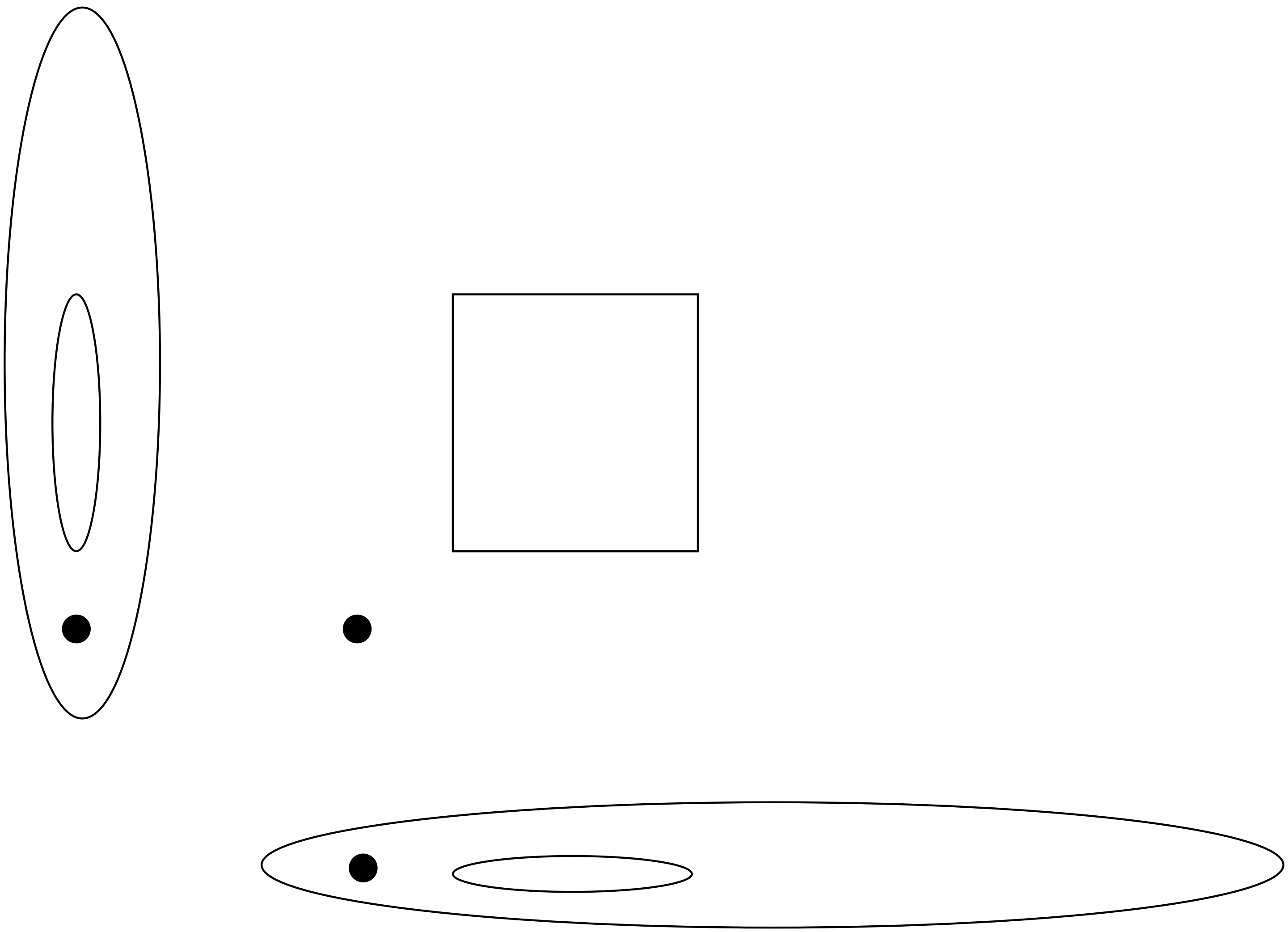
Proof much more complicated.

To treat this case we use ideas that helped to find an alternate proof (in the finite case) for McKenzie's result of unique prime factorization of connected, nonbipartite graphs with loops*.

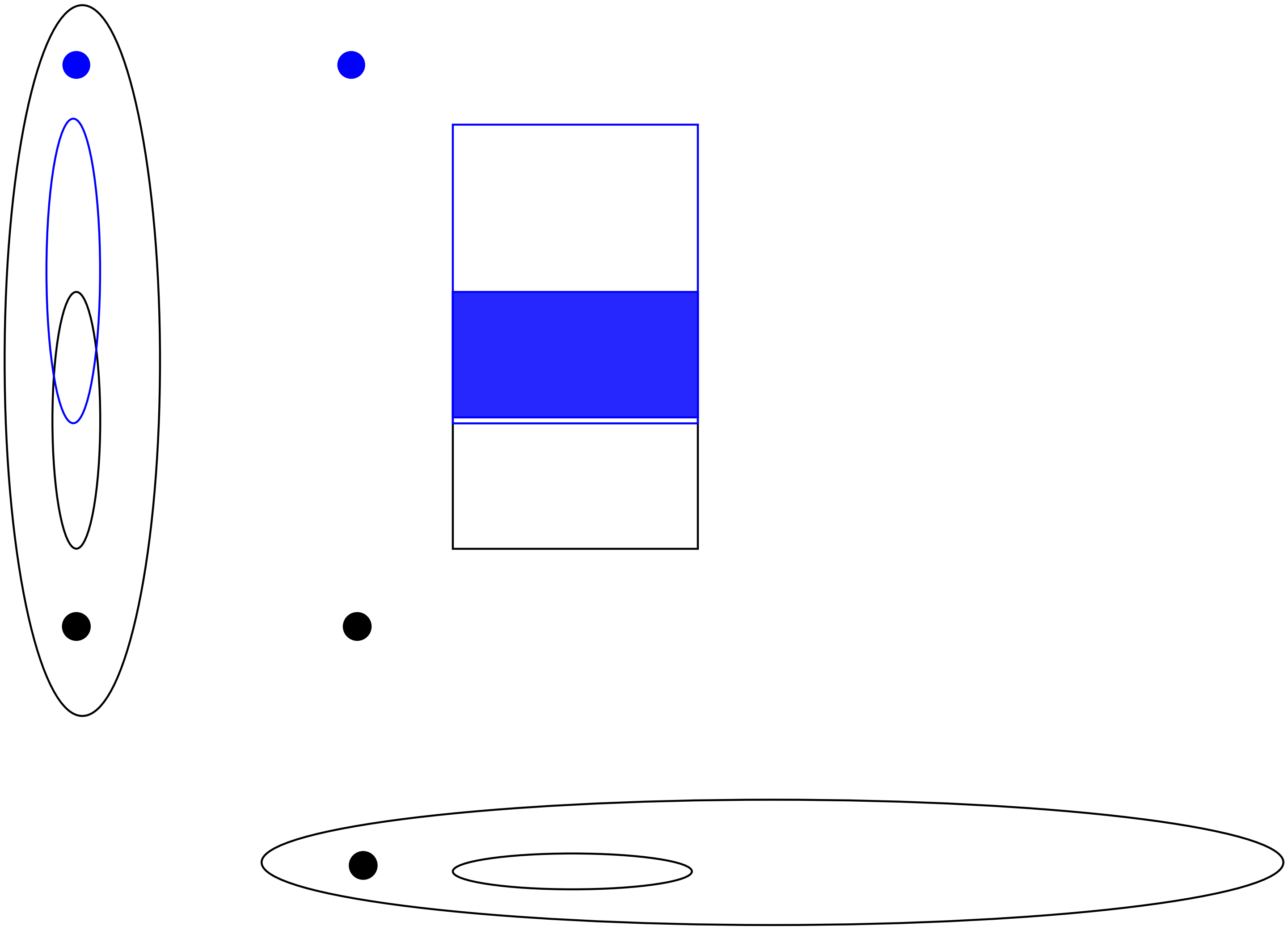
We try to find pairs of vertices that have the same projection into one of the factors in any factorization. We call such pairs Cartesian pairs of vertices.

*This method extends a method of Feigenbaum and Schäffer, Finding the prime factors of strong direct products in polynomial time, Discrete Math. 109 (1992), 77 - 102

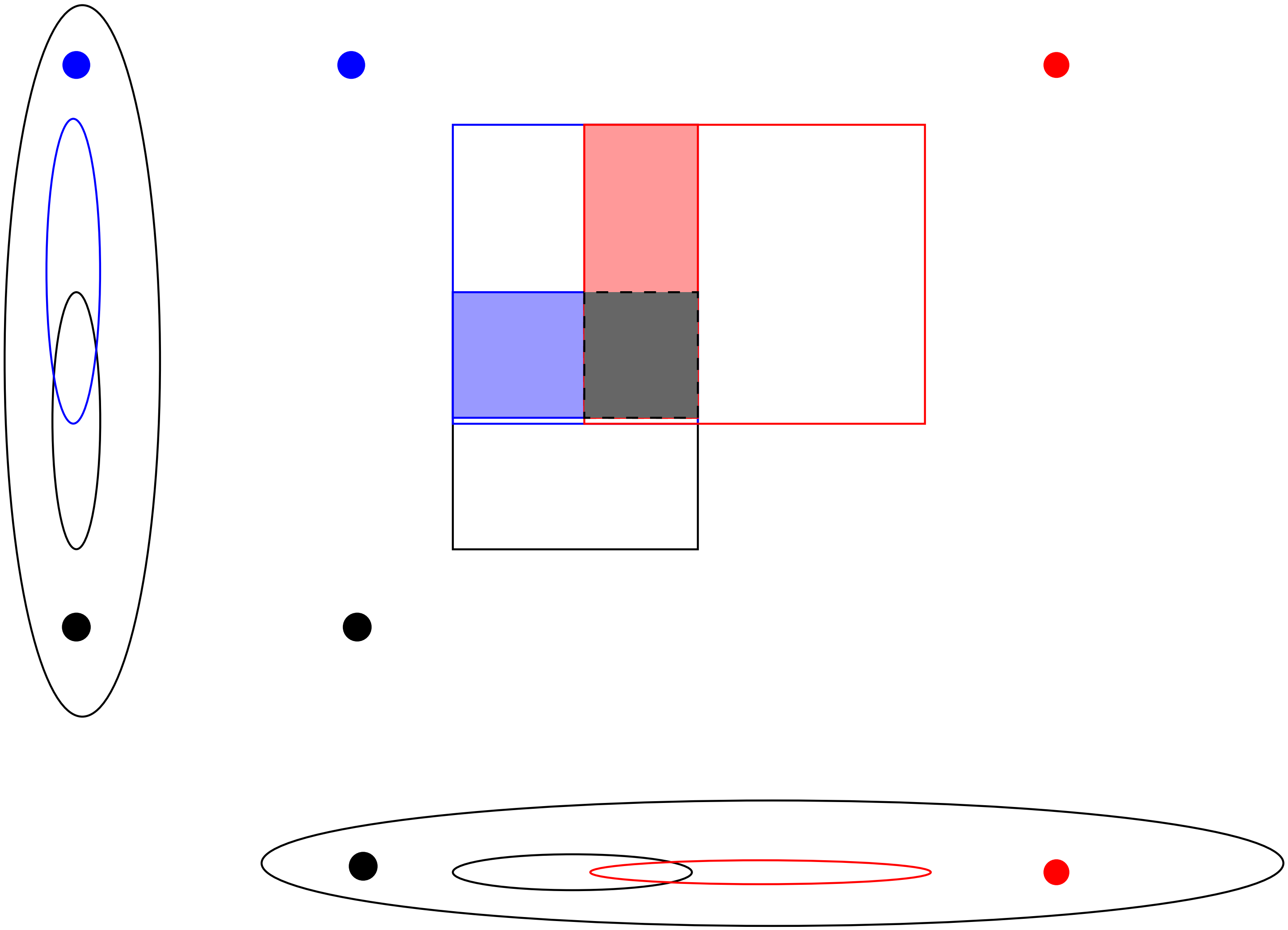
Neighborhoods of vertices in the product.



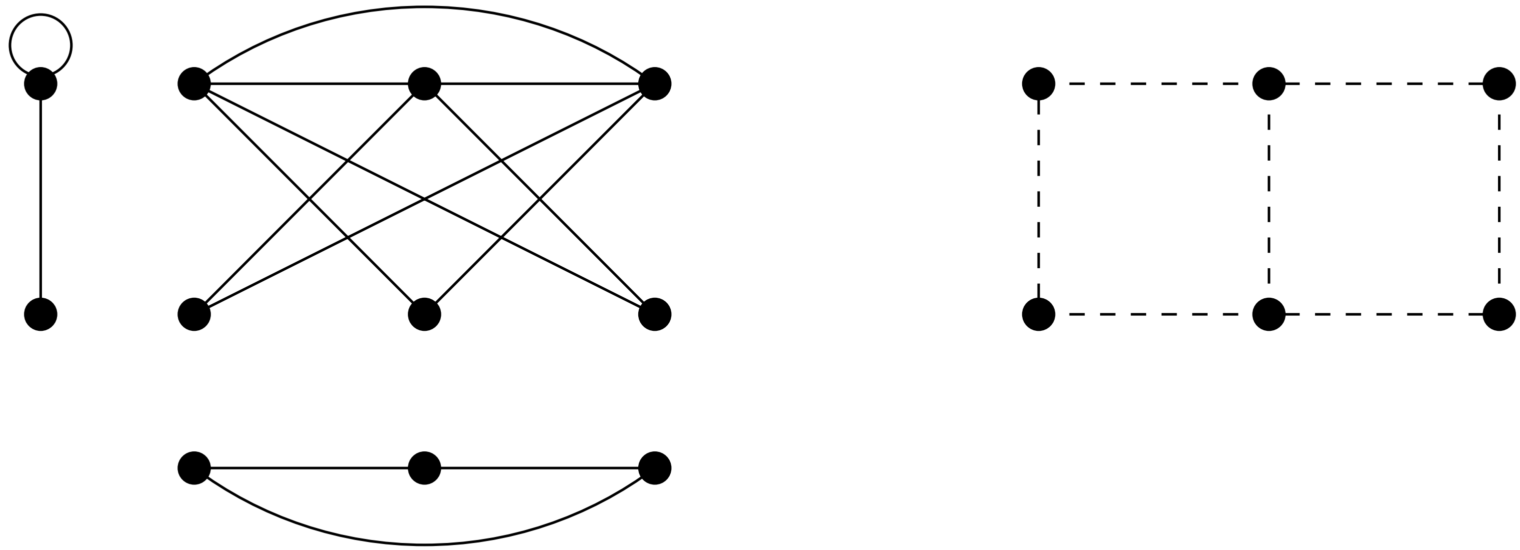
What characterizes vertices with the same projections?



Maximality conditions!



An example for the construction of Cartesian pairs of vertices:



We call this graph the Cartesian Skeleton or Cartesian core.

It is connected for non-bipartite graphs, disconnected for bipartite ones.

The Cartesian core is a Cartesian product. Any factorization of the original (thin) graph with respect to the direct product is a factorization of the Cartesian core*.

It might have more factors than the original direct product, but one can still use it to factor non-bipartite graphs, show uniqueness of the prime factor decomposition with respect to the direct product, also for graphs that are not thin, and to find a polynomial algorithm for the decomposition.

*Requires proof, nontrivial

In our case we take any $x \in X$ and use this method on every $N^i(x)$ to find candidates for Cartesian pairs. Those pairs that do not belong to all $N^i(x), i = 1, \ldots, d(x)$, for a given x are discarded.

This way one obtains a Cartesian skeleton for thin neighborhood systems in that any two vertices have different neighborhood systems.

The resulting Cartesian skeleton is then factored as a Cartesian product. This is well understood and can be done in polynomial, even linear time.

Then we proceed as before when we considered $\Gamma(X, \mathfrak{N})$. The advantage here is that we can do this for thin $(X, \mathfrak{N})$ where vertices are distinguished by their neighborhood systems and not only for those with thin $\Gamma(X, \mathfrak{N})$.

Similar arguments are needed to treat the case when two vertices x, y with the same neighborhood systems can be distinguished by an $N^i(z)$ that contains one of the vertices x and y, but not both.

Thus, we hope these ideas suffice to show that connected, thin neighborhood systems have unique prime factor decomposition.

Since $(X, \mathfrak{N})/R$ is connected and thin if is connected $\mathfrak{N}$-system – whereas $(X, \mathfrak{N})/R(\Gamma(X, \mathfrak{N}))$ may not be defined as an $\mathfrak{N}$-system at all – this would imply the following result.

Connected $\mathfrak{N}$-systems have unique prime factorizations with respect to the direct product of $\mathfrak{N}$-systems.