Products of edge-transitive graphs

with an interlude on the internet graph
and an appendix on the cardinal product

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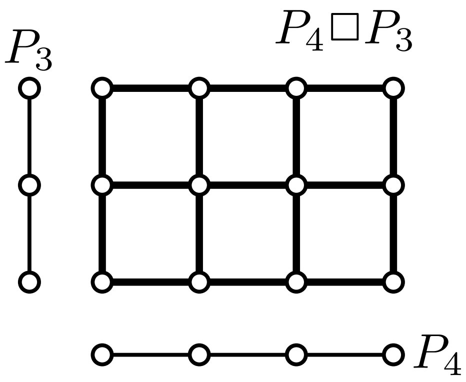
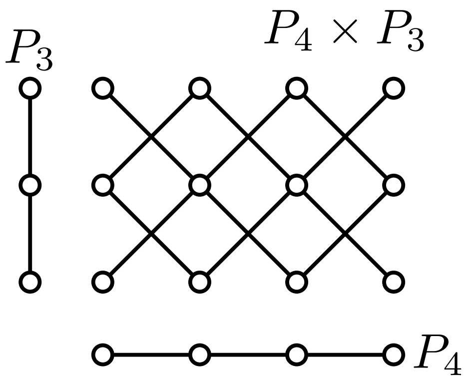
A graph $G = (V(G), E(G))$ is edge-transitive if the automorphism group $\mathrm{Aut}(G)$ acts transitively on $E(G)$.

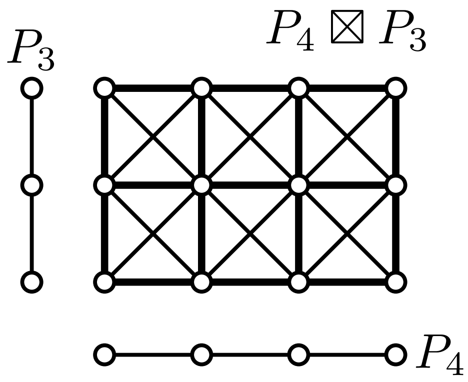
We will study edge-transitivity for the Cartesian, the direct, the strong, and the lexicographic product.



Cartesian product

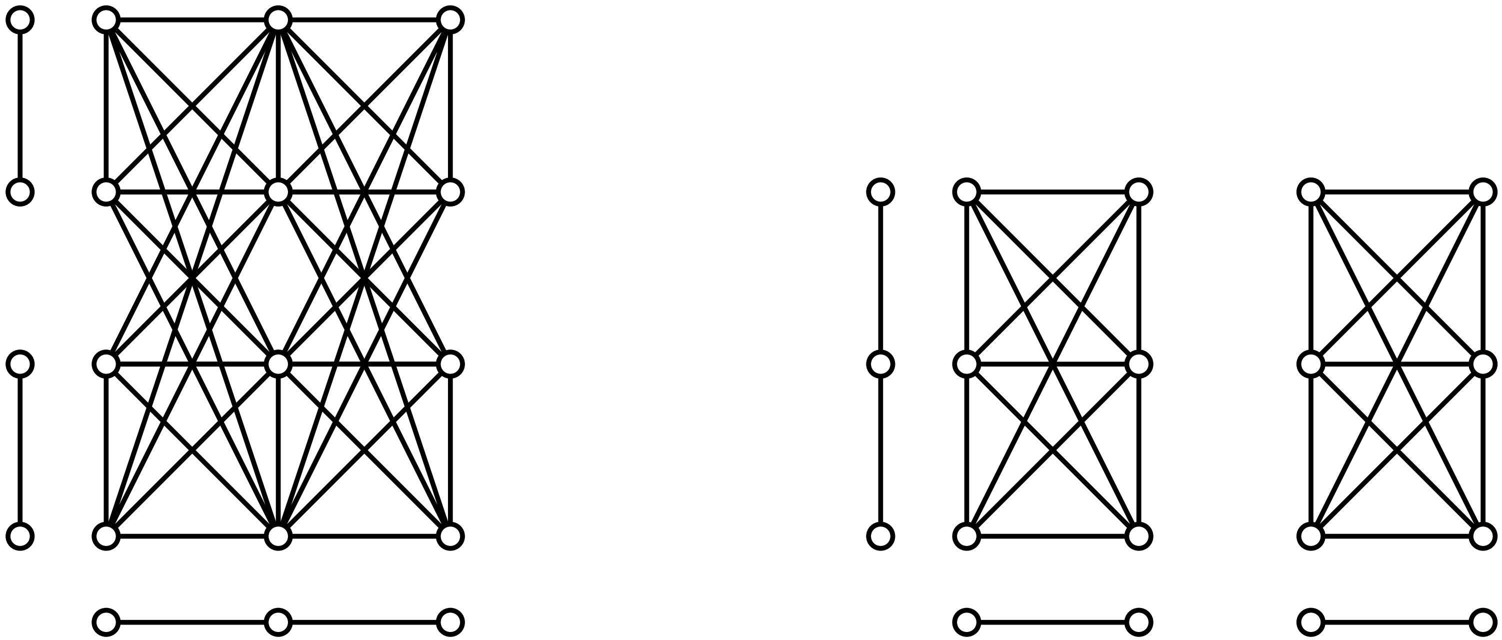


Direct product



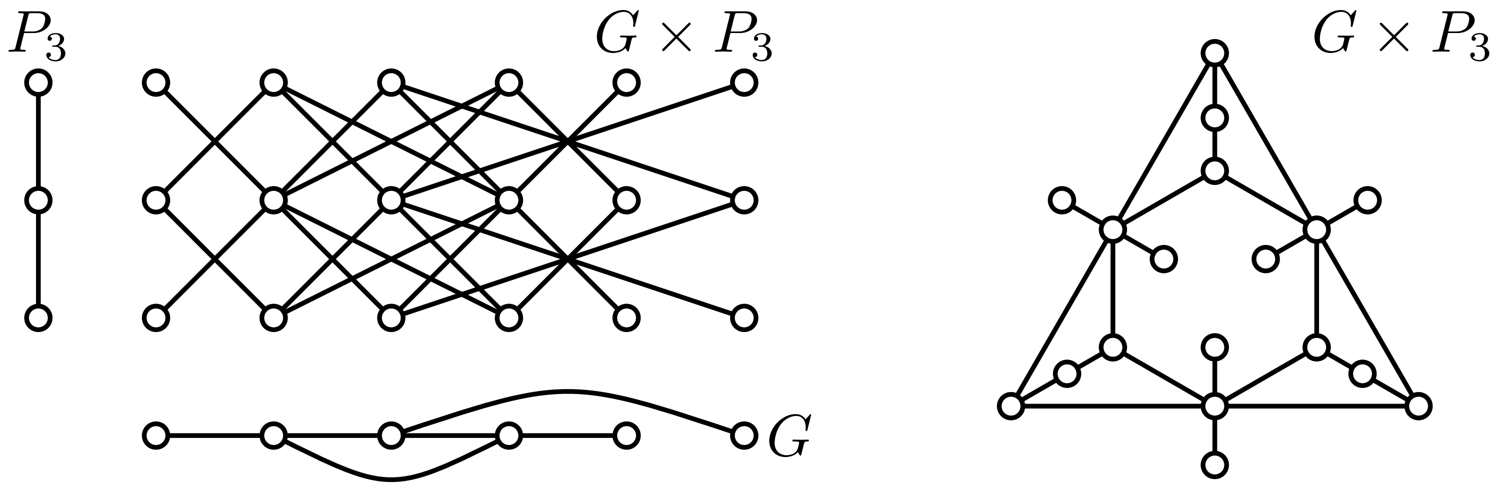
Strong product

Here is a picture for the lexicographic product:



This product is not commutative.

The direct product does not always look as one might expect a product to look like:



Interlude about the direct product and the internet graph

The direct product can be used to model large networks, networks like the internet graph.

Networks are studied for performance, reliability, stability, robustness, growth, self-organization, virus propagation and the like.

They have the following static properties: the power law degree distribution, the small-world property, self-similarity, and more.

They also have temporal properties. As a network grows it becomes denser and its diameter decreases. (Leskovec, Kleinberg, Faloutsos 2005):

$$|E(t)| \propto |V(t)|^a, \text{ where } 1 \leq a \leq 2 \,.$$

The adjacency matrix of the direct product of two graphs is the Kronecker product of the adjacency matrices of the factors.

The direct product is thus also called the Kronecker product.

Given G, then $A(G^{\times,k})$ is the kth power of $A(G)$ with respect to the Kronecker product.

They obey the static graph properties.

The small world property follows, because the diameter of the direct product is close to the maximum of the diameters of the factors. ("Six degrees of separation".)

The power law degree distribution says that the probability that a vertex has degree k is

$$P(k) \simeq ck^{-\gamma}.$$

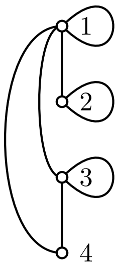
In our cases $2 < \gamma < 3$.

This causes the variance of the degrees to be large (about the size of the expected value)

This makes the graph in a sense scale-free.

Which is related to the fractal structure.

The fractal property is visualized by the next slide.



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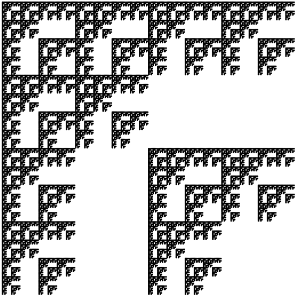


How do we get the power law?

The degree of a vertex $v \in V(G^{\times,k})$ is $d(p_1(v)) \cdots d(p_k(v))$.

If the degrees in G are $d_1, d_2, \ldots, d_n$,

then the degrees in $G^{\times,k}$ are $d_1^{i_1} d_2^{i_2} \cdots d_n^{i_n}$, where $\sum_{j=1}^{n} i_j = k$.

Hence the degree probabilities are proportional to $\binom{k}{i_1, i_2, \ldots, i_n}$.

Thus we have a multinomial degree distribution.

Now, a careful choice of the degrees of the vertices of G causes it to behave like a power law degree distribution.

Stochastic Kronecker graphs

Start with a square probability matrix $\mathcal{P}_1$ whose i, j-entry represents the probability that an edge joins vertex i to vertex j,

Compute the Kronecker k^{th} power $\mathcal{P}_k$.

Then (an instance of) a *stochastic Kronecker graph* is obtained from $\mathcal{P}_k$ by including an edge between two vertices with probability as given in $\mathcal{P}_k$.

But how do we do that? We have $O(N^2)$ pairs. We wish to generate in $O(E)$ time, that is in about $O(N)$ time.

But, the properties of the direct product of graphs (respectively the Kronecker product of matrices) are well understood.

Leskovets shows to fit them, by appropriate choice of $\mathcal{P}_1$, to real-world networks.

To be more precise, $\mathcal{P}_1$ can be chosen to fit such parameters as diameter, or the constants c and γ in the degree distribution power law.

Leskovec and Faloutsos, 2009, do this in linear time.

For example they showed that the matrix

$$\begin{pmatrix} .98 & .58 \\ .58 & .06 \end{pmatrix}$$

yields a Kronecker graph that fits the Internet (at the autonomous system level) fairly well.

If one can generate random networks with prescribed properties in linear time, then one can play with them.

For example, one can check virus propagation models, or the stability of networks with respect to attacks.

It turns out that the internet is very robust with respect to random damage, but not with respect to targeted attacks.

End of interlude

Vertex- versus edge-transitivity

All our products are vertex-transitive if and only if both factors are vertex-transitive.

But things are quite different with respect to edge-transitivity as Hammack, Imrich, Iranmanesh,Klavžar, Soltani have shown.

Not all products of edge-transitive graphs are edge-transitive.

Edge-transitive products may even have factors that are not edge transitive, or, they may be vertex- and edge-transitive, but the factors are only edge-transitive.

THANK YOU FOR YOUR ATTENTION

But, if you really wish to learn about edge-transitive products, then just continue reading.

The lexicographic product

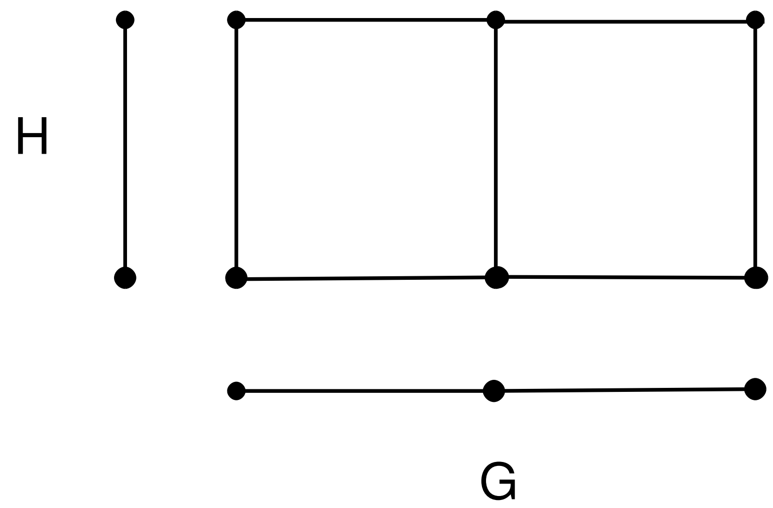
If G is connected, but not complete, then $G \circ H$ is edge-transitive iff G is edge-transitive and H is edgeless.

If G be nontrivial, complete, then $G \circ H$ is edge-transitive iff H is the product of a complete graph by an edgeless graph.

The Cartesian product

Recall that a Cartesian product is vertex-transitive iff all factors are vertex transitive.

But, the Cartesian product of two edge-transitive graphs need not be edge-transitive:



An edge-transitive graph need not be vertex transitive, see P_3.

If G is edge- and vertex transitive it is possible that G is not arc-transitive, that is, if ab is an edge, then there may be no automorphism that swaps a and b.

Such graphs are called half-transitive. An example is the Gray graph on 51 vertices.

Theorem (IIKS)* *Let G be connected, not $\Box$-prime.*

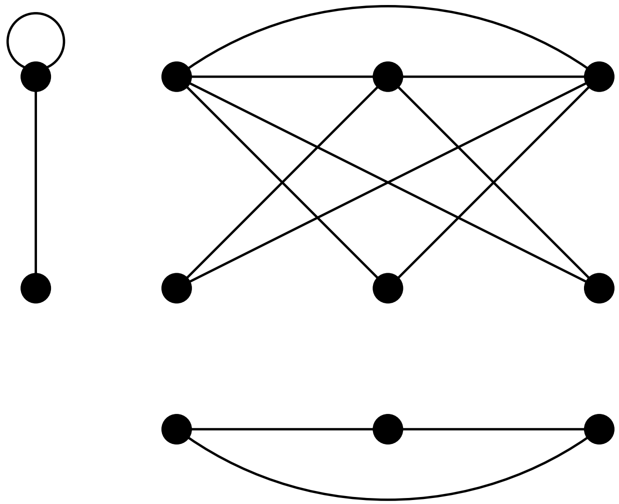
Then $G \cong H^k$, where H is connected, edge- and vertex transitive.

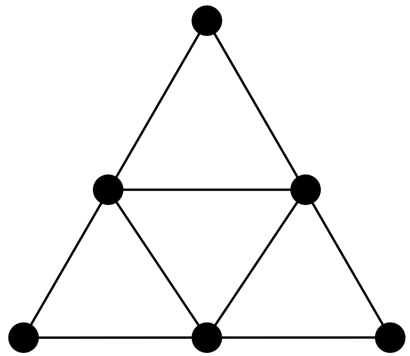
G is half-transitive iff H is half-transitive.

*Imrich, Iranmanesh,Klavžar,Soltani

The direct product*

For the direct product we also admit loops.





$$K_3 \times K \qquad = \qquad L$$

*All further results are by Hammack, Imrich, Klavžar.

Theorem *Suppose $A \times B$ is connected and non-bipartite. Then it is edge-transitive if and only if either*

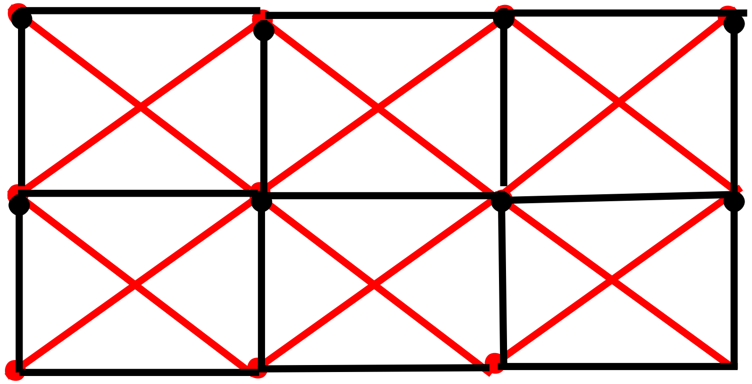
(i) both factors are edge-transitive and at least one is arc-transitive, or

(ii) one factor is edge-transitive (and non-trivial) and the other is a K_n with loops at every vertex.

Proposition *Suppose A has an odd cycle and B is bipartite. If both $A \times K_2$ and B are edge-transitive and one is arc-transitive, then $A \times B$ is edge-transitive.*

We conjecture that the converse also holds.

The strong product



Theorem *The strong product $G = A \boxtimes B$ of two connected, non-trivial graphs is edge-transitive if and only if both factors are complete.*

The weak Cartesian product

Given an index set I and graphs G_ι, $\iota \in I$, we define

$$G = \mathop{\Box}_{\iota \in I} G_\iota$$

on the vertex set consisting of all functions

$$x : \iota \to x_\iota \quad \text{with} \quad x_\iota \in V(G_\iota).$$

Two vertices x and y are adjacent if there exists a $\kappa \in I$ such that

$$x_\kappa y_\kappa \in E(G_\kappa) \quad \text{and} \quad x_\iota = y_\iota \quad \text{for} \quad \iota \in I \setminus \{\kappa\}.$$

For finite I this is the usual Cartesian product.

The Cartesian product $G = \Box_{\iota \in I} G_\iota$ of infinitely many non-trivial connected graphs is disconnected.

The connected components are called *weak Cartesian products*.

We denote the connected component containing $a \in V(G)$ by

$$\mathop{\Box}\limits_{\iota \in I}^{a} G_\iota.$$

$\Box^a_{\iota \in I} G_\iota = \Box^b_{\iota \in I} G_\iota$ if and only if a and b differ in only finitely many coordinates.

Notice that no connected graph is the Cartesian product of infinitely many prime graphs.

But, every connected graph is a weak Cartesian product of prime graphs, and the prime factors are uniquely determined. We say:

Every connected graph has a unique prime factor decomposition with respect to the weak Cartesian product*.

*I 1971, D.J. Miller 1970.

Differences to the finite case.

1. Products of infinitely many connected, non-trivial factors are disconnected.

2. Weak Cartesian products of connected, asymmetric graphs can be vertex-transitive*.

3. A connected, edge-transitive graph G that is not prime with respect to the Cartesian product is the Cartesian or weak Cartesian power of a connected, edge-transitive graph H.

As in the finite case G is vertex-transitive, but H need not be vertex-transitive†.

*I 1987.

†Hammack, I and Klavžar 20xx.

Case 3 can be more precisely described as follows:

Theorem *Let G be a connected, edge-transitive graph that is not prime with respect to the Cartesian product.*

Then G is the Cartesian or weak Cartesian power of a connected, edge-transitive graph H.

For vertex-transitive H the structure of G is described (on the next slide) by Lemma A, otherwise by by Lemma B.

In both cases G is vertex-transitive.

Lemma A *Let H be connected, edge- and vertex-transitive and $G = \Box_{\iota \in I}^{a} H_\iota$, where $H_\iota \cong H$ and $2 \leq |I|$. Then:*

(i) *G is also edge- and vertex-transitive.*
(ii) *G is half-transitive if and only if H is half-transitive.*

Lemma B *Let H be connected, edge-transitive but not vertex-transitive, bipartioned by its two vertex orbits V_1 and V_2.*

Let $G = \Box_{\iota \in I}^{a} H_\iota$, where $H_\iota \cong H$, $2 \leq |I|$. Suppose

(a) *infinitely many of the a_ι are in the vertex-orbit of G_ι corresponding to V_1, and*
(b) *infinitely many in the vertex-orbit of G_ι corresponding to V_2.*

Then G is edge-transitive (but only half-transitive) and vertex-transitive.

This covers edge-transitive products

But, there is more to say about the direct product.

If you are interested, continue reading.

More about the the categorical alias direct product

There is a seminal paper of R. McKenzie* from 1971 about the categorical product of (binary) relational structures.

As binary relations can be considered as directed graphs, we describe his results in the language of graphs.

Notice that the Cartesian product is not a product in the category sense!

*Ralph McKenzie, Cardinal multiplication of structures with a reflexive relation, Fundamenta Mathematicae 70 (1971), 59-101.

So let us have a look at the only real product, the categorical product of binary relations.

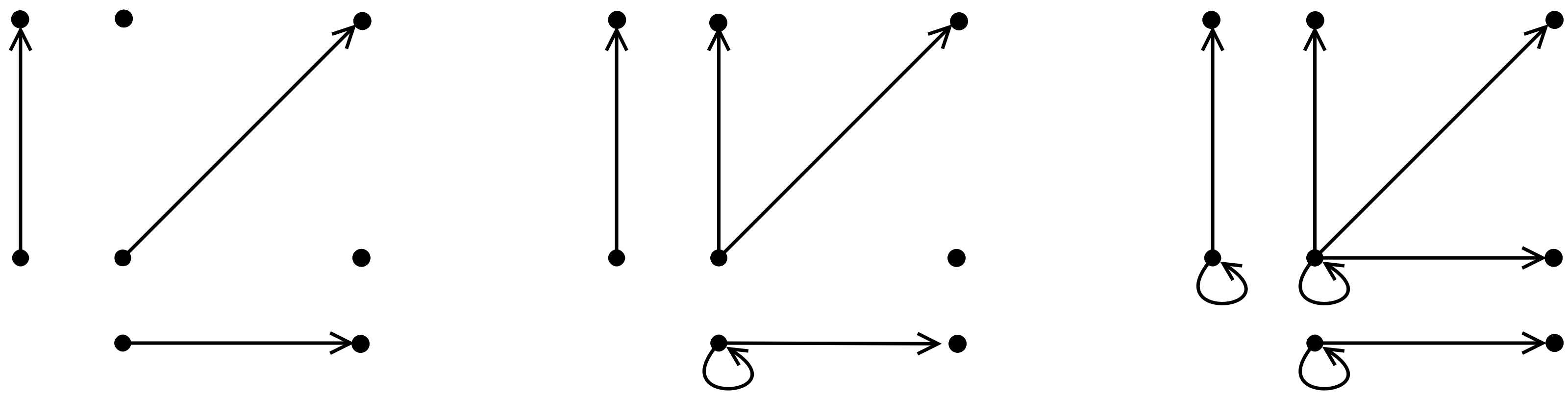
In our language it is the direct product of directed graphs G.

Given a directed graph G we write uRv if there is an arrow from u to v. The inverse relation $\bar{R}$ is then denoted $v\bar{R}u$.

The product $G \times H = \{V(G \times H), R_{G \times H}\}$ of two structures G and H is then defined by $V(G \times H) = V(G) \times V(H)$ and

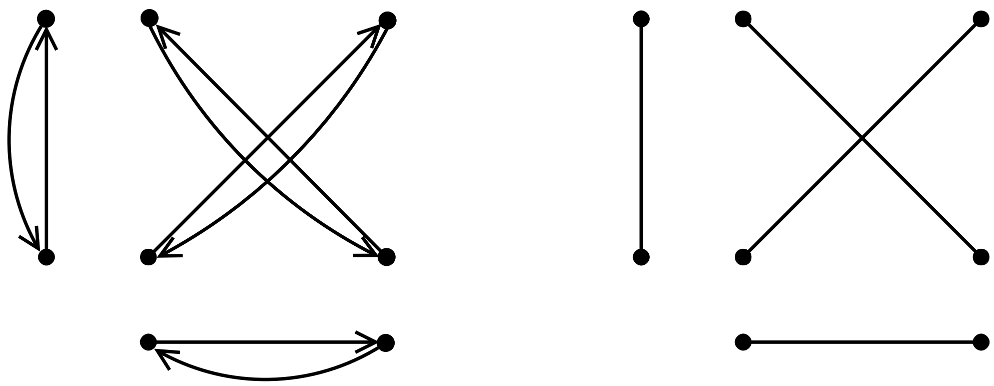
$$(g, h)\, R_{G \times H}\, (g', h') \text{ if } \; g\, R_G\, g' \text{ and } \; h\, R_H\, h'.$$

Three examples of the categorical product



It is commutative, associative, and the one-vertex graph with a loop is a unit.

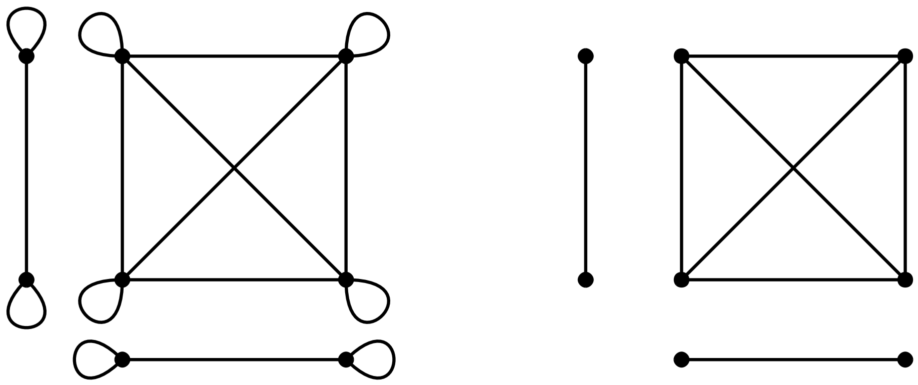
If the relation R is symmetric, that is, if $R = \bar{R}$, then we obtain the direct product.



Hence, the direct product of graphs is a special case of the categorical product of binary relations.

This makes it so difficult and interesting.

If we add loops to every vertex, multiply and delete the loops again, then we obtain the strong product.



Hence we can also consider the strong product as a special case of the categorical product of relations.

So what did Ralph McKenzie, a student of Bjarni Jónnson, prove? He extended the class of graphs for which the refinement property holds, answering a question that goes back to Alfred Tarski.

He proved, among many other results, that

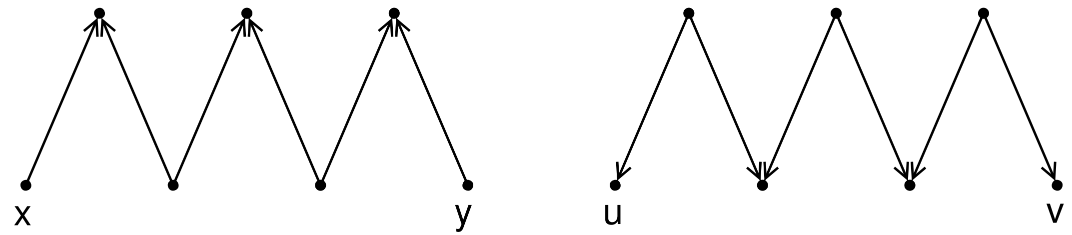
if both $R|\bar{R}$ and $\bar{R}|R$ are connected over $V(G)$

and if G has no distinct elements with the same set of neighbors*,

then the refinement property holds.

*We call such structures thin.

$R|\bar{R}$ and $\bar{R}|R$ connectedness are depicted in the following diagram:



For the direct product $R|\bar{R}$ and $\bar{R}|R$ is equivalent to non-bipartiteness.

For the strong product it simply implies connectedness.

The refinement property says that for any two isomorphic products

$$\prod_{\iota \in I} B_\iota \cong \prod_{\kappa \in J} C_\kappa$$

there exist structures $D_{\iota,\kappa}$ such that

$$B_\mu \cong \prod_{\kappa \in J} D_{\mu,\kappa} \quad \text{and} \quad C_\nu \cong \prod_{\iota \in I} D_{\iota,\nu}$$

for all $\mu \in I$ and $\nu \in J$.

What are the implications of this theorem?

1. It implies unique prime factorization if the refinement process does not continue indefinitely.

Hence, $R|\bar{R}$ and $\bar{R}|R$ connected (thin) structures have unique prime factorization if they are finite or

if they satisfy certain finiteness conditions, such as being locally finite or almost locally finite.

Many of the properties of finite Cartesian products have analogues for finite strong and direct products, in particular if they are thin.

Why was this not really taken up by graph theorists?

A. The proof of the refinement proof is non-algorithmic and leaves open the algorithmic side of factorizing finite graphs.

For the strong product a polynomial algorithm was found by Feigenbaum and Schäffer*.

Later their method was extended to the direct product†.

In both cases unique prime factorization was a byproduct.

*Feigenbaum and Schäffer 1992.
†Imrich 1998.

B. Bipartite graphs are not covered.

This corresponds to the case when R is symmetric and when the structure has two R-connected components.

There are numerous results in this area, many by Richard Hammack.

Little is known about factorizations of infinite bipartite graphs.

An exception is the infinite hypercube. It is the direct product of a K_2 and of a (non-unique) non-bipartite graph*.

*I and D. Rall, 2006.

C. There is no criterion for the existence of prime factorizations.

D. However, we discuss of products of prime graphs.

1. If G is $R|\bar{R}$ and $\bar{R}|R$ connected and the product $\prod_{\iota\in I} B_\iota$ of prime graphs, then this presentation is unique by the refinement property.

2. But, the product $G = \prod_{\iota\in I} B_\iota$ of prime graphs that are $R|\bar{R}$ and $\bar{R}|R$ connected need not be connected.

To see this observe that in a strong product

$$d_G(x, y) = \max_{\iota\in I}\{d_{B_\iota}(x_\iota, y_\iota)\}.$$

Hence strong products are disconnected if there is no bound on the diameters of the factors.

For the direct product the situation is slightly more complicated, but a bound on the diameters is still needed.

Is the presentation of every connected component unique?

If so, what can one say about the relation between the vertex transitivity of the factors and the product?

3. There is also the weak strong product $G^a = \prod_{\iota \in I}^a B_\iota$ of prime graphs.

G^a is spanned by all vertices in $G = \prod_{\iota \in I} B_\iota$ that differ from a given vertex $a \in V(G)$ in only finitely many coordinates.

Notice that G^a it is not a connected component of $G = \prod_{\iota \in I} B_\iota$.

If the B_ι are prime, is weak factorization of G^a unique?

Such products were recently investigated by Tardif et al., because they have the nice property that they can be vertex-transitive even when all the B_ι are asymmetric.

E. Is there a good sufficient condition for the existence of prime factorizations?

The Cartesian case was resolved because every edge was in a layer with respect to any factorization.

Can we at least find pairs of vertices - so called Cartesian pairs - that are in one and the same layer with respect to any factorization?

Then we could proceed as in the Cartesian case.

For the direct product the next figure indicates how to look for such pairs. The idea relies on that for the strong product of finite graphs*.

*Feigenbaum and Schäffer 1992.

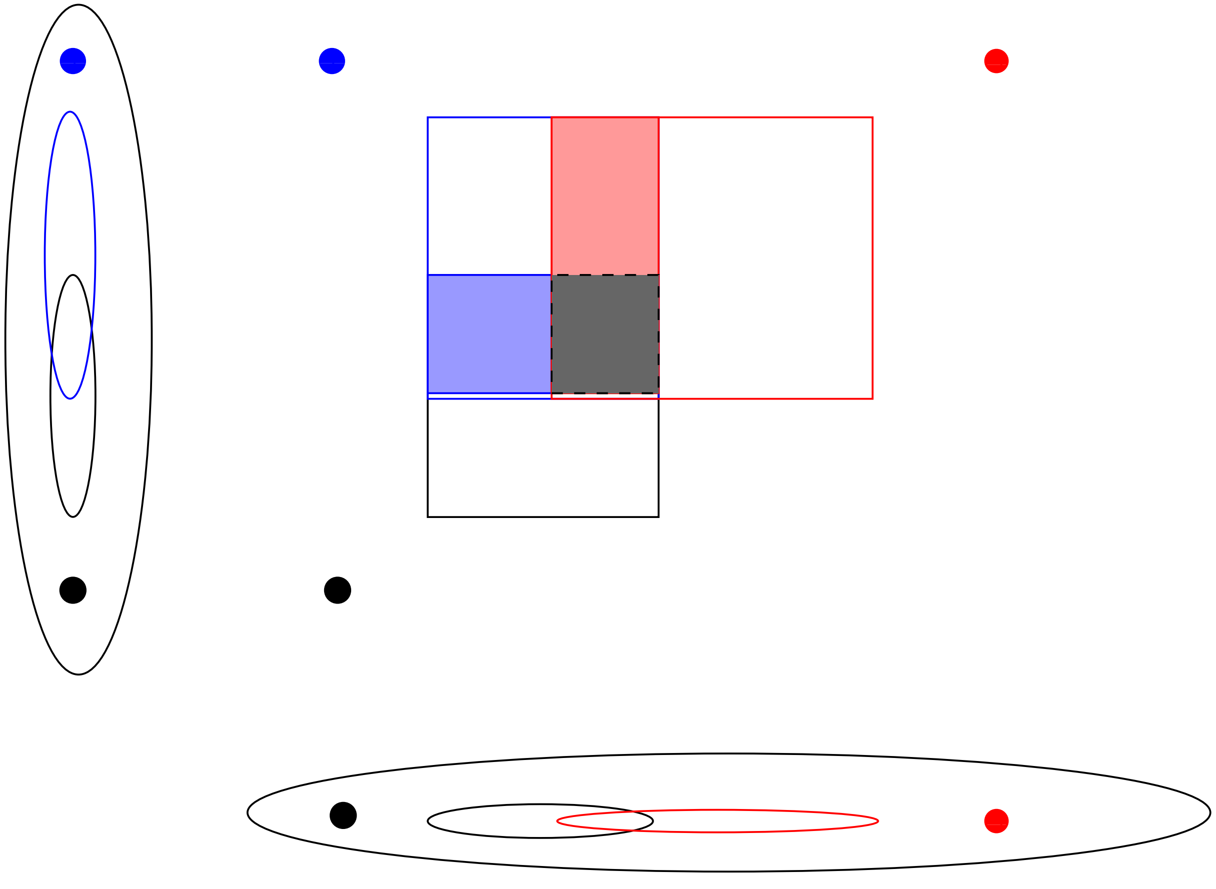


Figure by D. Rall.

F. Graphs with no prime factor decomposition.

There are graphs* G with $G \cong G^3$ but $G \ncong G^2$.

They cannot have a pfd.

Is there an easy example of a thin graph without pfd?

*Trnkova 1976.

Selection of open problems pertaining to infinite cardinal products that are thin and $R|\bar{R}$ and $\bar{R}|R$ connected

P1. Is the presentation of components of the product of connected prime graphs (when the product is disconnected) unique?

P2. Is the presentation of a connected graph as a weak strong product of prime graphs unique?

P3. Is there a criterion for the existence of prime factorizations of infinite graphs?

P4. Is there an easy example of a thin graph without pfd?

Finally, there is the class of products that are thin and connected, but not $R|\bar{R}$ and $\bar{R}|R$ connected.

What can one say about the structure of bipartite graphs that are direct products of finite or infinite graphs?

P5. If K_2 is a factor of G, is it unique?

That is, can G have a factorization where a prime bipartite graph $H \neq K_2$ is a factor?

If you got that far:

THANK YOU AGAIN!