Bubble Space on Hypergraphs

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WU Vienna · Austria

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Motivation

Question by Michael Hecht:
(many years ago in a pub in Bled)

- Can we compute **higher Betti numbers** for graphs?

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Simple undirected **Graph** $G = (V, E)$

- vertex set V
- edge set $E \subseteq \Big\{ \{u, v\} : u, v \in V \Big\} \subset \mathcal{P}(V)$

Betti numbers / Informally

First Betti number of a **topological space**:

maximum number of cuts that can be made
without dividing it into two separate pieces.

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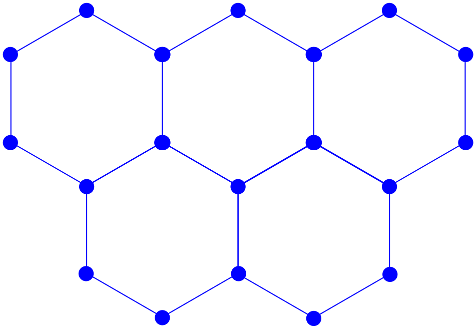

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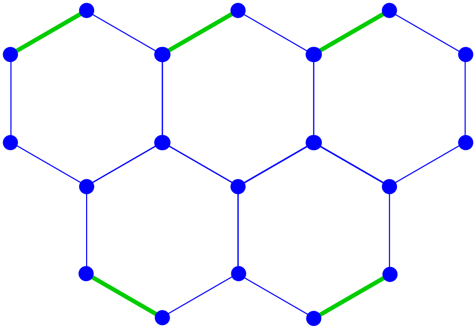

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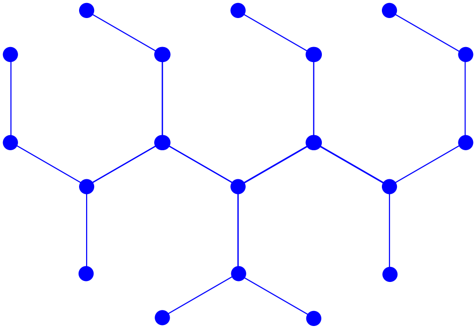

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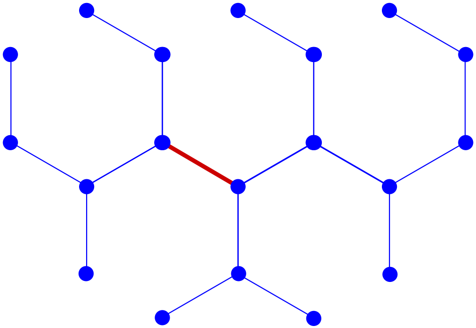

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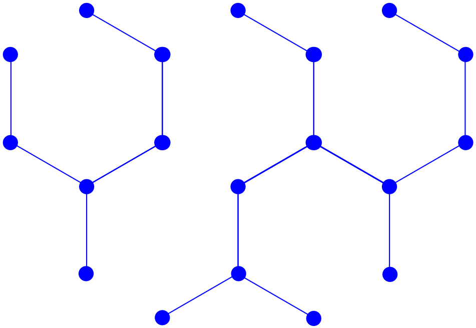

$b_1 = 5$

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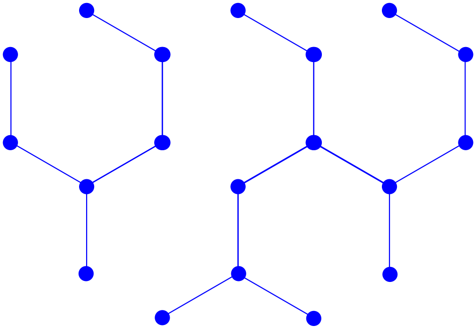

$b_1 = 5$

Geometric realization of G:
topological space where

Vertices $\longrightarrow$ points

Edges $\longrightarrow$ arcs that join vertices

Cycle Space

A **cycle** C in G is a closed path.
We identify C by its edge set, i.e., $C \in \mathcal{P}(E(G))$.
We can generalize cycles as arbitrary Eulerian subgraphs.

The set $\mathcal{C}(G)$ of all cycles forms a vector space over field $\mathbb{Z}_2$ with

$$S_1 \oplus S_2 = (S_1 \cup S_2) \setminus (S_1 \cap S_2)$$

$$1 \cdot S = S \qquad \text{and} \qquad 0 \cdot S = \emptyset$$

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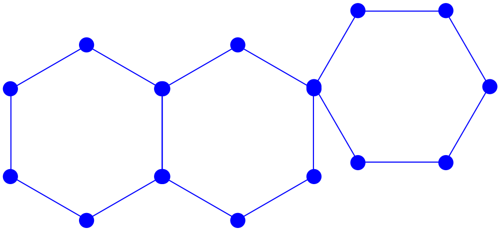

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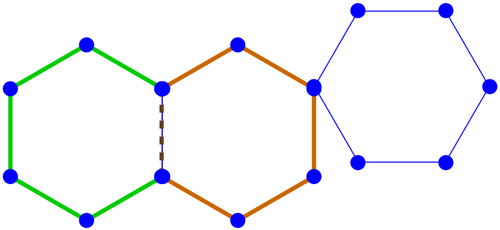

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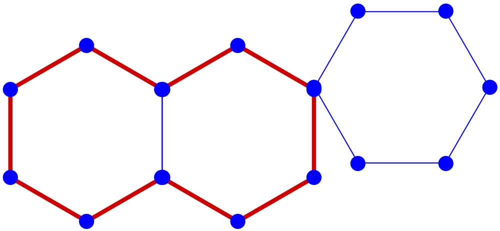

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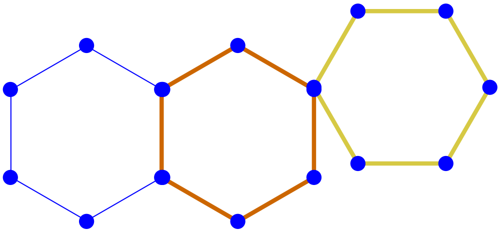

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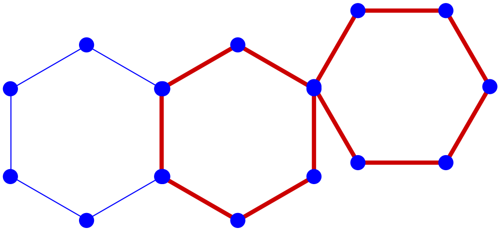

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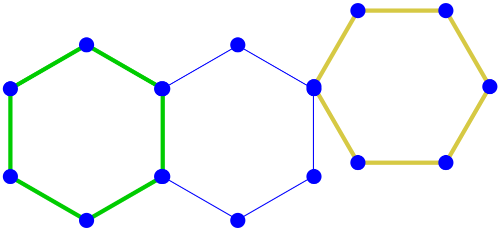

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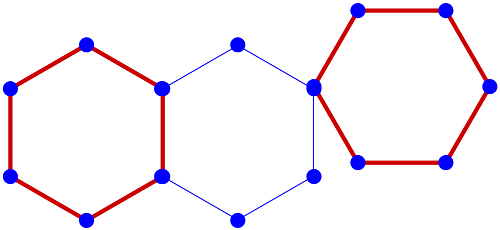

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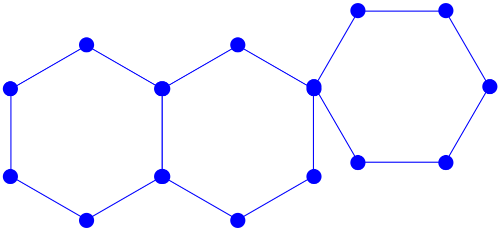
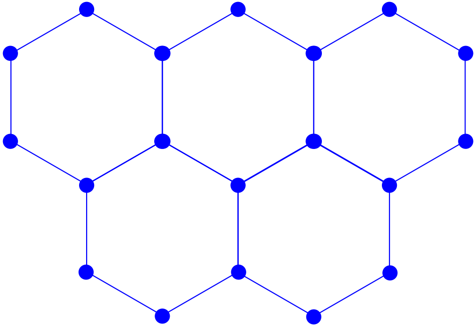


$$\begin{aligned} \dim(\mathcal{C}) &= |E| - |V| + 1 \\ &= 17 - 15 + 1 = 3 \end{aligned}$$

cyclomatic number

Cycle Space



$$\begin{aligned}\dim(\mathcal{C}) &= |E| - |V| + 1\\ &= 23 - 19 + 1 = 5\end{aligned}$$

CW complex

(1) Start with a finite discrete set X^0 of 0-cells.

(2) Inductively create n-skeleton X^n from X^{n-1} by attaching n-cells via continuous gluing maps $\partial_n D^n = S^{n-1} \to X^{n-1}$.

where

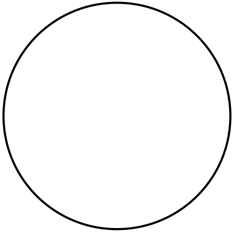
- **0-cell**: a single point
- **n-cell**: homeomorphic to open n-disc $D^n = \{\boldsymbol{x} \in \mathbb{R}^n \colon \|x\| < 1\}$

Skeleton X^n is a disjoint union of cells.

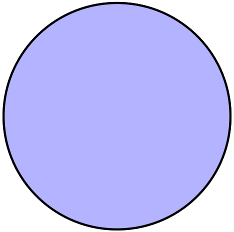
$$\emptyset = X^{-1} \subseteq X^0 \subseteq X^1 \subseteq X^2 \subseteq \cdots \subseteq X$$

CW complex

Circle:



Disc:



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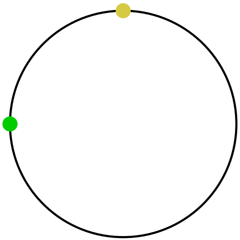
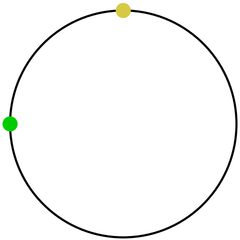


two **0-cells**

Disc:



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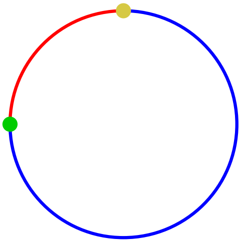

two **0-cells**
two **1-cells**

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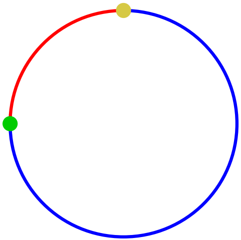

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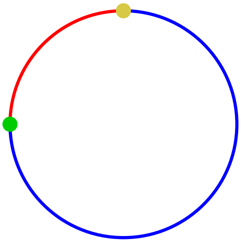

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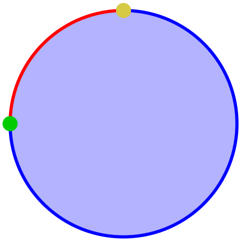
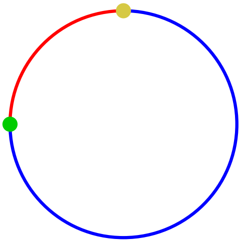


two **0-cells**
two **1-cells**
one **2-cell**

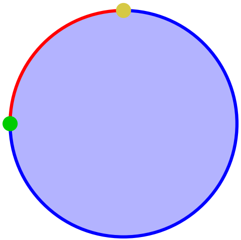
CW complex

Circle:



1-dimensional CW complex

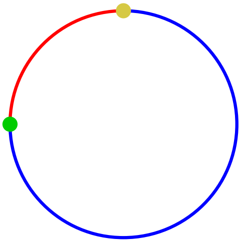
Disc:



2-dimensional CW complex

CW complex

Circle:



1-dimensional CW complex

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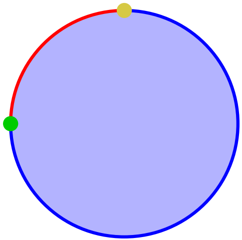

2-dimensional CW complex

The geometric realization of a graph is a 1-dimensional CW complex.

Betti numbers / More Formally

First Betti number of a **topological space**:

- **rank** of the first **homology group**.

nth Betti number of a **topological space**:

- **rank** of the nth **homology group**.

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CW complexes are a convenient tool for computing homology groups (and hence Betti numbers)

Chain complex

Let operator ∂_n assign the **boundary** for a n-dimensional region, e.g.,

$$\partial_3(D^3) = S^2 \qquad \text{but} \qquad \partial_3(D^2) = \emptyset$$

Also observe

$$\partial_3(D^3) = S^2 \qquad \text{and} \qquad \partial_2(\partial_3(D^3)) = \partial_2(S^2) = \emptyset$$

We get a chain

$$\cdots \longrightarrow X^{n+1} \xrightarrow{\partial_{n+1}} X^n \xrightarrow{\partial_n} X^{n-1} \longrightarrow \cdots$$

with

$$\partial_n \circ \partial_{n+1} = 0$$

Homology Group

The set of cells can by described by Abelian groups.
(Similar to cycle space.)

- cycle group

$$Z_n = \ker \partial_n$$

- boundary group

$$B_{n+1} = \operatorname{im} \partial_{n+1}$$

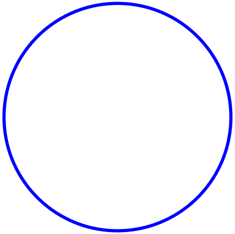
- nth homology group $H_n(X)$

$$H_n = Z_n / B_n = \ker \partial_n / \operatorname{im} \partial_{n+1}$$

Sets without boundary that are not the boundary of a set: **holes**

Homology Group

Circle:



$$\begin{aligned} Z_1 &= \mathbb{Z} \\ B_2 &= \emptyset \\ H_1 &= \mathbb{Z}/\emptyset = \mathbb{Z} \\ &\dim(H_1) = 1 \end{aligned}$$

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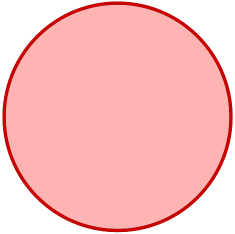

$$\begin{aligned} Z_1 &= \mathbb{Z} \\ B_2 &= \mathbb{Z} \\ H_1 &= \mathbb{Z}/\mathbb{Z} = 1 \\ &\dim(H_1) = 0 \end{aligned}$$

Boundary Operator on Graphs

Observe:

- 2-cells in geometric realizations can be identified by (connected) cycles.
- $\mathcal{P}(V)$ forms a vector space.
- Boundary of a union of edges is a subset of V.
- Cycle are exactly those with $\bigoplus_{c_i \in C} c_i = \emptyset$.

Boundary operator

$$\partial\colon \mathcal{P}(\mathcal{P}(V)) \to \mathcal{P}(V), \quad S \mapsto \partial S = \bigoplus_{s_i \in S} s_i$$

$C \subseteq \mathcal{P}(V)$ is a cycle $\quad \Leftrightarrow \quad \partial C = \emptyset$

Hypergraphs

A **simple** undirected **hypergraph** consists of

- **vertex** set V
- **hyperedges** $E \subseteq \mathcal{P}(V) \setminus \{\emptyset\}$

We have

- **boundary** operator

$$\partial\colon \mathcal{P}(\mathcal{P}(V)) \to \mathcal{P}(V), \quad S \mapsto \partial S = \bigoplus_{s_i \in S} s_i$$

- **cycle space**[1]

$$\mathcal{C}(G) = \ker \partial \subseteq \mathcal{P}(E)$$

[1] In literature a cycle in a hypergraph is defined as sequence of hyperedges

Matroid

Vector space $\mathcal{P}(E)$ forms a **matroid**:

- A is **linearly dependent** if it contains a cycle.
- **elementary cycle**: minimal linearly dependent set
  Graph: a connected cycle where all degrees are 2.

Previous work by Döring (1975) and Oubiña (1974)

CW Complex on Hypergraph

Geometric realization of a graphs is a 1-dimensional CW complexes with

- vertex set V as its 0-skeleton and
- arcs as 1-cells,
- but no 2-cells. $\Rightarrow \quad b_2(G) = 0$

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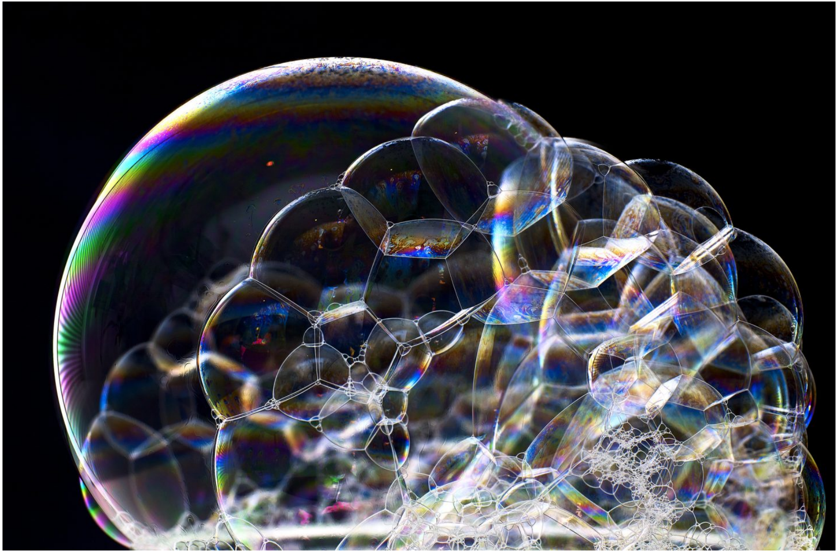
Idea:

- Attach 2-cells to geometric realization. $\Rightarrow X^2 \qquad b_2(G)$
- Attach n-cells to geometric realization. $\Rightarrow X^n \qquad b_n(G)$

Issue:

- Where do we attach n-cells?
- How can we describe this procedure?

Bubbles



Source: Getty Images

2-Cells on Graphs

- 0-cell: vertex in $V(G)$
- 1-cell: edge in $E(G)$
  – identified by its boundary
  – an element in $\mathcal{P}(V)$.
- 2-cell:
  – identified by its boundary
  – identified by (elementary) cycles:
  an element in $\mathcal{C}(G) = \ker\partial \subseteq \mathcal{P}(E)$.

Select appropriate subset $E^2(G) \subseteq \mathcal{C}(G) \in \mathcal{P}(\mathcal{C}(G))$ (How?)

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New hypergraph

$$G_1 = \big(E(G), E^2(G)\big)$$

3-Cells on Graphs

- 3-cells: bubbles
  – identified by its boundary
  – identified by (elementary) cycles:
  an element in $\ker\partial \subseteq \mathcal{P}(E^2)$.

  Select appropriate subset $E^3(G) \subseteq \ker\partial \in \mathcal{P}(\ker\partial)$ (How?)

New hypergraph

$$G_2 = \left(E^2(G), E^3(G)\right)$$

Edge Complex

(C1) $E^{-1} = \emptyset$

(C2) $E^0 = V(G)$

(C3) $E^1 = E(G) \subseteq \mathcal{P}(V(G)) = \ker \partial_0 = \mathcal{P}(E^0)$

(C4) $E^{n+1} \subseteq \ker \partial_n \subseteq \mathcal{P}(E^n)$ for $n \geq 1$,

$$\partial_0 \colon \mathcal{P}(E^0) \to \mathcal{P}(E^{-1}) = \{\emptyset\}$$

$$\partial_n \colon \mathcal{P}(E^n) \to \mathcal{P}(E^{n-1}),$$
$$\{e^n_1, \ldots, e^n_l\} \mapsto \partial(\{e^n_1, \ldots, e^n_l\}) = \bigoplus_{i=1}^{l} e^n_i .$$

Edge Complex

(C1) $E^{-1} = \emptyset$

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(C3) $E^1 = E(G) \subseteq \mathcal{P}(V(G)) = \ker \partial_0 = \mathcal{P}(E^0)$

(C4) $E^{n+1} \subseteq \ker \partial_n \subseteq \mathcal{P}(E^n)$ for $n \geq 1$,

$$\partial_0 \colon \mathcal{P}(E^0) \to \mathcal{P}(E^{-1}) = \{\emptyset\}$$

$$\begin{aligned} \partial_n \colon \mathcal{P}(E^n) &\to \mathcal{P}(E^{n-1})\,, \\ \{e_1^n, \dots, e_l^n\} &\mapsto \partial\big(\{e_1^n, \dots, e_l^n\}\big) = \bigoplus_{i=1}^{l} e_i^n\,. \end{aligned}$$

$$\cdots \longrightarrow \mathcal{P}(E^{n+1}) \xrightarrow{\partial_{n+1}} \mathcal{P}(E^n) \xrightarrow{\partial_n} \mathcal{P}(E^{n-1}) \longrightarrow \cdots$$

with $\partial_n \circ \partial_{n+1}(C) = \emptyset$

Higher Order Betti Numbers

n**th order cycle space**:

$$\mathcal{C}_n(G) = \ker \partial_n \subseteq \mathcal{P}(E^n)$$

n**th Betti number**:

$$b_n(G) = \dim\left(\mathcal{C}_n(G)\right)$$

Higher Order Betti Numbers

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Sequence of hypergraphs

$$G_n = (E^n, E^{n+1})$$

with

$$\mathcal{C}_n(G) = \mathcal{C}(G_n)$$

How? – Selector

We are in urgent need of a **selector** (**choice function**)

$$\mathcal{E}\colon G_{n-1} = (E^{n-1}, E^n) \mapsto E^{n+1} \quad \subseteq \ker \partial_n \subseteq \mathcal{P}(E^n)$$

such that

(S1) $\mathcal{E}$ is universal, i.e., it does not depend on n and G.

(S2) All elements in the image of $\mathcal{E}$ are circuits in matroid $\mathcal{P}(E^n)$.

Recursion for G_n

$$G_n = (E^n, E^{n+1}) = \big(E^1(G_{n-1}), \mathcal{E}(G_{n-1})\big)$$

and

$$E^k(G_n) = E^{k+j}(G_{n-j}) = E^{k+n}(G)$$

Selector

nth cycle space $\mathcal{C}_n(G)$ heavily depends on selector $\mathcal{E}$.

So we have to use notations

$$\mathcal{C}_{n,\mathcal{E}}(G) \qquad \text{and} \qquad b_{n,\mathcal{E}}(G) = \dim\left(\mathcal{C}_{n,\mathcal{E}}(G)\right)$$

and call $b_{n,\mathcal{E}}(G)$ the the nth Betti number w.r.t. selector $\mathcal{E}$.

Possible Selectors

- All **elementary cycles**:

  $b_n(G)$ explodes for increasing n.

- Only those cycles that have strictly **larger support** than each of its edges:

  $b_n(G) = 0$ for all $n \geq |V(G)|$ but is sensible against edge splitting

  ... and $b_k(G)$ can still be huge for $k < n$.

- Minimal length cycles:

  Length is a metric concept and not a purely topological one.
  Hence it is extremely sensible against edge splitting.

- Non-included cycles

- ???

The Big Question

- What do these Betti numbers **tell us** about a (hyper) graph?

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References I

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