Bipartite edge frustration of nut graphs

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Spectra of graphs

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```
sage: g = Graph('JqGWGCA?GA_')
sage: g.adjacency_matrix()
[0 1 1 0 0 0 0 0 0 0 0]
[1 0 0 1 0 0 0 0 0 0 0]
[1 0 0 0 1 0 0 0 0 0 0]
[0 1 0 0 0 1 0 0 0 0 0]
[0 0 1 0 0 1 0 0 0 0 0]
[0 0 0 1 1 0 1 0 0 0 0]
[0 0 0 0 0 1 0 1 1 0 0]
[0 0 0 0 0 0 1 0 0 0 1]
[0 0 0 0 0 0 1 0 0 1 0]
[0 0 0 0 0 0 0 0 1 0 1]
[0 0 0 0 0 0 0 1 0 1 0]
sage: g.spectrum()
[2.290776209081233?,
 1.878024362938573? + 0.?e-90*I,
 1.195212497036078?,
 1,
 0.618033988749895?,
 0.4539862891288374?,
 -0.920002013080433?,
 -1,
 -1.618033988749895?,
 -1.677763504260575?,
 -2.220233840843714?]
sage: 
```

$\sigma(G)$

Nut graphs

G is a nut graph if $0 \in \sigma(G)$ with multiplicity 1, the corresponding eigenvector is full.

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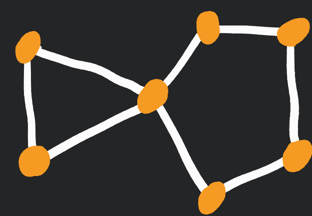
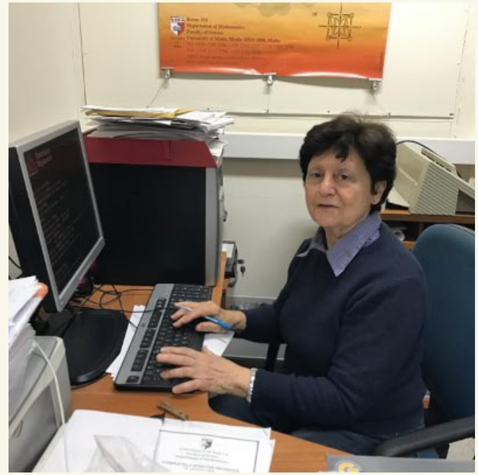
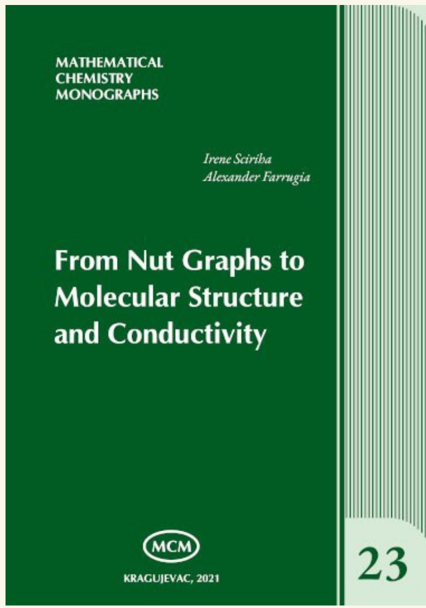
```
sage: g = Graph('FxGGg')
sage: g.spectrum()
[2.472833908995256?,
 1.462598422974775?,
 0.618033988749895?,
 0,
 -1,
 -1.618033988749895?,
 -1.935432331970030?]
sage: g.eigenspaces()
[(0,
  Vector space of degree 7 and dimension 1 over Rational Field
  User basis matrix:
  [ 1  1 -1 -1 -1  1  1]),
 (-1,
  Vector space of degree 7 and dimension 1 over Rational Field
  User basis matrix:
  [ 1 -1  0  0  0  0  0]),
 (a2,
  Vector space of degree 7 and dimension 1 over Number Field in a2 with defining polynomi
al x^2 + x - 1
  User basis matrix:
  [  0   0   0   1  -1 -a2  a2]),
 (a3,
  Vector space of degree 7 and dimension 1 over Number Field in a3 with defining polynomi
al x^3 - 2*x^2 - 4*x + 7
```



Nut graphs

Coined in 1998 by Gutman & Sciriha.



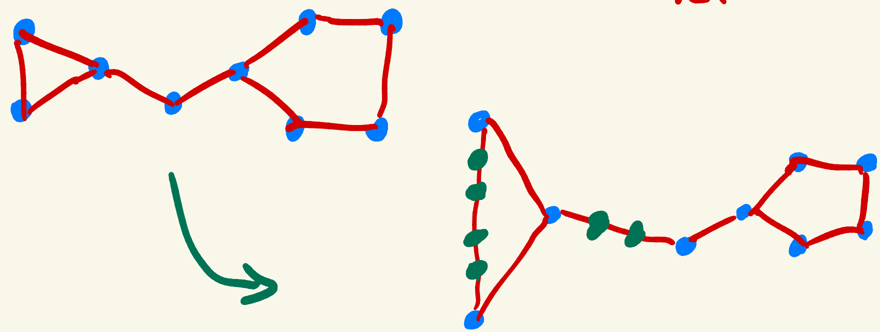


Why study nut graphs?

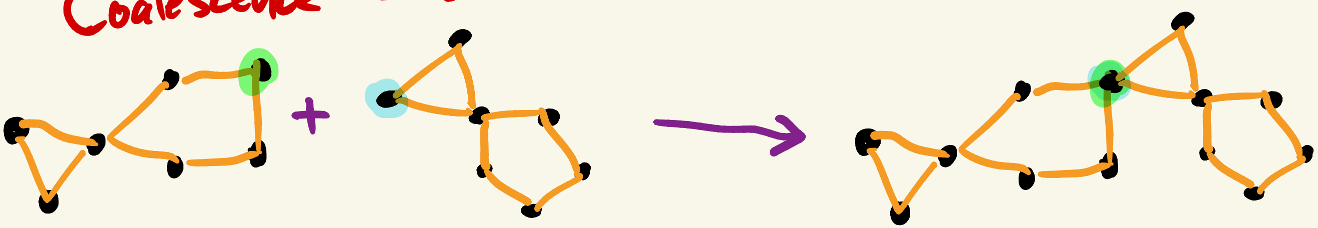
Basic properties & constructions

- connected
- leafless
- non-bipartite

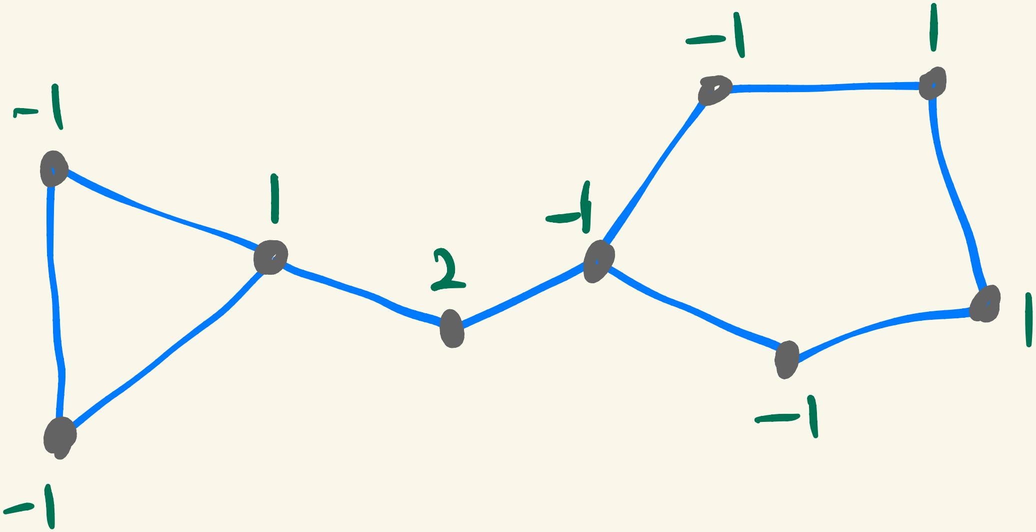
Subdivision cons.



Coalescence cons.



The Local Condition



$$\boxed{\sum_{u \in N(v)} x(u) = 0}$$

in general:

$$\sum_{u \in N(v)} x(u) = \lambda x(v)$$

House of Graphs

House of Graphs

houseofgraphs.org/meta-directory/nut

The House of Graphs

Nut graphs

A *nut graph* is a graph of at least 2 vertices whose adjacency matrix has nullity 1 (i.e., rank n-1 where n is the order of the graph) and for which the non-trivial kernel vector does not contain a zero.

The graph lists are currently only available in 'graph6' format. The larger files are compressed with gzip. Please refer to [1] for more information on how these nut graphs were obtained.

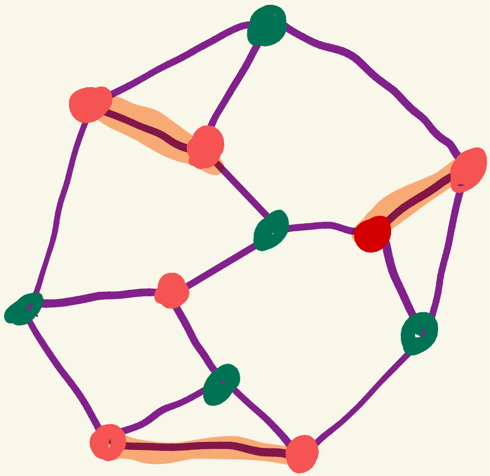
The following lists are available:

- Nut graphs
- Chemical nut graphs
- Nut graphs among cubic polyhedra
- Nut fullerenes
- Regular nut graphs

Nut graphs

Vertices	Girth ≥ 3	Girth ≥ 4	Girth ≥ 5	Girth ≥ 6	Girth ≥ 7	Girth ≥ 8	Girth ≥ 9
0-6	0	0	0	0	0	0	0
7	3	0	0	0	0	0	0
8	13	0	0	0	0	0	0
9	560	0	0	0	0	0	0
10	12551	0	0	0	0	0	0

Bipartite edge frustration



An edge is **frustrated** w.r.t. a given bipartition $V(G) = V_1 \sqcup V_2$ if both endvertices are in the same set of bipartition.

$\varphi(G)$ ... min # edges over all bipartitions of G

Known bound for $\varphi(G)$

Erdős, 1965: $\varphi(G) \le \left\lceil \frac{m}{2} - \frac{n}{4} \right\rceil$, G without isolated vertices

Thomassen, 2006: $\varphi(G) \le \left\lfloor \frac{7n}{24} + \frac{7}{6} \right\rfloor$, G planar, cubic, 3-connected, triangle free

Došlić & Vukičević, 2007:

$\varphi(G) \ge 6$ if G fullerene

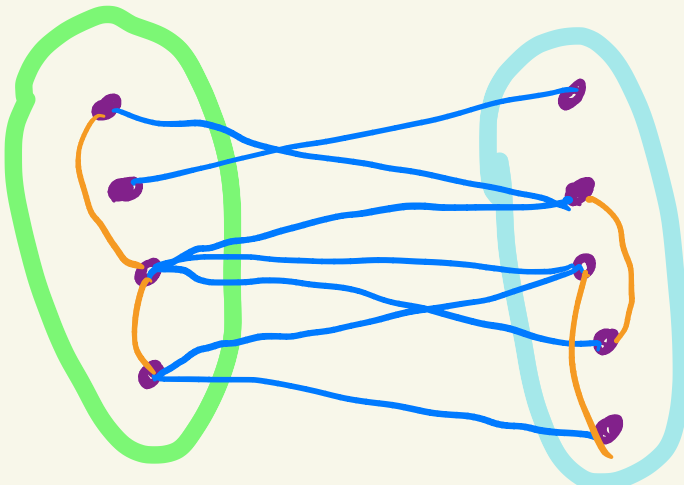
$\varphi(G) \ge 12$ if G IPR-fullerene

Conjecture:

$\varphi(G) \le \sqrt{\frac{12}{5} n}$

fullerene

Min Frustration ≡ Max Cut



Yannakakis, 1978: NP-hard in general

Hadlock, 1975: polynomial for planar graphs

Main theoretical result

Thm. Let G be a nut graph.
Then $\varphi(G) \geq 2$.

Proof. G bipartite, $V(G) = U \sqcup W$, $u_1, u_2 \in U$

$w \in \ker(G + u_1u_2)$

$$A(G+u_1u_2) = \begin{bmatrix} P & B \\ B^T & 0 \end{bmatrix}$$

$$A(G+u_1u_2)\,w = 0$$

$$\Rightarrow \quad Px + By = 0$$

$$B^T x = 0$$

$$w = \begin{bmatrix} x \\ y \end{bmatrix}$$

Proof cont'd.

$$Px + By = 0 \quad \Rightarrow \quad x^T P x + x^T B y = 0$$

$$B^T x = 0 \qquad x^T B = (B^T x)^T = 0$$

$$\Rightarrow \quad x^T P x = 0$$

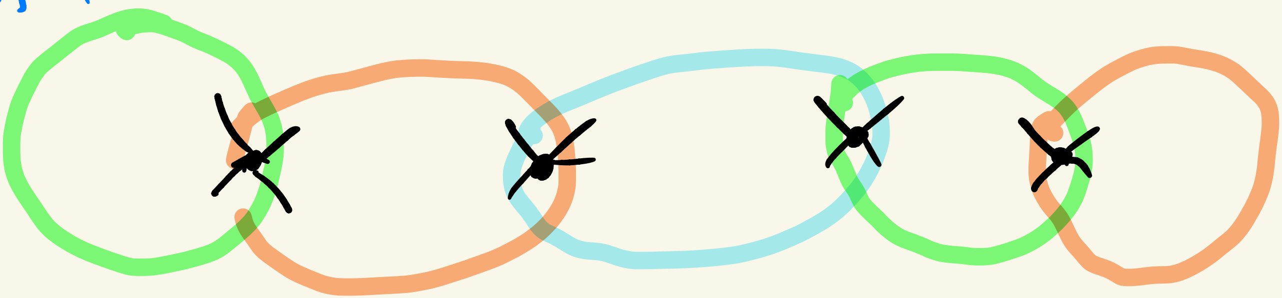
$$2x(u_1)x(u_2) = 0 \qquad \Rightarrow \quad x(u_1) = 0 \text{ or } x(u_2) = 0$$

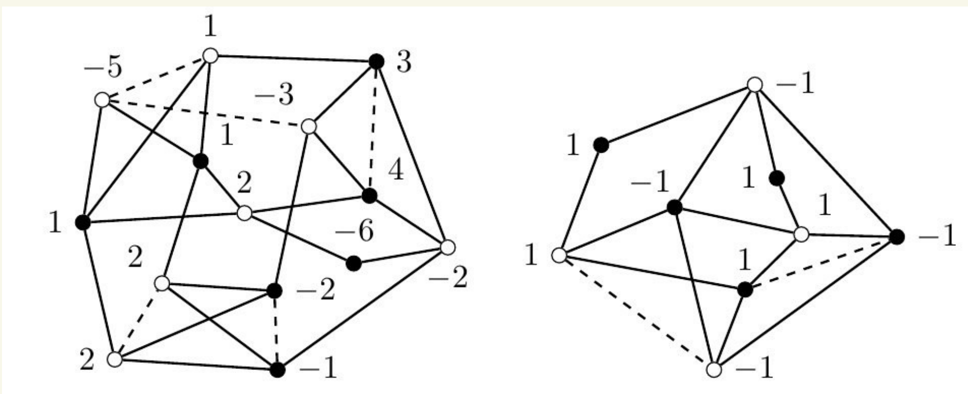
$\blacksquare$

Extra results (one of them)

Thm. For every even $k \geq 2$ there exists a quartic unit graph G with $\varphi(G) = k$.

Proof idea.





(a) G_1, $\varphi(G_1) = 5$ (b) G_2, $\varphi(G_2) = 2$

Computational Results

... coming soon ...

Hvala!

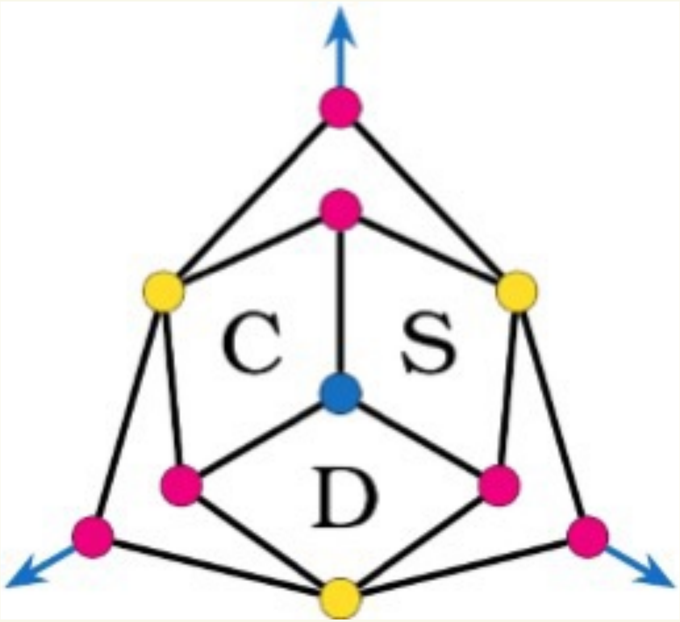
Danke!

Dziękuję!

Cảm ơn!

Thank you!

Gracias!



https://csd11.si