Abstention Causes Bifurcations in Two-Party Voting Dynamics

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Abstract

In a recent dynamical systems model of platform adaptation in spatial voting models, Miller and Stadler (*J. Econ. Dyn. & Control* 1998, in press) showed that, assuming concave voter utility functions and complete participation, there is a globally stable equilibrium located at the mean voter position. Here we show that abstention depending on the distances between voters and platforms may lead to bifurcations rendering the mean voter equilibrium unstable. We find up to seven equilibria, up to four of which are local attractors for the platform dynamics.

1. Introduction

Spatial voting models are widely used in Economics and Political Science. They describe the interactions of two classes of agents: *voters* and *candidates* or *parties*. Work on this subject can be traced back as far as to Hotelling's work in the 1920s; the core of the theory was developed in the 1940s and 1950s by Smithies (1941), Downs (1957) , and Black (1958). For recent reviews see (Enelow & Hinich, 1984a; Enelow & Hinich, 1990). Only very rarely, however, was research focussed on its dynamical aspects.

The importance of adaptive processes in economic theory, on the other hand, has long been recognized (see, for example, Malthus 1798). Recently, Miller and Stadler (1998) indroduced a dynamical system describing the platform adaptation of two opportunistic political parties. Surprisingly, these authors found globally stable equilibria under rather general assumptions despite the general lack of stable equilibria in the classical, (game theoretic) approaches (Davis *et al.*, 1972; Kramer, 1973; McKelvey & Ordeshook, 1976; McKelvey, 1979; Plott, 1967). Their model, which is expressed in terms of coupled ordinary differential equations, was derived from earlier computer simulations that showed a similar tendency towards stability (Kollman *et al.*, 1992; Kollman *et al.*, 1996).

A potential short-comming of (Miller & Stadler, 1998) is the restriction to complete participation. In this contribution we will relax this assumption and expand the game dynamics of platform adaptation to a scenario in which voters choose between participating in the election (i.e., voting for one of the competing parties) and abstention.

Formal modeling of voter turnout in the rational choice tradition (Aldrich, 1993) has typically not been very successful in explaining many aspects of turnout behavior. Green and Shapiro (1994) summarize much of the work in this area. A less restrictive model was proposed recently by Irons (1997) . It also assumes explicitly that voters weigh the costs and benefits of voting but takes into account that these costs and benefits may depend on the policy alternatives in a more general manner. We shall pursue a similar approach in this paper.

This contribution is organized as follows: In section 2 we introduce the voting dynamics, we briefly review the stability results of Miller and Stadler and we generalize them by taking non-policy values into account. The differential equations are modified to accomodate two different models of voter turnout in section 3, where we present the main result: abstention may cause bifurcations of the global dynamics, and in particular, it may render the mean voter equilibrium unstable.

2. Adaptive Platform Dynamics

2.1. The Basic Model

The key element in spatial voting models is the idea that voters as well as candidates are seen as points in an abstract metric space, the *issue space*. It describes all the factors which are of concern to the voter. As usual in the literature, we assume that the issue space is the Euclidean space $\mathbb{R}^I$, the dimension of which is given by the number I of different issues. Each voter has a particular opinion on every issue to be voted on, and therefore a given stake or interest in the outcome of the vote that leads her to vote as she does. A voter's position in issue space is referred to as the voter's *ideal point* and will be denoted by the vector $x_v \in \mathbb{R}^I$. As in (Miller & Stadler, 1998) we assume that the voter ideal points do not change with time. The active players in our model are the two parties, or candidates, who are described by their positions $y^1 \in \mathbb{R}^I$ and $y^2 \in \mathbb{R}^I$. Parties will be allowed to modify their positions adaptively in order to gain more votes.

An adaptive agents model of spatial voting consists of four ingredients: The first two are utility functions for voters and parties, respectively. Then there is a rule that the parties use to update their platforms, and finally we need to specify the mechanism by which a voter decides to abstain from the election or to vote for a particular party.

The utility of a party $p = 1, 2$ is simply its share of votes E_p or the fraction of votes W_p that it is ahead of its competitor. Clearly, $W_1 = -W_2 = E_1 - E_2$. Obviously, E_p depends explicitly on the platform positions of both parties and implicitly also on the voter positions and the mechanism by which a voter decides to vote for a particular party or not to vote at all. Since the voters' characteristics are fixed parameters, it suffices to write $E_1(y^1, y^2)$, etc. The fraction of votes $E_p(y^p|\ldots)$, with the position of the other party fixed, has been termed the *electoral landscape* (Kollman *et al.*, 1997; Kollman *et al.*, 1998) of party p. The landscape metaphor is a common model in adaptive search. Applications to political science include (Axelrod, 1986) and (Axelrod & Bennet, 1993). In the context of biological evolution it goes back to Sewall Wright ((1932). It is not hard to verify that the vote landscapes in a two-party one-issue model are always unimodal. This is not necessarily true for more than one issue or more than two parties. The relationship between the distribution of voters' strengths and the slopes and positions of peaks on an electoral landscape is investigated in (Kollman *et al.*, 1997).

Opportunistic parties try to maximize their share of votes without regard of an ideological position. Our basic assumption is that each party tries to increase its share of votes by small corrections to its platform y. Each party attempts to increase its own utility by means of steepest ascent in the variables under its own control, i.e., by locally optimizing its own platform under the assumption of a fixed position of the platforms of all other parties. In other words, it follows the gradient of its electoral landscape. The other player, of course, reacts to the changes in the first party's platform and adjusts her position. Assuming that these platform adjustments are being conducted continuously, (guided by, say, opinion polls), the most natural ansatz for the dynamics of platform adjustment (Miller & Stadler, 1998) is the differential equation

$$\dot{y}^p = \nabla_{y^p} E_p(y^p|\ldots), \qquad p = 1, 2 \tag{1}$$

This dynamics corresponds to simultaneous hill-climbing of each party on its own electoral landscape. However, a party's vote landscape is constantly changing due to the movements of all other parties in issue space. The platform dynamics, equ.(1), is a dynamical system "living" on the *phase space* $\mathbb{R}^{2I}$. It will sometimes be necessary to directly refer to vectors in phase space. In the following, we will use the notation $x, y \in \mathbb{R}^I$ for positions (vectors) in issue space and $\vec{y} \in \mathbb{R}^{2I}$ for vectors in phase space.

The *voter dissatisfaction* function $u_v(y^p)$ measures the utility of a particular political platform y^p, $p = 1, 2$ for voter v. For the moment, we assume that a voter's utility depends only on the distance between the voter's ideal point and the platform's position in issue space, i.e., it does not depend on the name of the party. The most common ansatz (Enelow & Hinich, 1984a) is the weighted squared Euclidean distance

$$u_v(y^p) = -(y^p - x_v)\mathbf{S}_v(y^p - x_v), \tag{2}$$

where $\mathbf{S}_v$ is a positive definite matrix of strength factors that determine how much voter v cares about each of the issues. A voter's ideal point is of course the (unique) maximum of its dissatisfaction function u_v. Therefore we will assume throughout this paper that $u_v : \mathbb{R}^I \mapsto \mathbb{R}$ satisfies

$$(y_j - x_{vj})\frac{\partial u_v}{\partial y^j}(y) < 0 \tag{3}$$

for all $y \neq x_v$, relaxing thereby the assumption of quadratic voter utilities.

This situation is complicated however by the existence of certain *non-policy issues*, such as a candidate's age, religion, and gender, on which parties have no influence. Non-policy issues do not appear as coordinates of issue space. They remain fixed during campaigns and are commonly modelled as additive constants m_v^p in the voter dissatisfaction functions:

$$\tilde{u}_v^p(y^p) = u_v(y^p) + m_v^p \tag{4}$$

Instead of specifying the individual position of each voter v it may be more realistic and/or more convenient to introduce the density $\rho(x)$ of voters in issue space. It is normalized in the usual way: $\int_{\mathbb{R}^I} \rho(x)dx = 1$. Instead of $u_v(y)$ we now need to specify the utility function $u(y,x)$ of a platform y for a voter with position x. We remark that we can use the δ-distribution to translate the "discrete" model into the form of the continuous model by setting

$$\rho(x) = \frac{1}{V} \sum_{v} \delta(x - x_v)\,. \tag{5}$$

The voting mechanism is most conveniently described by probabilities $\mathcal{P}_v^p$ that voter v votes for the party p. In spatial voting models, these probabilities depend only on the utilities $\tilde{u}_v^p$ of the various parties for voter v. While the ideas of probabilistic voting (for a general review see (Coughlin, 1990)) and locally restricted strategy searches in such models (Coughlin & Nitzan, 1981; Samuelson, 1984) have been widely discussed, here we assume that parties do not start at an identical status quo platform, and that their ability to maximize voter support is limited to climbing the local gradient of their respective utility functions.

If the voter's knowledge is infinitely accurate, v will simply vote for the party that provides the largest utility, i.e., $\mathcal{P}_\infty^1 = 1$ if party 1 yields the largest utility for voter v, $\mathcal{P}_\infty^1 = 0$ if the utility for party 2 is larger; if both parties yield the same utility for voter v, then $\mathcal{P}_\infty^1 = \frac{1}{2}$. It is convenient to use the notation

$$\tilde{z}_v = \tilde{u}_v^1(y^1) - \tilde{u}_v^2(y^2) = u_v(y^1) - u_v(y^2) + m_v^1 - m_v^2 = z_v + \delta_v \tag{6}$$

for the utility difference. Then v votes for party 1 if $\tilde{z}_v > 0$, for party 2 if $\tilde{z}_v < 0$ and she flips a coin if $\tilde{z}_v = 0$. In a model with V voters, the expected fraction of votes for the two parties is given by

$$\begin{aligned} E_1(y^1, y^2) &= \frac{1}{V} \sum_{v=1}^{V} \mathcal{P}(\tilde{z}_v) \\ E_2(y^1, y^2) &= \frac{1}{V} \sum_{v=1}^{V} \mathcal{P}(-\tilde{z}_v) \end{aligned} \tag{7}$$

It is quite reasonable, however, to assume that in mass elections most voters will lack complete information about the parties' true positions, and therefore a voter might make a wrong decision by voting for a party that does not yield the largest utility for her. Following (Miller & Stadler, 1998), $\mathcal{P}(\tilde{z}_v)$ is the probability that voter v chooses platform 1 if the utility difference $\tilde{z}_v$ against the opposing platform 2 is $\tilde{z}_v$. We shall refer to $\mathcal{P}(z)$ as the *response function* of the voters. It is a sigmoidal function approaching 0 for $z \to -\infty$ and 1 for $z \to +\infty$.

While almost the entire literature on spatial voting assumes complete knowledge and hence works with the discontinuous function $\mathcal{P}_\infty$, we shall have to relax this assumption throughout this contribution. In order to ensure that the solutions of the dynamical system are unique, we assume that $\mathcal{P} : \mathbb{R} \to [0,1]$ is a twice continuously differentiable approximation of $\mathcal{P}_\infty$ (Miller & Stadler, 1998). For the sake of mathematical tractability we shall assume throughout this work that $\mathcal{P}$ does *not* depend on the individual voter v. Valid response functions are for instance

$$\begin{aligned} \mathcal{P}(z) &= \tfrac{1}{2}(1+\tanh(az)) \quad = 1/(1+\exp(-2az)) \\ \mathcal{P}(z) &= \tfrac{1}{2}(1+\operatorname{erf}(az)) \end{aligned} \tag{8}$$

The slope of the sigmoidal response function expresses how critically voters react in case there are alternatives to choose between. A steep slope $\mathcal{P}'(0)$ of the function $\mathcal{P}$ at 0 indicates a strong division of voters for either party whenever the platforms are different.

Complete participation implies that the response function $\mathcal{P}$ must satisfy $\mathcal{P}(\tilde{z}_v) + \mathcal{P}(-\tilde{z}_v) = 1$, since each voter v always votes either for one or the other party. It follows immedidately that $E_2(y^1, y^2) = 1 - E_1(y^1, y^2)$. In two-party models it is customary to assume that party p's utility is given by the fraction W_p of votes that it is ahead of its rival:

$$\begin{aligned} W_1(y^1,y^2) &= E_1(y^1,y^2) - E_2(y^1,y^2) = 2E_1(y^1,y^2) - 1 \\ W_2(y^1,y^2) &= E_2(y^1,y^2) - E_1(y^1,y^2) = 2E_2(y^1,y^2) - 1 \end{aligned} \tag{9}$$

where the last equalities assume complete participation. Equ.(9) implies $\dot{W}_p = 2\dot{E}_p$, and we conclude immediately that, in the case of complete participation, the platform dynamics does not depend on whether W_p or E_p is used to model the party payoff (except for the obvious rescaling of the time axis).

2.2. The Simplest Case: Complete Participation, No Non-Policy Values

We briefly review here the 2-party system with V voters, I issues, and complete participation, that was first described in (Miller & Stadler, 1998). More explicitly, the model reads in component-wise notation

$$\begin{aligned} \dot{y}_j^1 &= \frac{1}{V}\sum_v \mathcal{P}'(z_v)\partial_j u_v(y^1) \\ \dot{y}_j^2 &= \frac{1}{V}\sum_v \mathcal{P}'(-z_v)\partial_j u_v(y^2) \end{aligned} \tag{10}$$

where $\partial_j u_v(y^p)$ denotes the j-th component of the vector $\nabla u_v(y^p)$. Note that $\mathcal{P}'(-z_v) = \mathcal{P}'(z_v)$. Using W_p instead of E_p as the party payoff function merely results in an additional factor 2 in equ.(10), and hence does not change the dynamics.

It is not hard to verify that the manifold $\mathcal{H}_2 = \left\{\vec{y} \in \mathbb{R}^{2I} | y^1 = y^2\right\}$ on which z_v vanishes is invariant under the game dynamics (10). The system is symmetric under exchange of the platform indices, and hence fixpoints outside $\mathcal{H}_2$, if they exist at all, have to come in pairs: if $(\hat{a}, \hat{b})$ is a fixed point of (10), so is $(\hat{b}, \hat{a})$. Furthermore, the box $\mathcal{B}$ spanned by the extremal voter positions is forward invariant and all orbits are eventually bounded within $\mathcal{B}$. Introducing the *average voter utility*

$$U(y) = \frac{1}{V}\sum_v u_v(y) \tag{11}$$

we find that the dynamics (10) reduces to the gradient system $\dot{y} = \mathcal{P}'(0)\nabla U(y)$ within the manifold $\mathcal{H}_2$. We shall call a fixed point $(\hat{x}, \hat{x}) \in \mathcal{H}_2$ a *trivial fixed point* of the voting dynamics (10). Note that the trivial fixed points are thus the critical points of U.

The trivial fixed point is *globally* stable provided all voter utility functions u_v are concave. If this condition is relaxed, bifurcations may occur. For later reference we recall the Jacobian matrix of (10) at a trivial fixed point:

$$\mathbf{J}(y, y) = \mathcal{P}'(0) \begin{pmatrix} \mathbf{H}(y) & 0 \\ 0 & \mathbf{H}(y) \end{pmatrix}, \tag{12}$$

where $\mathbf{H}(y)$ is the Hessian matrix of the average voter utility $U(y)$. It has the entries

$$\mathbf{H}_{jk}(y) = \frac{\partial^2}{\partial y_j \partial y_k} U(y)\,. \tag{13}$$

The stability of a fixed point on $\mathcal{H}_2$ is therefore determined by the curvature of the average voter dissatisfaction $U(y)$. The stable trivial fixed points are therefore the maxima of $U(y)$.

2.3. Non-Policy Values

In this section we extend the results of (Miller & Stadler, 1998) by investigating the effect of non-policy values, i.e., we consider voter dissatisfaction functions of the form (4), $\tilde{u}_v^p(y^p) = m_v^p + u_v(y^p)$, where $u_v(y^p)$ is the policy-dependent utility and m_v^p is a constant that depends on the voter and (the label of) the party, but not on the party's platform position. The difference in the dynamics stems from the fact that $\tilde{z}_v = z_v + \delta_v$ replaces z_v in equ.(10). Note that $\nabla \tilde{u}_v^p(y) = \nabla u_v(y)$. In order to show that $\mathcal{H}_2$ is still invariant we observe that $(y^1, y^2) \in \mathcal{H}_2$, i.e., $z_v = 0$, implies

$$\dot{y}^p = \frac{1}{V} \sum_v \mathcal{P}'(\delta_v) \nabla u_v(y^p)\,, \tag{14}$$

and hence $\dot{y}^1 - \dot{y}^2 = 0$. The stability of a trivial fixed point is determined by its Jacobian (which can be obtained from equ.(12) by subsituting $z_v + \delta_v$ for z_v and subsequently setting $z_v = 0$):

$$\mathbf{J}(y, y) = \mathbf{Q}(y) \otimes \begin{pmatrix} 1 & -1 \\ 1 & -1 \end{pmatrix} + \mathbf{H}(y) \otimes \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\,. \tag{15}$$

where we use the abbreviations

$$\begin{aligned} \mathbf{Q}_{kl}(y) &= \frac{1}{V} \sum_v \left[\mathcal{P}''(\delta_v) - \mathcal{P}''(-\delta_v)\right] \partial_k u_v(y) \partial_j u_v(y)\,, \\ \mathbf{H}_{kl}(y) &= \frac{1}{V} \sum_v \left[\mathcal{P}'(\delta_v) + \mathcal{P}'(-\delta_v)\right] \partial_k \partial_j u_v(y)\,. \end{aligned} \tag{16}$$

The first term in equ.(15) is nilpotent since $\begin{pmatrix} 1 & -1 \\ 1 & -1 \end{pmatrix}$ is nilpotent. The stability of a trivial equilibrium is therefore determined by the eigenvalue of $\mathbf{H}(y)$ alone. This matrix can be regarded as the Hessian of the potential function

$$\Psi(y) = \frac{1}{V} \sum_{v} \left[\mathcal{P}'(\delta_v) + \mathcal{P}'(-\delta_v)\right] u_v(y) \,, \tag{17}$$

which determines the dynamics on $\mathcal{H}_2$ since $\dot{y} = \nabla\Psi(y)$ on this invariant manifold. Comparing eqns.(17) and (11), we find that Ψ is merely a weighted version of the average voter utility $U(y)$. The conclusions of ref. (Miller & Stadler, 1998) thus remain valid also in the presence of non-policy issues. In the remainder of this paper we will therefore neglect this complication.

3. Complete Participation and Abstention

3.1. Modelling Voter Participation

In general, the voters face a decision between participating in the election (i.e., voting for one of the competing parties) and abstention. The literature on voter turnout is extensive, see e.g. (Aldrich, 1993). Formal modeling of voter turnout in the rational choice tradition has typically not been very successful in explaining many aspects of turnout behavior. Green and Shapiro (1994) summarize much of the work in this area. A less restrictive model was proposed recently by J. S. Irons (1997). It also assumes explicitly that voters weigh the costs and benefits of voting but allows that these costs and benefits depend on the policy alternatives in a more general manner.

In the context of the dynamical models presented here it will be more convenient to recast the turnout behavior in terms of the probabilities $\Theta_v(\vec{y})$ that voter v participates in the election as a function of the platform positions. For the sake of tractability we shall restrict ourselves to very simple heuristic examples of $\Theta_v(\vec{y})$ instead of incorporating a detailed model. It is then always possible to write the expected fraction of votes E_k in the form $\sum_v \Theta_v(\vec{y})E_k^\circ(\vec{y})$, where $E_k^\circ(\vec{y})$ corresponds to a model with complete participation.

We assume that voter v will not participate in the election if her utility difference $u_v(y^1)-u_v(y^2)$ is very small, which is in particular the case if the platform positions of the two parties are (almost) the same, or if the utilities of both parties are very small. In order to incorporate abstention into the 2-party model discussed in the previous section, we first consider the case where the probability for *non-voting* is a (universal) function μ of the utility difference z_v, section 3.2. It seems reasonable to assume that the non-voting probability $\mu(z)$ is a non-increasing function of $|z|$. A special case, in which μ is a step function, has been considered by Irons (1997).

In a more realistic model, section 3.3, we assume that a voter will abstain if the platforms are far away from her ideal point. For simplicity we set:

$$\begin{aligned} E_1(y^1,y^2) &= \frac{1}{V}\sum_v P_0(z_v)\Psi(u_v(y^1)) \\ E_2(y^1,y^2) &= \frac{1}{V}\sum_v P_0(-z_v)\Psi(u_v(y^2)) \end{aligned} \tag{18}$$

where $\Psi(u)$ is an increasing function of u. Recall that the voter utility u is negative and decreasing with distance from the voter's ideal point.

In the case of incomplete participation we obtain different dynamical systems depending on whether E_p or W_p is chosen. The resulting models may exhibit qualitatively different behavior; an example will be discussed in the following section.

3.2. Abstention Depending on Voter Utility Differences

We require that the non-voting probability $\mu(z)$ satisfies the following conditions:

(i) $\mu(z)$ is symmetric, i.e. $\mu(z) = \mu(-z)$. This means that the parties seem exchangeable to a person who does not vote at all.

(ii) $\mu(z)$ is decreasing for $z > 0$.

(iii) $\mu(z)$ is twice differentiable. (We make this assumption for technical simplicity.)

As an immediate consequence, $\mu'(0) = 0$ and $\mu''(0) \le 0$ by definition. Complete participation corresponds to setting $\mu(z) = 0$.

Using W as the party utility function, as in equ.(2.3), the dynamics can be written in the form

$$\begin{aligned} \dot{y}^1 &= \frac{1}{V}\nabla_{y^1}\sum_{v=1}^{V}(1-\mu(z_v))(2P_0(z_v)-1) \\ \dot{y}^2 &= \frac{1}{V}\nabla_{y^2}\sum_{v=1}^{V}(1-\mu(z_v))(1-2P_0(z_v)) \end{aligned} \tag{19}$$

and the Jacobian at the trivial fixed point becomes

$$\mathbf{J} = 2{P_0}'(0)(1-\mu(0))\begin{pmatrix} \mathbf{H}(y) & 0 \\ 0 & \mathbf{H}(y) \end{pmatrix} \tag{20}$$

As in section 2.2., $\mathbf{H}(y)$ denotes the Hessian matrix of the average voter utility $U(y)$, equ.(13). The mean voter equilibrium remains stable also with incomplete participation, as long as $\mathbf{H}(y)$ is negative definite, since $1-\mu(0) > 0$.

If we use E instead of W, the dynamics is given by

$$\begin{aligned} \dot{y}^1 &= \frac{1}{V}\nabla_{y^1}\sum_{v=1}^{V}(1-\mu(z_v))P_0(z_v) \\ \dot{y}^2 &= \frac{1}{V}\nabla_{y^2}\sum_{v=1}^{V}(1-\mu(-z_v))P_0(-z_v) \end{aligned} \tag{21}$$

The Jacobian at the trivial fixed point thus becomes

$$\mathbf{J} = \frac{1}{2}\mu''(0)\mathbf{C}(y)\otimes\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} + P_0'(0)(1-\mu(0))\begin{pmatrix} \mathbf{H}(y) & 0 \\ 0 & \mathbf{H}(y) \end{pmatrix} \tag{22}$$

where we have introduced the abbreviation

$$\mathbf{C}_{kj}(y) = \frac{1}{V}\sum_{v=1}^{V}\partial_k u_v(y)\partial_j u_v(y)\,, \tag{23}$$

and $\mathbf{H}(y)$ is again the Hessian of the average voter satisfaction $U(y)$, equ.(13).

If (y, y) is a fixed point on $\mathcal{H}_2$ we have necessarily $(1/V)\sum_v \partial_k u_v(y) = \partial_k U(y) = 0$. Therefore we can write

$$\mathbf{C}_{kj}(y) = \frac{1}{V}\sum_{v=1}^{V}\big(\partial_k u_v(y) - \partial_k U(y)\big)\big(\partial_j u_v(y) - \partial_j U(y)\big)\,. \tag{24}$$

Thus the matrix $\mathbf{C}(y)$ is the covariance matrix of the components of the vectors $\nabla u_v(y)$ with the the average being taken over all voters v, and hence $\mathbf{C}(y)$ is non-negative definite for any fixed point $(y,y) \in \mathcal{H}_2$. Therefore, the eigenvalues corresponding to the first part of $\mathbf{J}$ are 0 and $\mu''(0)\lambda_j$, resp., where λ_j is an eigenvalue of $\mathbf{C}$, and therefore positive[1]. In the case of strictly concave voter utility functions, all eigenvalues of the Hessian $\mathbf{H}$ are negative. Since the sum of two negative definite matrices is again negative definite, all eigenvalues of $\mathbf{J}$ are also negative and the mean voter equilibrium is stable.

3.3. Abstention Depending on Voter Utilities

Let us now turn to a model in which the probabily for participation depends explicitly on the utilities of the two parties for each voter. From equ.(18) we derive the following voting dynamics:

$$\begin{aligned}
\dot{y}^1 &= \frac{1}{V}\sum_v \left[P_0'(z_v)\Psi(u_v(y^1)) + P_0(z_v)\Psi'(u_v(y^1))\right]\nabla u_v(y^1)\\
\dot{y}^2 &= \frac{1}{V}\sum_v \left[P_0'(-z_v)\Psi(u_v(y^2)) + P_0(-z_v)\Psi'(u_v(y^2))\right]\nabla u_v(y^2)
\end{aligned}$$

$$\begin{aligned}
\dot{y}^1 &= \int_{-\infty}^{\infty} \left[P_0'(u(y^1,x)-u(y^2,x))\Psi(u(y^1,x)) + P_0(u(y^1,x)-u(y^2,x))\Psi'(u(y^1,x))\right]\\
&\qquad \times\ \nabla u(y^1,x)\,\rho(x)dx\\
\dot{y}^2 &= \int_{-\infty}^{\infty} \left[P_0'(u(y^2,x)-u(y^1,x))\Psi(u(y^1,x)) + P_0(u(y^2,x)-u(y^1,x))\Psi'(u(y^2,x))\right]\\
&\qquad \times\ \nabla u(y^2,x)\,\rho(x)dx
\end{aligned} \tag{25}$$

in the case of discrete and continuous voter distributions, respectively. Again this dynamical system remains unchanged under permutation of the platform indices.

[1] Strictly speaking covariance matrices are only positive semi-definite. However, eigenvalues 0 occur only in the degenerate cases where the voter positions are confined to a hyper-plane in issue space. We will neglect this complication in the following.

As in the other two-party models, $\mathcal{H}_2$,i.e., $z_v = 0$, is invariant. The Jacobian at a trivial fixed point for a discrete voter distribution is

$$\begin{aligned}
\frac{\partial \dot{y}_j^1}{\partial y_k^1} &= \frac{1}{V}\sum_v \Big[[2P_0'(0)\Psi'(u_v(y)) + \frac{1}{2}\Psi''(u_v(y))]\partial_k u_v(y)\partial_j u_v(y) \\
&\qquad + [P_0'(0)\Psi(u_v(y)) + \frac{1}{2}\Psi'(u_v(y))]\partial_k\partial_j u_v(y)\Big] \\
\frac{\partial \dot{y}_j^1}{\partial y_k^2} &= \frac{1}{V}\sum_v -\mathcal{P}'(0)\Psi'(u_v(y))\partial_k u_v(y)\partial_j u_v(y) \\
\frac{\partial \dot{y}_j^2}{\partial y_k^1} &= \frac{1}{V}\sum_v -\mathcal{P}'(0)\Psi'(u_v(y))\partial_k u_v(y)\partial_j u_v(y) \\
\frac{\partial \dot{y}_j^2}{\partial y_k^2} &= \frac{1}{V}\sum_v \Big[[2P_0'(0)\Psi'(u_v(y)) + \frac{1}{2}\Psi''(u_v(y))]\partial_k u_v(y)\partial_j u_v(y) \\
&\qquad + [P_0'(0)\Psi(u_v(y)) + \frac{1}{2}\Psi'(u_v(y))]\partial_k\partial_j u_v(y)\Big]
\end{aligned} \tag{26}$$

The corresponding expression for a continuous voter distribution is obtained by replacing $\frac{1}{V}\sum_v$ with $\int \ldots \rho(x)dx$. The dynamics on $\mathcal{H}_2$ is given by the differential equation:

$$\dot{y} = \frac{1}{V}\sum_v \Big[P_0'(0)\Psi(u_v(y)) + \frac{1}{2}\Psi'(u_v(y))\Big]\nabla u_v(y). \tag{27}$$

Little can be said in the general case. In particular, the expressions become rather unhandy for discrete voter distributions. We will therefore illustrate the behavior of equ.(25) using two simple continuous voter distributions $\rho(x)$. We shall restrict ourselves to the one-issue case. Furthermore, we assume the Enelow-Hinich-type voter utility function $u(y,x) = -(y-x)^2$ and a voting probability of the form $\Psi(u_v(y)) = \exp(\mu u_v(y))$ with $\mu > 0$, which is exponentially decreasing with the distance between voter and platform.

Let us consider the Gaussian case (with mean 0 and variance σ^2) first:
$\rho(x) = \frac{1}{\sigma\sqrt{2\pi}}\exp(\frac{-x^2}{2\sigma^2})$. Writing equ.(27) in continuous form, we obtain

$$\dot{y} = \frac{1}{\sigma\sqrt{2\pi}}\int_{-\infty}^{\infty}[P_0'(0) + \frac{1}{2}\mu]\exp\left(-\frac{x^2}{2\sigma^2} - \mu(y-x)^2\right)(-2)(y-x)\,dx. \tag{28}$$

It is easy to verify that this is of the form $\dot{y} = -Cy\exp(-\mu y^2)$ with a positive constant C. Thus the mean voter point $y = 0$ is the only fixed point. It is obviously stable within $\mathcal{H}_2$. The Jacobian at $y = 0$ is readily computed:

$$\mathbf{J} = \frac{4\mu P_0'(0)}{2\sqrt{2}\sigma a^3}\begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix} + \frac{2\mu^2}{2\sqrt{2}\sigma a^3}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - \frac{2P_0'(0)+\mu}{\sqrt{2}\sigma a}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \tag{29}$$

where $a = \sqrt{\frac{2\sigma^2\mu+1}{2\sigma^2}}$. The eigenvalues of $\mathbf{J}$ are

$$\begin{aligned} \lambda_1 &= \frac{1}{2\sqrt{2}\sigma a^3}\left[4P_0'(0)(3\mu - a^2) + 2\mu(\mu - a^2)\right] \\ \lambda_2 &= \frac{1}{2\sqrt{2}\sigma a^3}\left[(4P_0'(0) + 2\mu)(\mu - a^2)\right] \end{aligned} \tag{30}$$

The second eigenvalue, λ_2, is always negative since $\mu - a^2 = -1/(2\sigma^2)$. For the first eigenvalue, λ_1, a bifurcation occurs at $P_0'(0) = \frac{\mu(a^2-\mu)}{2(3\mu-a^2)}$. The mean voter fixed point is therefore stable if $\mu \le 1/(4\sigma^2)$ independent of $P_0'(0)$. For $\mu > 1/(4\sigma^2)$, it is stable for small values of $P_0'(0)$. The mean voter equilibrium becomes unstable if

$$P_0'(0) > \frac{\mu}{8\sigma^2\mu - 2} \tag{31}$$

in this case.

If we modify the model by using the expected difference in votes, W_p, as a party payoff function, we obtain the seemingly more complicated dynamical sysem

$$\begin{aligned} \dot{y}^1 &= \int_{-\infty}^{\infty} \nabla_{y^1} P_0(u(y^1,x) - u(y^2,x))\Psi(u(y^1,x)) - P_0(u(y^2,x) - u(y^1,x))\Psi(u(y^2,x))\,\rho(x)dx \\ \dot{y}^2 &= \int_{-\infty}^{\infty} \nabla_{y^2} P_0(u(y^2,x) - u(y^1,x))\Psi(u(y^2,x)) - P_0(u(y^1,x) - u(y^2,x))\Psi(u(y^1,x))\,\rho(x)dx \end{aligned} \tag{32}$$

Of course $\mathcal{H}_2$ is again invariant. Choosing Ψ, the voter distribution ρ, and the voter utility u as above, we find that, again, the mean voter equilibrium $y = 0$ is the only fixed point on $\mathcal{H}_2$. As above this fixed point is stable within the invariant manifold $\mathcal{H}_2$. The Jacobian at the mean voter equilibrium is

$$\mathbf{J} = \frac{(4P'(0) + \mu)(\mu - a^2)}{a^3\sigma\sqrt{2}}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \tag{33}$$

Table 1. Bifurcation at the Trivial Equilibrium

Distribution	Party Utility Function $E(y^1, y^2)$ equ.(25)	$W(y^1, y^2)$ equ.(32)
Gaussian	$P_0'(0) = \frac{\mu}{8\sigma^2\mu - 2}$	no bifurcation
Uniform	$P_0'(0) = \frac{\mu e^{-\mu}}{2\left[\sqrt{\frac{\pi}{\mu}}\mathrm{erf}\,(\sqrt{\mu}) - 3e^{-\mu}\right]}$	no bifurcation

and hence it is always stable.

Qualitatively similar results were obtained under the assumption of a uniform voter distribution in the interval $[-1, 1]$: The mean voter equilibrium is the only fixed point in $\mathcal{H}_2$, which is again invariant under the dynamics. In the model with E_p as party payoff function, a bifurcation at the mean voter equilibrium occurs at

$$P_0'(0) \geq \frac{\mu e^{-\mu}}{2\left[\sqrt{\frac{\pi}{\mu}}\mathrm{erf}\,(\sqrt{\mu}) - 3e^{-\mu}\right]} \tag{34}$$

given that

$$\sqrt{\frac{\pi}{\mu}}\mathrm{erf}\,(\sqrt{\mu}) \geq 3e^{-\mu}\,, \quad \text{i.e., if} \qquad \mu > \mu^* \approx 0.5861855742\,. \tag{35}$$

In the model with W_p as party payoff function, the mean voter equilibrium is again unconditionally stable.

Not much can be said in general about the dynamics of equ.(25). We expect a globally stable mean voter equilibrium for small values of μ and $P'(0)$ by analogy with the continuous cases discussed above. It is shown in (Stadler, 1998) that the voting dynamics with a continuous voter distribution can be approximated using discrete voter distributions in such a way, that the difference of the resulting vector fields is a (regular) perturbation. Hence we may draw voter ideal positions x_v from the voter distribution $\rho(x)$ and use the discrete version of equ.(25) as an approximation of the continuous version.

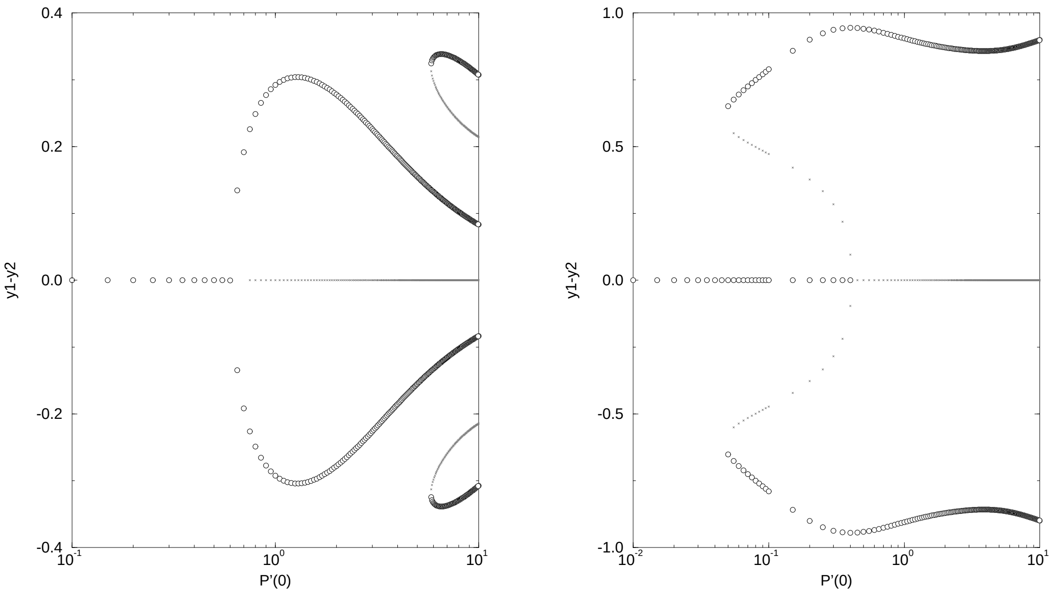

Figure 1: Bifurcation diagram for equ.(25). Parameters: 100 voters uniformly distributed in $[-1, 1]$, $u(y, x) = -(y - x)^2$, $\Psi(u) = \exp(-\mu u)$, $P(z) = (1 + \exp(az))^{-1}$. Stable fixed points are indicated by black circles, small gray dots denote saddle points. For small values of μ there is a supercritical pitchfork bifurcation (l.h.s., $\mu = 1$). For large values of μ one finds a saddle-node bifurcation (which is doubled due to the symmetry of the model) and a sub-critical pitchfork bifurcation (r.h.s., $\mu = 5$).

In order to study the dynamics outside the invariant manifold $\mathcal{H}_2$, numerical bifurcation diagrams have been computed, see fig. 1. The fixed point coordinates are calculated using Broyden's multidimensional secant method (Broyden, 1965), as implemented in *Numerical Recipes* (Press *et al.*, 1986). The eigenvalues of the Jacobian are computed by first reducing $\mathbf{J}$ to Hessenberg form and then applying the QR algorithm. 500 to 5000 independent initial values, uniformly distributed in the box $\mathcal{B}$ have been used in order to ensure that all fixed points are found. Computations were performed on a `Pentium` PC running `Linux`.

Bifurcation diagrams, among them the examples shown in fig. 1, show that a variety of different bifurcations occur in equ.(25). Of course, the exact parameter values of the bifurcation points depend quite strongly on the details of the underlying voter distribution. Different (random) choices of the voter positions are found to lead to qualitatively the same bifurcation diagrams, though the numerical values are different.

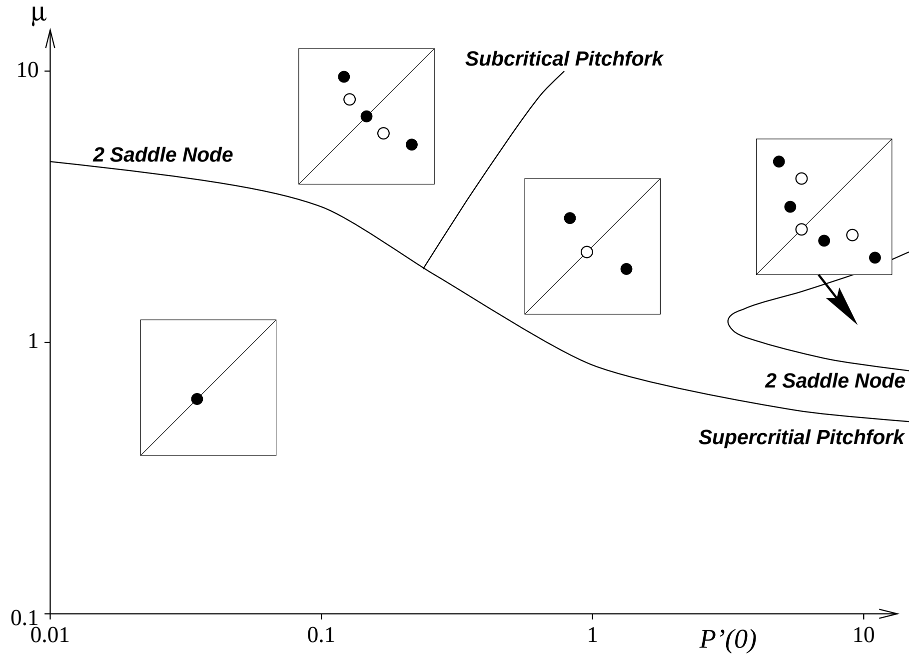

Figure 2: Schematic phase diagram for two-party models with utility-dependent abstention, equ.(25).
Parameters: $u(x, y) = -(y - x)^2$, $\Psi(u) = \exp(-\mu u)$, $P(z) = (1 + \exp(az))^{-1}$, 100 voters uniformly distributed in $[-1, 1]$.
The regions of the phase diagram are labeled by schematic phase portraits indicating the forward invariant box $\mathcal{B} = [-1, 1]^2$, the invariant manifold $\mathcal{H}_2$ (diagonal line), stable fixed points (full circles) and saddle points (open circles). We find 4 different phase portraits with up to 7 equilibria. For small $P'(0)$ and small μ there is a single stable equilibrium on $\mathcal{H}_2$ close to the mean voter position. The exact locations of the bifurcations depend on the voter distribution.

The schematic phase diagram in fig. 2 was obtained from a series of 1-parameter bifurcation diagrams (with μ fixed and parameter $P'(0)$). Within the range of the numerical survey ($\mu, P'(0) \leq 10$) we encounter four different generic phase portraits. For small values of both μ and $P'(0)$ there is only a single equilibrium on $\mathcal{H}_2$ which is stable. An increase of μ gives rise to bifurcations. With μ large enough, for small values of $P'(0)$ we find two saddle node bifurcations which are related by symmetry. As $P'(0)$ increases, first a saddle-node and then a subcritical pitchfork bifurcation occur. As a consequence, up to 7 equilibria were found. Within the parameter range of the numerical survey there is no Hopf bifurcation.

4. Discussion

We have extended the differential equations (Miller & Stadler, 1998) describing the adaptive behavior of the platforms of two parties in a spatial voting model to incorporate the possibility of voter abstention. Extending earlier results for the complete participation case, we showed that the mean voter equilibrium remains asymptotically stable if non-policy issues are taken into account.

The probability that a given voter votes for a particular party depends on the utility difference between both parties for the given voter in such a way that the party yielding the higher utility is the one that is more likely to receive the vote. Abstention is modeled in two different ways: we first assume that voting becomes uninteresting if the utility differences between the parties are small. This is in particular the case if the platform positions of the two parties are similar. We find that such a modification of the voting probabilities does not change the qualitative behavior of the platform dynamics, i.e., the mean voter equilibrium remains stable.

An alternative approach assumes that the probability for participating in the election decreases with the distance of the platforms from a voter's ideal point. In this case we found that the qualitative behavior of the platform dynamics depends strongly on whether the expected fraction of votes that a parties receives, or the expected fraction of votes that a party is ahead of its competitor is used as the party payoff function. With expected differences, there are no bifurcations, while using the expected fraction of votes we observe that the mean voter fixed point becomes unstable for critical voters and participation probabilities that drop sufficiently fast with the distance between the voter and the platforms.

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