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Adaptive Platform Dynamics in Multi-Party Spatial Voting

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ABSTRACT. We explore the dynamics of multiple competing political parties under spatial voting. Parties are allowed to modify their positions adaptively in order to gain more votes. The parties in our model are opportunistic, that means they try to maximize their share of votes regardless of any ideological position. Each party makes small corrections to its current platform in order to increase its own utility by means of steepest ascent in the variables under its own control, i.e. by locally optimizing its own platform.

We show that in models with more than two parties bifurcations at the trivial equilibrium occur if only the voters are critical enough, that is, if they respond strongly to small changes in relative utilities. A numerical survey in a three-party model yields multiple bifurcations, multi-stability, and stable periodic attractors that arise through Hopf bifurcations. Models with more than two parties can thus differ substantially from the two-party case, where it has been shown that under the assumptions of quadratic voter utilities and complete voter participation there is always a globally stable equilibrium that coincides with the mean voter position.
KEYWORDS: spatial voting model, adaptive platform dynamics, electoral landscape, bifurcation, stability analysis

1. Introduction

The importance of adaptive processes in economic theory has long been recognized, see, for example, Malthus (1798). In the field of Political Science studies on spatial voting theory can be traced back as far as to Hotelling's work in the 1920s. The core of the theory was developed in the classical works by Smithies (1941), Downs (1957), and Black (1958).

Spatial voting theory describes the interactions of two classes of agents: *voters* and *candidates* or parties (Enelow and Hinich, 1984b). A key element in this type of models is the assumption that political issues can be quantified and therefore voters as well as candidates can be reprensented by points in an abstract metric space, the *issue space*. Usually one regards the issue space as the (Euclidean) vector space $\mathbb{R}^I$ that is spanned by the I issues. Each voter v has a position x_v in issue space, called her *ideal point* that represents her opinion on each issue. Similarly, each party or candidate p is described by the *platform position* $y^p \in \mathbb{R}^I$. Spatial theory is an attractive framework to analyze choice because Euclidean geometry is easy to visualize and the language of politics itself is replete with spatial references, such as the terms “left”,“right”, “moving left”, etc.

Extensive empirical analyses have been performed on U.S. elections by Weisberg and Rusk (1970, 1972), Rabinowitz (1976), Poole and Rosenthal (1984), Enelow and Hinich (1984b, 1984a), Enelow (1988), and Endersby and Hinich (1992). Applications to congressional voting include work by MacRae (1958, 1970), Krehbiel and Rivers (1988, 1990), and Poole and Rosenthal (1989, 1991). In these studies data from the American National Election Study, in particular so-called “thermometer scores” (for which a respondent is asked to gauge support or disapproval for a particular candidate on a numerical scale), are used to construct spatial maps of voter and candidate positions.

Empirical tests have generally supported spatial election theory but the estimation methods employed to produce the spatial representations of voters have raised serious statistical issues which have not been fully resolved. One of these issues is determining the number of dimensions, which is rather difficult because the number of estimated parameters increases with the number of dimensions (Poole *et al.*, 1992).

A troubling result of voting theory is the general lack of stable equilibria once the assumption of a symmetric ideal point distribution is relaxed. Under simple preference-based voting it has been shown that in election theory, a pure strategy equilibrium for the candidates becomes quite rare when the election concerns two or more issues (Plott, 1967; Davis *et al.*, 1972; Kramer, 1973). This discovery has its counterpart in the committee-voting theory when voting takes place over multidimensional policy alternatives. There exists rarely a policy alternative that cannot be defeated in a majority vote. An even more disappointing discovery has been made in the absence of a majority rule equilibrium: the majority preference relation may engulf the entire outcome in one gigantic cycle (McKelvey and Ordeshook, 1976; McKelvey, 1979). This discovery caused some theorists to despair of ever being able to predict candidate behavior or committee-voting outcomes (Riker, 1980).

For two-party systems, computer simulations by Kollman, Miller, and Page (1992, 1996) and a differential equations model by Miller and Stadler (1998) showed, however, that an adaptive dynamics approach to spatial voting restores stable outcomes to a certain extent, at least in the long run. In this contribution, their results are generalized to political systems with an arbitrary number of parties.

2. Adaptive Platform Dynamics

2.1. Voter Utility Functions

Viewed in simplest spatial terms, the voter will cast her vote for the candidate "closest" to her. This concept is formalized by the *utility function* $u_v(y)$ which measures the degree of agreement of a particular candidate or political platform position y with voter v's ideal point. Naturally, this utility function has its unique maximum at the voter's ideal point, and it decreases with the distance from x_v. In the context of spatial voting models, sometimes the term dissatisfaction function instead of utility function is used.

Davis *et al.* (1970) broadened the scope of the spatial voting literature by including weights or strengths in a multidimensional model. Throughout their book, Enelow and Hinich (1984b) therefore use a quadratic voter utility function of the form

$$u_v(y^p) = m_v^p - (y^p - x_v)\mathbf{S}_v(y^p - x_v)\,. \tag{2.1}$$

In general $\mathbf{S_v}$ is a non-negative definite matrix. In most applications, however, one assumes that it is diagonal, in which case the entries $(\mathbf{S}_v)_{ii} = s_{vi}$ are the *strength factors* s_{vi} that measure how strongly voter v feels about issue i. The additive term m_{vp} is the so-called *non-policy value* of party p to voter v. Non-policy values, which we will neglect in this contribution, measure all those aspects of a candidate or party that are not subject to adaptation such as age, religion, or gender. Completely indifferent voters, for who $s_{vi} = 0$ for all issues, do not influence the expected outcome of elections and can therefore be neglected in our model. For a more general voter utility function, we may interpret the curvatures $s_{vk} = -\frac{1}{2}\frac{\partial^2 u_v}{\partial {y_k}^2}(x_v)$ at a voter v's ideal point as her strength factors. Off-diagonal elements in $\mathbf{S}_v$ or mixed second derivatives signify correlations among the various issues as perceived by voter v.

Some voters base their decisions among candidates entirely on a single issue, such as abortion, civil rights, or foreign policy. Such a voter would attach weight 1 to a single issue and 0 to all other issues. Most voters probably distribute their strengths s_{vi} more evenly. The correlation between ideal points and strength may be used to characterize different types of voters as in a model by Kollman *et al.* (1997):

$$\begin{array}{rlc} s_{vi} = & 1 - |x_{vi}| & \text{centrist} \\ s_{vi} = & |x_{vi}| & \text{extremist} \\ s_{vi} = & 1/2 & \text{uniform} \end{array} \tag{2.2}$$

These authors consider *centrist* voters which place more weight on issues on which they have moderate views, *extremist* voters placing more weight on issues on which they have extreme views, and *uniform* voters with equal weights on every issue. In their model, strength is simply a function of ideal points.

2.2. Party Payoffs and Electoral Landscapes

The basic assumption of all spatial voting models is that each voter will vote for the party that yields the largest value of $u_v(y^p)$. However, the information about a platform position as well as her own location in issue space will be known to the individual voter only with a certain accuracy (Enelow and Hinich, 1984b, chapter 7). We model this behavior following Miller and Stadler (1998) by introducing the probability $\mathcal{P}$ that voter v chooses platform (party, candidate) 1 given the utility differences between the platform positions of all involved parties. We refer to $\mathcal{P}$ as the *response function* of the voter. For the sake of mathematical tractability we shall assume throughout this work that the function $\mathcal{P} : \mathbb{R} \to [0,1]$ does *not* depend on the individual voter v. In the case of two parties $\mathcal{P}$ will of course be a sigmoidal function.

If the voters' decisions are perfect in the sense that they always match the utilities, then $\mathcal{P}_\infty = 1$ if party 1 has the largest utility for voter v. If m parties have the same largest utility as party 1, then $\mathcal{P}_\infty = 1/m$, since voter v will vote for each of these m parties with the same probability $1/m$; otherwise $\mathcal{P}_\infty = 0$. Almost the entire literature on spatial voting assumes complete knowledge and hence works with the discontinuous function $\mathcal{P}_\infty$. Here we shall assume that $\mathcal{P}$ is a continuously differentiable approximation of $\mathcal{P}_\infty$.

The outcome of an election, that is the fraction of votes that a party receives, determines the party's *payoff.* More precisely, the payoff of party p is the *expected*

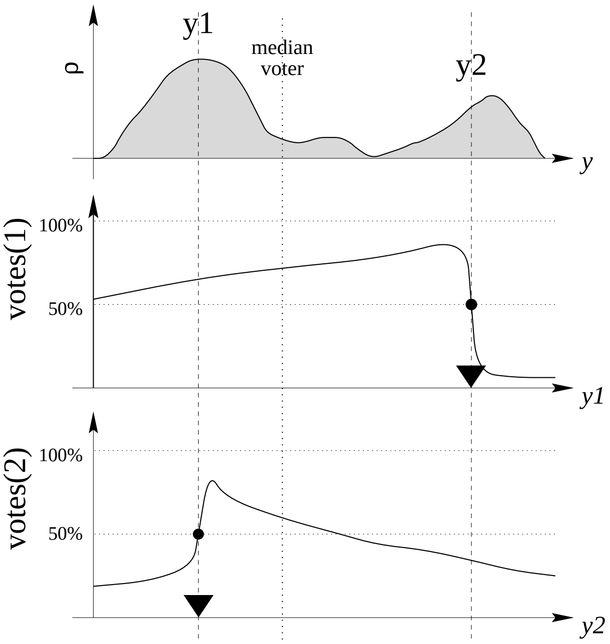

Figure 1. Voter distribution and electoral landscapes.
The upper part of the figure shows the voter distribution ρ in a 1-dimensional issue space. The two parties occupy at a given time the positions y^1 and y^2, respectively. The lower portion of the plot shows the expected number of votes, for parties 1 and 2 as a function of their own platforms, given that the other party stays at its current position. We call these curves the "electoral landscapes" perceived by the two parties. The shape of the electoral landscapes can be explained by the fact that party 1 receives the votes of all voters to the left of $(y_1 + y_2)/2$ (apart from the uncertainties introduced by $\mathcal{P}$). Hence Party 1 could receive more votes if it took a position closer to party 2, and conversely, party 2 could increase its share by moving closer to party 1.

fraction of votes

$$\begin{aligned} &E_p(y^p|y^1,\ldots,y^{p-1},y^{p+1},\ldots,y^P) = \\ &\quad \tfrac{1}{V}\textstyle\sum_v \mathcal{P}(d_v(y^p,y^1),\ldots,d_v(y^p,y^{p-1}), \\ &\quad\;\; d_v(y^p,y^{p+1}),\ldots,d_v(y^p,y^P),d_v(y^1,y^2),d_v(y^1,y^3),\ldots,d_v(y^1,y^P), \\ &\quad\;\; d_v(y^2,y^3),\ldots,d_v(y^2,y^P),\ldots,d_v(y^{P-1},y^P))\,, \end{aligned} \tag{2.3}$$

i.e., the sum over the probabilities with which the voters choose party p. Here $d_v(y^k, y^l) = u_v(y^k) - u_v(y^l)$ is the utility difference of the platforms y^k and y^l for voter v. The surface $E_p(y^p|\ldots)$, with the positions of all other parties fixed, has been termed the *electoral landscape* of party p (Kollman *et al.*, 1997;

Kollman *et al.*, 1998), see Figure 1. The landscape metaphor is a common model in adaptive search. Applications to political science include (Axelrod, 1986) and (Axelrod and Bennet, 1993). In the theory of biological evolution it goes back to Wright (1932), see also the book by Kauffman (1993).

It is not hard to see that the vote landscapes in a two-party one-issue model are always unimodal, see Figure 1. This is not necessarily true for more than one issue or more than two parties. There is a relationship between the distribution of voters' strengths and the slopes and positions of peaks on an electoral landscape (Kollman *et al.*, 1997). The electoral landscapes defined in equ.(2.3) depend in addition on the slope of the sigmoidal function $\mathcal{P}$.

Instead of specifying the individual position of each voter v it may be more realistic and/or more convenient to introduce the density $\rho(x)$ of voters in issue space. It is normalized in the usual way: $\int_{\mathbb{R}^I} \rho(x)dx = 1$. Instead of $u_v(y)$ we now need to specify the utility function $u(y,x)$ of a platform y for a voter with position x. Furthermore, we may assume that strength is a function of the ideal point, as defined in equ.(2.2). In analogy with the definition of d_v we set $d^x(y^k, y^l) := u(y^k, x) - u(y^l, x)$, where $u(y,x)$ denotes the utility of party platform y for a voter with ideal point x. For instance, we have

$$u(y,x) = -\sum_{i=1}^{I} s_i(x)(y_i - x_i)^2 \tag{2.4}$$

where the strength factor is a function of the platform position as for instance in equ.(2.2). The expected outcome of an election is then

$$\begin{aligned} &E_p(y^p | y^1, \dots, y^{p-1}, y^{p+1}, \dots y^P) = \\ &\int_{\mathbb{R}^I} \mathcal{P}\left(d_x(y^p,y^1),\dots,d_x(y^p,y^{p-1}),d_x(y^p,y^{p+1}),\dots,d_x(y^p,y^P),\dots\right) \rho(x)\, dx\,. \end{aligned} \tag{2.5}$$

A more general model could be constructed by introducing a joint distribution $\rho(x,s)$ of voter positions x and strengths s. We remark, finally, that we may translate the "discrete" model (2.3) into the "continuous" form by defining

$$\rho(x) = \frac{1}{V}\sum_{v} \delta(x - x_v)\,. \tag{2.6}$$

Here $\delta(\,.\,)$ denotes the δ-distribution. It can be shown that the dynamics of platform adaptation is essentially the same whether a continuous or a discrete voter distribution is assumed (Stadler, 1998b). More precisely, if a continuous density $\rho(x)$ decreases sufficiently fast with $|x|$, than it can be approximated by a finite discrete voter distribution in such a way that the differences in the electoral landscapes E_p amount to a (regular) perturbation in the sense of (Hirsch and Smale, 1974, chap. 16). Hence we can expect the same qualitative behavior of spatial voting models, equ.(2.7) below, with both discrete and continuous voter distributions.

2.3. Platform Dynamics

The emphasis of our model is on an *adaptive dynamical system*, whereby global consequences emerge from locally adapting candidates. The inspiration for this model comes from the computational results of Kollman, Miller, and Page (1992, 1996). They found that parties following simple locally adaptive rules rapidly converged toward common platforms. While the ideas of probabilistic voting (for a general review see (Coughlin, 1990)) and locally restricted strategy searches in such models (Coughlin and Nitzan, 1981; Samuelson, 1984) have been widely discussed, here we assume that parties do not start at an identical status quo platform, and that their ability to maximize voter support is limited to climbing the local voting gradient.

Our approach is based on the assumption that each party p tries to increase its share of votes by small corrections to its platform y^p, see Figure 2. Each party attempts to increase its own utility by means of steepest ascent in the variables under its own control, i.e., by locally optimizing its own platform under the assumption of a fixed position for the platforms of all other parties. The other players of course react to the changes in our party's platform and adjust their positions. Assuming that these platform adjustments are being conducted continuously, guided by, say, opinion polls, we argue that

$$\dot{y}^p = \nabla_{y^p} E_p(y^1, ..., y^P), \qquad p = 1, \ldots, P \tag{2.7}$$

is a plausible functional form for the dynamics of platform adjustment. This dynamics corresponds to simultaneous hill-climbing of each party on its own electoral landscape. However, a party's vote landscape is constantly changing due to the movements of all other parties in issue space. This type of game dynamics was introduced by Miller and Stadler (1998) for a two-party spatial voting model. Note, however, that the dynamics here depends on the fraction of votes, not on the number of votes. As a consequence, the velocity of adaptation does not depend on the size of the electorate.

The platform dynamics is a dynamical system, equ.(2.7), "living" on the *phase space* $\mathbb{R}^{PI}$. It will sometimes be necessary to directly refer to vectors in phase space. In the following, we will use the notation $x, y \in \mathbb{R}^I$ for positions (vectors) in issue space and $\vec{y} \in \mathbb{R}^{PI}$ for vectors in phase space.

It seems interesting to compare the dynamics of party platforms, equ.(2.7), with some more standard models of game dynamics. Probably the most widely used type of game dynamics is the *replicator equation* (Hofbauer and Sigmund,

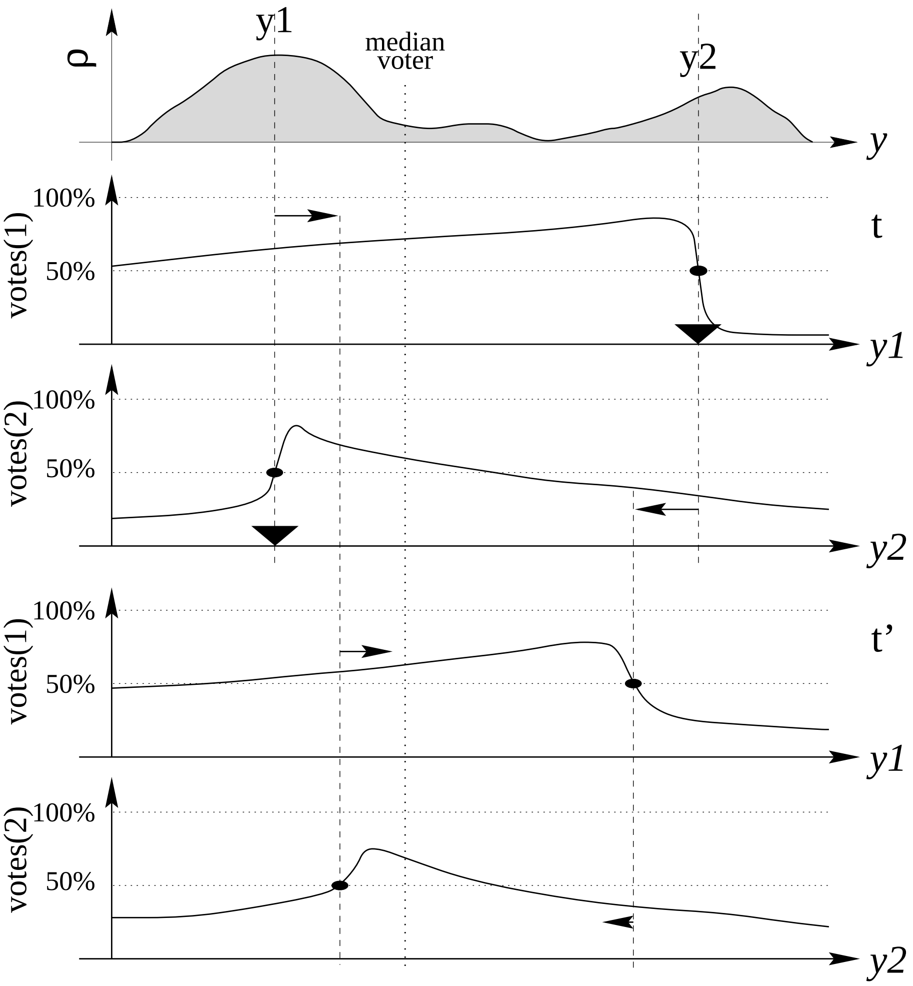

Figure 2. The topmost part of the figure shows again the voter distribution ρ in a 1-dimensional issue space. At time t, both parties will alter their positions following the gradient of their own expected election outcome in issue space. By changing its position y^2, however, party 2 changes the vote landscape for party 1 at time t', and *vice versa*. The "vote landscapes" thus change at the same time scale at which the parties try to hill-climb on them.

1988). One assumes a (usually finite) set of pure strategies (but see e.g. the work of Bomze (1988,1990,1991) and Zeeman (1981) for the infinite case). The multi-population version of the replicator equation (Taylor, 1979; Weibull, 1995) might be an alternative starting point for a model of platform adaptation. In this picture the platform of party p would be represented as a superposition of "pure" positions, the weights of which would become the dynamical variables. Such an approach, however, feels much less natural than the simple gradient dynamics. In particular, the choice of a set of pure strategies would be rather artificial in the context of spatial voting theory.

Hofbauer and Sigmund (1990) consider the evolution of an essentially monomorphic population under the assumption that a small number of mutants y that are very similar to the consensus x test out alternatives. Let $E(y,x)$ be the fitness (payoff) of such a mutant in a monomorphic x-population. *Adaptive Dynamics* assumes that the whole population moves into the direction of the most promising mutant: $\dot{x} = \nabla_y E(y,x)\big|_{y=x}$. In contrast to replicator equations, this model is not restricted to a concentration simplex. It describes, however, the time evolution of a single population rather than the coevolution of two populations. A multi-population version, however, might be useful in a more detailed model of the mechanism by which a party modifies its platform.

3. Two Parties

We briefly review here the 2-party system with V voters, I issues, and complete participation, that was first described in (Miller and Stadler, 1998). We will write P_0 instead of $\mathcal{P}$ in the special case of a model with two parties and complete participation of voters, i.e., $P_0(z_v) + P_0(-z_v) = 1$. Explicitly, the model reads in component-wise notation

$$\dot{y}_j^1 = \frac{1}{V}\sum_v P_0{}'(z_v)\partial_j u_v(y^1) \qquad \dot{y}_j^2 = \frac{1}{V}\sum_v P_0{}'(-z_v)\partial_j u_v(y^2) \tag{3.1}$$

where $\partial_j u_v(y^p)$ denotes the j-th component of the vector $\nabla u_v(y^p)$. Note that $P_0{}'(-z_v) = P_0{}'(z_v)$.

It is not hard to verify that the manifold $\mathcal{H}_2 = \left\{\vec{y} \in \mathbb{R}^{2I} | y^1 = y^2\right\}$ on which z_v vanishes is invariant under the game dynamics (3.1). The system is symmetric under exchange of the platform indices, and hence fixed points outside $\mathcal{H}_2$, if they exist at all, have to come in pairs: if $(\hat{a},\hat{b})$ is a fixed point of equ.(3.1), so is $(\hat{b},\hat{a})$. Furthermore, the box $\mathcal{B}$ spanned by the extremal voter positions is forward invariant and all orbits are eventually bounded within $\mathcal{B}$. Introducing the *average voter utility*

$$U(y) = \frac{1}{V} \sum_v u_v(y) \tag{3.2}$$

we find that equ.(3.1) reduces to the gradient system $\dot{y} = P_0{}'(0)\nabla U(y)$ within the manifold $\mathcal{H}_2$. We shall call a fixed point $(\hat{x}, \hat{x}) \in \mathcal{H}_2$ a *trivial fixed point* of the voting dynamics (3.1). Note that the trivial fixed points are thus the critical points of U. The trivial fixed point is *globally* stable provided all voter utility functions u_v are concave. If this condition is relaxed, bifurcations may occur (Stadler, 1998a). For later reference we recall that the Jacobian matrix of equ.(3.1) at a trivial fixed point is of the form:

$$\mathbf{J}(y, y) = P_0{}'(0) \begin{pmatrix} \mathbf{H}(y) & 0 \\ 0 & \mathbf{H}(y) \end{pmatrix}, \tag{3.3}$$

where $\mathbf{H}(y)$ is the Hessian matrix of the average voter utility $U(y)$. It has the entries

$$\mathbf{H}_{jk}(y) = \frac{\partial^2}{\partial y_j \partial y_k} U(y) \tag{3.4}$$

The stability of a fixed point on $\mathcal{H}_2$ is therefore determined by the curvature of the average voter dissatisfaction $U(y)$. The stable trivial fixed points are therefore the maxima of $U(y)$. In particular, there is a unique equilibrium, which is globally stable, if the voter utility functions are concave (Miller and Stadler, 1998). Recently it was shown that a large level of abstention may cause bifurcations at the trivial equilibrium (Stadler, 1998a).

4. Multiparty Response Functions

It is rather straightforward to generalize equ.(3.1) to more than two parties. In fact, eqns.(2.3) and (2.7) already describe the dynamical system. The only ingredient that still requires some more discussion is the response function $\mathcal{P}$, i.e., the way in which voters compare the utilities of different platforms in a multiparty system.

In the following we use the shorthand

$$d^v_{pq} = d_v(y^p, y^q) = u_v(y^p) - u_v(y^q)\,. \tag{4.1}$$

for the difference in utility of the platforms y^p and y^q for voter v. Note that $d^v_{pq} = -d^v_{qp}$. We assume that the response function $\mathcal{P}$ depends on all pairwise utility comparisons. We set

$$\mathcal{P}(d^v_{12}, d^v_{13}, \ldots, d^v_{1P}; d^v_{23}, d^v_{24}, \ldots, d^v_{P-1,P}) \tag{4.2}$$

for the probability that voter v votes for party 1. This function must satisfy a number of symmetry requirements because the names of the parties are irrelevant. Thus for any permutation π of the party names that leaves 1 invariant, i.e., $\pi(1) = 1$ we have

$$\begin{aligned}\mathcal{P}(d_{12}, d_{13}, \ldots, d_{1P}; d_{23}, d_{24}, \ldots, d_{2P}, d_{34}, \ldots, d_{P-1,P}) = \\ \mathcal{P}(d_{\pi(1)\pi(2)}, d_{\pi(1)\pi(3)}, \ldots, d_{\pi(1)\pi(P)}; d_{\pi(2)\pi(3)}, d_{\pi(2)\pi(4)}, \ldots \\ \ldots, d_{\pi(2)\pi(P)}, d_{\pi(3)\pi(4)}, \ldots, d_{\pi(P-1),\pi(P)})\end{aligned} \tag{4.3}$$

For short, we write equ.(4.3) as $\mathcal{P}(\pi \circ \mathsf{d}) = \mathcal{P}(\mathsf{d})$ for all permutations with $\pi(1) = 1$. The notation $\pi\circ$ means that the permutation acts on all party labels. In the same vain, equ.(2.7) must remain unchanged under a relabeling of the parties, i.e., the probability that voter v votes for party p is simply $\mathcal{P}(\pi \circ \mathsf{d})$ where π is any permutation satisfying $\pi(p) = 1$. A more explicit form of equ.(2.7) is therefore:

$$\dot{y_k}^p = \frac{1}{V} \sum_v \sum_{q \neq p}^{P-1} \partial_q \mathcal{P}(\pi \circ \mathsf{d}) \partial_k u(y^p, x_v)\,. \tag{4.4}$$

where π is a permutation satisfying $\pi(p) = 1$, and $\partial_q \mathcal{P}$ denotes the partial derivative of $\mathcal{P}$ with respect to its q-th argument. It is clear that equ.(4.4) has indeed full permutation symmetry.

In the following we will oftentimes need partial derivatives of the form $\frac{\partial \mathcal{P}}{\partial y_k^p}$. Permutation symmetry implies that it is sufficient to explicitly calculate the partial derivatives

$$\frac{\partial \mathcal{P}}{\partial y_k^1}\,, \qquad \frac{\partial^2 \mathcal{P}}{\partial y_k^1 \partial y_j^1} \qquad \text{and} \qquad \frac{\partial^2 \mathcal{P}}{\partial y_k^1 \partial y_j^2}\,. \tag{4.5}$$

All other partial derivatives can be obtained using a permutation π of the indices.

The I-dimensional surface $\mathcal{H} = \{y | y^1 = y^2 = \ldots = y^P\}$ is invariant, as are of course all manifolds on which two or more platforms coincide. As a consequence of the symmetry, fixed points that are not located on any of the invariant manifolds must have multiplicity $P!$. As in the two-party case, we call a fixed point within $\mathcal{H}_P$ *trivial*. The manifold $\mathcal{H}$ is characterized by $\mathsf{d} = \mathsf{o}$. The relevant partial derivatives of $\mathcal{P}$ on $\mathcal{H}$ are therefore determined by only three parameters:

$$\mathcal{P}' = \partial_1 \mathcal{P}(\mathsf{o})\,, \qquad \mathcal{P}'' = \partial_1 \partial_1 \mathcal{P}(\mathsf{o})\,, \qquad \text{and} \quad \hat{\mathcal{P}} = \partial_1 \partial_2 \mathcal{P}(\mathsf{o})\,. \tag{4.6}$$

In addition, it is not hard to verify $\partial_P \mathcal{P}(\mathsf{o}) = 0$ and $\sum_{p=1}^{P-1} \partial_p \partial_P \mathcal{P}(\mathsf{o}) = 0$.

Furthermore, it seems natural to require that the probability of voting for party 1 increases with increasing values of d_{1q}, $q \neq 1$ if all other differences remain unchanged. Thus we require that the *monotonicity condition*

$$\partial_q \mathcal{P}(\mathsf{d}) > 0 \tag{4.7}$$

is satisfied for $q = 1, \ldots, P$. As a consequence of condition (4.7), the orbits are bounded within the PI-dimensional box $\mathcal{B}^P$, where $\mathcal{B}$ is the I-dimensional box spanned by the extreme voter positions. For a formal proof we refer to (Stadler, 1998b).

Finally $\sum_{p=1}^{P} \mathcal{P}_p = 1$ if all voters participate in the election. In this contribution we shall mostly be concerned with *complete participation*.

Let us now consider an explicit example of a multiparty response function. Let $P_0 : \mathbb{R} \to [0,1]$ be a symmetric *sigmoidal* function. That is, P_0 is monotonically increasing on $\mathbb{R}$, $P_0(d)$ approches 1 for $d \to \infty$ and 0 for $d \to -\infty$, P_0 is concave for $d > 0$ and convex for $d < 0$, and $P(d) + P(-d) = 1$. Note that P_0 may be regarded as a response function in a two-party model with complete participation. Now consider

$$\mathcal{P}(\mathsf{d}) = \frac{\prod_{p=2}^{P} P_0(d_{1p}^v)}{\sum_{i=1}^{P} \prod_{p \neq i}^{P} P_0(d_{ip}^v)} \tag{4.8}$$

It is not hard (but a bit tedious) to verify that $\mathcal{P}$ has the required symmetries, and satisfies the monotonicity condition (Stadler, 1998b). It corresponds to complete participation by construction of the denominator. A short calculation yields the three parameters describing the behavior of $\mathcal{P}$ near the origin o:

$$\mathcal{P}' = \frac{2}{P} P_0'(0)\,, \qquad \mathcal{P}'' = 0\,, \qquad \hat{\mathcal{P}} = \frac{4(P-1)}{P^2} P_0'(0)^2\,. \tag{4.9}$$

5. Multiparty Dynamics

5.1. Dynamics on $\mathcal{H}$

While it seems to be impossible to explicitly compute the coordinates of equilibria outside $\mathcal{H}$, let alone their stabilities, a complete analysis of the behavior of equ.(4.4) is feasible. First we observe

$$\dot{y}^p = \frac{1}{V} \sum_v \left[\sum_{q \neq p} \partial_q \mathcal{P}(\mathsf{o}) \right] \nabla u_v(y^p) = (P-1)\mathcal{P}' \, \nabla U(y^p) \tag{5.1}$$

The equilibria on the manifold $\mathcal{H}$ are thus exactly the critical points of the average voter utility $U(y)$, independent of the number P of parties. Consequently, for concave voter utility functions there is a single trivial equilibrium which is the ω-limit of all trajectories starting in $\mathcal{H}$.

5.2. Alliances

It is easy to verify that the linear manifolds $\mathcal{H}_{pq}$ on which two parties p and q have the same platform are invariant. We may interpret this situation as an alliance formed by the parties p and q. The platform dynamics, however, does not reduce to a $P-1$ party model in this case: Each party obtains a fraction $1/P$ of the votes on any point of $\mathcal{H}$, hence the pq-alliance would receive a share of $2/P$ of the total number of votes in a P-party setting, while it would receive only $1/(P-1)$ in a "true" $P-1$ party model.

In some cases, namely when all alliances are formed by the same number of parties, it is sufficient to regard the reduced model. As an examples we show that two alliances composed of two parties each behave just like a two-party system. Without losing generality, let us assume $y^1 = y^3$ and $y^2 = y^4$. With

$$\mathcal{P}_1(\mathsf{d}) = \mathcal{P}(d_{12}, d_{13}, d_{14}, d_{23}, d_{24}, d_{34}) = \frac{P_0(d_{12})P_0(d_{13})P_0(d_{14})}{\sum_{i=1}^{P}\prod_{p\neq i}^{P} P_0(d_{ip})} \tag{5.2}$$

on the surface $\mathcal{H}^2{}_4 := \{y \in \mathbb{R}^{4I}\,|y^1 = y^3, y^2 = y^4\}$ we have the following expressions for the probabilities of voting for the four parties:

$$\begin{aligned} \mathsf{p}_1 = \mathsf{p}_3 = \quad & \frac{P_0(z)^2}{2\big(P_0(z)^2 + P_0(-z)^2\big)} & = \tfrac{1}{2}\tilde{\mathcal{P}}(z) \\ \mathsf{p}_2 = \mathsf{p}_4 = \quad & \frac{P_0(-z)^2}{2\big(P_0(z)^2 + P_0(-z)^2\big)} & = \tfrac{1}{2}\tilde{\mathcal{P}}(-z) \end{aligned} \tag{5.3}$$

where $z = d_{12}$. A short computation verifies that $\tilde{\mathcal{P}}(z)$ is indeed a sigmoidal function whenever $P_0(z)$ is sigmoidal, and hence the four-party system reduces to a two-party system with a modified response function. For the details see (Stadler, 1998b).

5.3. Stability of the Trivial Equilibrium

The stability of a fixed point is determined by its Jacobian matrix. A direct computation for an arbitrary position on $\mathcal{H}$ yields

$$\begin{aligned} \mathbf{J} \quad = \quad & \Big[\mathcal{P}'' + (P-2)\hat{\mathcal{P}}\Big]\,\mathbf{C}(y) \otimes \begin{pmatrix} P-1 & -1 & -1 & \dots & -1 \\ -1 & P-1 & -1 & \dots & -1 \\ \vdots & \vdots & \vdots & & \vdots \\ -1 & -1 & -1 & \dots & P-1 \end{pmatrix} \\ + \quad & [(P-1)\mathcal{P}']\,\mathbf{H}(y) \otimes \begin{pmatrix} 1 & 0 & 0 & \dots & 0 \\ 0 & 1 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \dots & 1 \end{pmatrix} \end{aligned} \tag{5.4}$$

where the matrices $\mathbf{C}(y)$ and $\mathbf{H}(y)$ are defined component-wise as follows:

$$\begin{aligned} \mathbf{C}_{kj}(y) &= \tfrac{1}{V}\textstyle\sum_{v=1}^{V} \partial_k u_v(y)\partial_j u_v(y) \\ \mathbf{H}_{kl}(y) &= \tfrac{1}{V}\textstyle\sum_{v=1}^{V} \partial_k\partial_l u_v(y) \end{aligned} \tag{5.5}$$

$\mathbf{H}(y)$ is simply the Hessian of the average voter utility U. In order to interpret $\mathbf{C}(y)$ we first note that $(1/V)\sum_v \nabla u_v(y) = \nabla U(y) = 0$ if and only if y is the location of a fixed point in $\mathcal{H}$. Thus $\mathbf{C}$ is the covariance matrix of the components of the vectors $\nabla u_v(y)$ at a fixed point of (4.4). Hence $\mathbf{C}$ is non-negative definite on any fixed point. For continuous voter distributions $\rho(x)$ we obtain a completely analogous representation:

$$\begin{aligned} \mathbf{C}_{kl}(y) &= \textstyle\int_{\mathbb{R}^I} \partial_l u(y,x)\partial_k u(y,x)\,\rho(x)dx \\ \mathbf{H}_{kl}(y) &= \textstyle\int_{\mathbb{R}^I} \partial_k\partial_l u(y,x)\,\rho(x)dx \end{aligned} \tag{5.6}$$

Again $\mathbf{H}$ is the Hessian of U, and $\mathbf{C}_{kl}$ can again be interpreted as a covariance, this time of the x-dependent function $\partial_l u(y,x)$ and $\partial_k u(y,x)$ with y fixed at the fixed point on $\mathcal{H}$.

The tensor product structure of the Jacobian, equ.(5.4) allows us to express the spectrum of $\mathbf{J}$ in terms of the matrices $\mathbf{C}(y)$ and $\mathbf{H}(y)$. The second factor in the first line is of the form $P\mathbf{I}-\mathbf{L}$, where $\mathbf{I}$ is the identity matrix and $\mathbf{L}$ is the matrix which has 1 in every entry. It is well-known that the eigenvalues of $\mathbf{L}$ are P (with multiplicity 1) and 0 (with multiplicity $P-1$). Therefore, the eigenvalues of $P\mathbf{I}+(-\mathbf{1})\mathbf{L}$ are P (with multiplicity $P-1$) and 0 (with multiplicity 1).

In order to simplify the interpretation we shall restrict ourselves to the response function equ.(4.8). Equ.(5.4) then becomes

$$\mathbf{J} = \frac{4(P-1)(P-2)}{P^2} P_0'(0)^2 \mathbf{C}(y) \otimes (P\mathbf{I}-\mathbf{L}) \;+\; \frac{2(P-1)}{P} P_0'(0)\mathbf{H}(y)\otimes \mathbf{I} \tag{5.7}$$

Thus the first line of equ.(5.4) is always non-negative definite, while the second line has the stability properties of $\mathbf{H}(y)$. We observe that the positive definite part is proportional to $P_0'(0)^2$ while the second term is only proportional to $P_0'(0)$. Thus, if the slope of the sigmoidal function P_0 is large enough, the trivial fixed point will inevitably become unstable. On the other hand, for sufficiently small values of $P_0'(0)$ the stability of a trivial fixed point is determined by the spectrum of $\mathbf{H}$ which in turn is determined by the local behavior of the average voter utility function U. Thus, if $P_0'(0)$ is small enough, a trivial equilibrium is stable if and only if it is a maximum of U. In the following sub-section we shall consider a few special cases in some more detail.

The bifurcation parameter $P_0'(0)$ deserves some discussion. In models with two parties, a steep slope of the sigmoidal response function P_0 at 0, i.e. at the point where the party platforms coincide, indicates a strong division of voters for either party whenever the platforms are different. Thus, in general $P_0'(0)$ expresses how critical the voters are in case the party platforms are not exactly equal.

5.4. Enelow-Hinich Models

If the voter utility functions are chosen as in the Enelow-Hinich model (2.1) an explicit analysis of bifurcations at the trivial equilibrium becomes feasible. We consider two cases: (1) a discrete voter distribution and constant strenght factors $s_{vi} = s$ for all v and i, and (2) a continuous voter distribution (either Gaussian or uniform) with a single issue and position-dependent strength factors, equ.(2.2).

In the first case we have $\mathbf{H} = -2s\mathbf{I}$ and $\mathbf{C} = 4s^2\mathbf{V}$ where

$$\mathbf{V}_{ij} := \frac{1}{V}\sum_v (x_{vi} - \bar{x}_i)(x_{vj} - \bar{x}_j) \tag{5.8}$$

is the covariance matrix of the voter distribution. Of course, this is still true if we consider a continuous voter distribution. The largest eigenvalue of the Jacobian is therefore

$$\begin{aligned} \lambda_{\max} &= \tfrac{4P(P-1)(P-2)4s^2\mathbf{r}}{P^2} P_0'(0)^2 - \tfrac{4s(P-1)}{P} P_0'(0) \\ &= \tfrac{4s(P-1)}{P} P_0'(0)\big(4(P-2)s\mathbf{r}P_0'(0) - 1\big). \end{aligned} \tag{5.9}$$

where $\mathbf{r}$ is the spectral radius of the covariance matrix $\mathbf{V}$. In case of a single issue, this is simply the variance σ^2 of the voter distribution. The average voter equilibrium therefore becomes unstable when $P_0'(0)$ exceeds the critical value $p^* = \frac{1}{4(P-2)s\mathbf{r}}$. For $P = 2$, i.e. two parties, there is no bifurcation, $\lambda_{\max} < 0$ in this case.

As a second example we consider the influence of position-dependent strength factors on the bifurcation values. We consider only the simplest case: a single issue, voter utility functions of the form $u(y, x) = -s(x)(y - x)^2$, and a voter distribution $\rho(x)$ that is symmetric around the origin, $\rho(x) = \rho(-x)$. Furthermore we require $s(x) = s(-x)$. Obviously, $\hat{y} = 0$ is the unique mean voter fixed point in this case. We consider the three examples of position dependent strength functions introduced in equ.(2.2). For

uniform voters: $s(x) = 1/2$, we find $\mathbf{C} = \mathrm{var}(x)$ and $\mathbf{H} = -1$.

extremist voters: $s(x) = |x|$, we find $\mathbf{C} = 4\mathrm{curt}(x) = 4\int_{-\infty}^{+\infty} x^4\rho(x)dx$ and $\mathbf{H} = -2\int_{-\infty}^{+\infty} |x|\rho(x)dx$

centrist voters: $s(x) = \max[1 - |x|, 0]$, we have to assume that the support of $\rho(x)$ is contained in $[-1, 1]$ in order to obtain “pretty” equations. With this additional assumption we find $\mathbf{H} = \int_{-\infty}^{+\infty} |x|\rho(x)dx - 2$ and a rather complicated expression for $\mathbf{C}$.

Table 1. Position Dependent Strength Factors.
The bifurcation at the trivial fixed point of a 3-party Enelow-Hinich model with position dependent strength factors depends quite strongly on the model for the strength factors, equ.(2.2). We assume a uniform voter distribution on $[-\alpha, \alpha]$ with $\alpha \leq 1$ and use the response function $\mathcal{P}$ defined in equ.(4.8).

	uniform	extremist	centrist
$\mathbf{H}$	-1	$-\alpha$	$\alpha - 2$
$\mathbf{C}$	$\alpha^2/3$	$(4/5)\alpha^4$	$(4/3)\alpha^2 - 2\alpha^3 + (4/5)\alpha^4$
p^*	$(3/2)\alpha^{-2}$	$(5/8)\alpha^{-3}$	$-\mathbf{H}/(2\mathbf{C})$

Suppose $\rho(x) = \begin{cases} 1/(2\alpha) & |x| \leq \alpha \\ 0 & |x| > \alpha \end{cases}$, i.e., the uniform distribution on $[-\alpha, \alpha]$. The values of $\mathbf{H}$, and $\mathbf{C}$ for the three models of the strength functions are compiled in Table 1.

Using the response function (4.8) we obtain

$$\lambda_{\max} = \frac{4}{3} P_0'(0)\mathbf{H} + \frac{8}{3} P_0'(0)^2 \mathbf{C} \tag{5.10}$$

from (4.4) with $P = 3$. The explicit values of the bifurcation points $p^* = -\mathbf{H}/(2\mathbf{C})$ are given in Table 1. Note the strong dependence of the bifurcation points on the model for the strength factors.

5.5. Numerical Analysis

In this section we report a few numerical results obtained for a three party model with a single issue, simple quadratic voter utility functions, and the response function $P(z) = (1 + \tanh(az))/2$. We use the criticality $P'(0)$ as bifurcation parameter.

There is a large number of bifurcation points in the three party model. For small and moderate values of $P'(0)$ we find the same sequence of bifurcations with different voter distributions. The sequence of eight bifurcations listed in Table 2 were identified in the two examples shown in Figure 3 as well as in all instances that have been investigated.

Bifurcation diagrams, Figure 3, show the location of the equilibria of equ.(2.7). The phase space $\mathbb{R}^{\nVdash}$ is projected onto the single coordinate

$$x^* = \frac{y^1 + 2y^2 + 5y^3}{8} \tag{5.11}$$

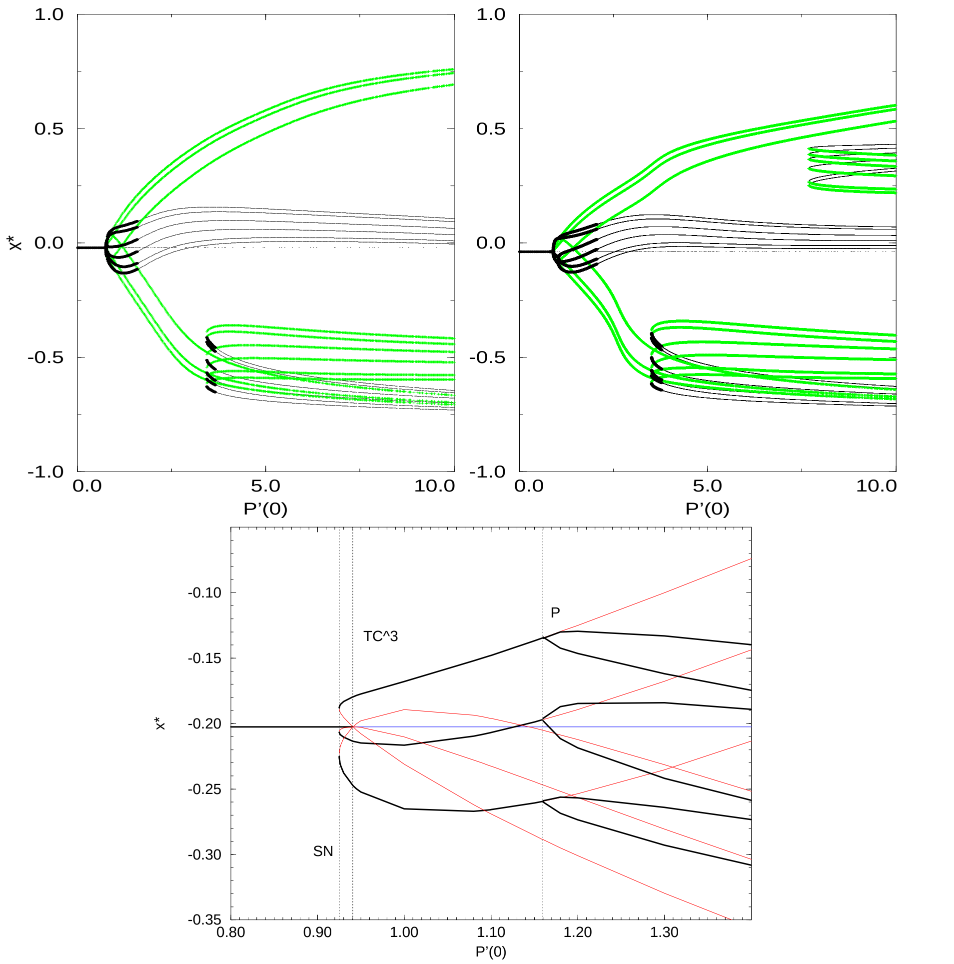

Figure 3. Numerical bifurcation diagrams of three-party one-issue Enelow-Hinich models. Stable fixed points appear as bold black lines. Two types of saddle points were found: Saddles with one unstable direction appear in bold grey, while those with two unstable manifolds correspond to thin black lines (or small dots).
Upper left: 100 voters with ideal points uniformly distributed in $[-1, 1]$.
Upper right: 20 voters with ideal points uniformly distributed in $[-1, 1]$. Beyond $P'(0) \approx 17$ the program seems to produce spurious solutions in regions of the phase space where the gradients become very small.
Note that the two diagrams differ only by the voter distribution.
Below: Detail of a bifurcation diagram for the 3-party 1-issue Enelow-Hinich model with 20 voters with different random number seeds.

Table 2. Bifurcations in the Three-Party Model.
Bifurcation points from the two cases shown in Figure 3 are listed here.

Bifurcation Type	Multi-plicity	100 voters (fig. 3 l.h.s.)	20 voters (fig. 3 r.h.s.)
Saddle-Node	3	0.76225	0.905
deg. Transcritical	1	0.7627	0.910
Pitchfork, supercrit.	3	0.7785	1.05
Hopf, supercrit.	6	1.59	2.06
Saddle-Node	6	3.43	3.51
Hopf, supercrit.	6	3.69	3.78
Saddle-Node	6	15.45	7.685
Saddle-Node	6	18.75	14.0

in order to break the symmetry of the model and to make all fixed points visible.

The first bifurcation is a saddle node bifurcation with the invariant planes $\mathcal{H}_{pq}$. Hence it occurs with multiplicity 3. For slightly larger values of $P'(0)$ we find a degenerate transcritical bifurcation in which 3 unstable fixed points (that are located in the in the planes $\mathcal{H}_{pq}$ and have a single unstable manifold) "collide" with the (stable) mean voter fixed point. After the bifurcation the mean voter is unstable as well (with two unstable manifolds).

At higher values of $P'(0)$ the stable equilibria within $\mathcal{H}_{pq}$ undergo pitchfork bifurcations producing six stable fixed points and three saddle points, one within each of the three planes $\mathcal{H}_{pq}$. The coordinates of the six stable equilibria are related by permutation symmetry. As $P'(0)$ increases further these sinks undergo supercritical Hopf bifurcations, thereby giving rise to stable limit cycles.

At even larger values of the bifurcation parameter we find a sixfold saddle node bifurcation followed by a supercritical Hopf bifurcation. At large values of $P'(0)$ the system does not contain stable fixed points. A number of saddle node bifurcations produce additional saddle points as $P'(0)$ increases. In this regime we also find large stable limit cycles, such as the example in Figure 4.

Note also, that the bifurcation diagrams displayed in Figure 3 differ only by the voter distribution. We observe, therefore, that the qualitative dynamics is very sensitive to the details of the voter distribution at least for large values of the voter criticality parameter $P'(0)$.

6. Conclusions

We have generalized the spatial voting approach to multi-party systems. A dynamical system has been derived that describes the adaptation of the party

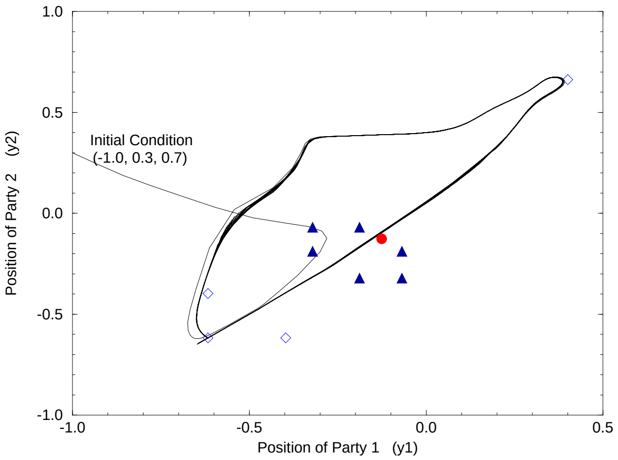

Figure 4. A large limit cycle for $P'(0) = 5$, $V = 20$ voters, and a single issue. Fixed points are indicated by symbols: $\diamond$ indicates saddle points with two equal coordinates (there are 2 sets, any two of them are projected onto the same point in the y^1/y^2 plane). A full circle indicates the mean voter point, $\triangle$ denotes saddle points with S_3-symmetry, i.e., not on any plane $\mathcal{H}_{pq}$.

platforms based on the following assumptions: (i) parties are opportunisitic in the sense that they only desire to maximize their share of votes; (ii) voters do not change their ideal points and utility functions during the campaign; (iii) the platform adaptation is a gradual process (driven for instance by opinion polls) and does not allow large jumps in issue space.

In most cases we assume that the voter utility functions are independent of the party labels; then the dynamical system has S_P-permutation symmetry. The linear manifolds where two or more party platforms coincide are invariant. Parties occupy a common platform position when they form alliances for the purpose of an upcomming election, a common practice in many European countries. A basic property of the adaptive platform dynamics considered in this contribution is that all trajectories eventually converge into the box spanned by the extremal voter positions.

An equilibrium on the linear manifold where all party platforms coincide is referred to as the *trivial* equilibrium. It is known that the trivial equilibrium is globally stable for the adaptive dynamics in two party models with concave voter utilities. This is not true in general in multi-party systems even for quadratic utilities.

We find that the crucial parameter is how critical voters are, that is how much small changes in the relative utilities influence the voter's behavior. In this present model, a voter's "criticality" is measured by the maximum slope of the multi-dimensional sigmoidal response function that determines the probabilities with which a voter chooses a particular party given the set of all utility differences.

For a system with three parties, a number of explicit examples, with both discrete and continuous distributions of voter ideal points in issue space are discussed. We obtain similar results in all cases: For small slopes, the trivial equilibrium is globally stable. As the slope increases, we observe a cascade of bifurcations leading to an increasing number of locally stable fixed points, and then to a situation in which there are no stable equilibria at all. Critical voters are thus capable of keeping the platform positions of different parties well separated. In a substantial fraction of the parameter space, as the slope increases further, there are Hopf-bifurcations and numerically we demonstrate the existence of stable limit cycles, Figure 4.

The model considered here is the simplest way of treating multiparty systems in the context of spatial voting theory. A number of effects have been deliberately neglected that are most likely of great importance in a more realistic setting: voters do not respond to the parties' campaigns by modifying their ideal points and/or strength factors here, there is complete participation, and we have ignored non-policy values. Furthermore the possibility to form particular coalitions after the election influences a party's campaign as well as the behavior of many voters. Computer simulations, however, rather than analytical approaches like the one presented here, seem to be a more appropriate way of addressing these issues in future investigations.

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