Abstention in Dynamical Models of Spatial Voting

Bärbel M.R. Stadler [a,1]

[a] *Institut für Theoretische Chemie und Molekulare Strukturbiologie*
Universität Wien
Währingerstraße 17, A-1090 Wien, Austria
`baer@tbi.univie.ac.at`

Abstract

We consider a model of platform adaptation in spatial voting focussing on the effect of abstention on the stability of the mean voter equilibrium. Two distinct approaches for modeling abstention are explored: (1) voters abstain if party platforms are too similar to each other and (2) voters abstain if both party platforms are far away from their ideal points.

Key words: Spatial Voting Models, Adaptive Dynamics, Abstention

1 Introduction

Voting models are a common tool in the fields of Economics and Political Science. In the language of politics, spatial references (e.g. "moving right, left, centrist") are intuitively used to describe political issues, platforms or voters' ideologies. The theory of spatial voting is based on the assumption that political issues as well as voter preferences can be quantified. Early work on this subject was done by Hotelling in the 1920s [1]; the core of the theory was developed in the 1940s and 1950s by Smithies [2], Downs [3], and Black [4]. For recent reviews see [5, 6]. A statistical physics approach to voting can be found in [7]. Only very rarely was research focussed on its dynamical aspects.

A dynamical system describing the platform adaptation of two opportunistic political parties is analyzed in [8], based upon earlier computer simulations

[1] Supported in part by the Austrian *Fonds zur Förderung der Wissenschaftlichen Forschung* Proj. No. 13887-MOB

that showed a similar tendency towards stability [9, 10]. A generalization to multi-party systems is introduced in [11]. Globally stable equilibria were found in the dynamical models under rather general assumptions despite the general lack of stable equilibria in the game theoretic approaches [12, 13, 14, 15].

In this work we focus on the influence of abstention on the platform dynamics of a two-party model, generalizing previous work in this direction [16].

2 The Model

The basic idea of this model is that voters and candidates are seen as points in the Euclidean space $\mathbb{R}^I$, the dimension of which is determined by the number I of different issues. It is assumed that the voter is rational in the sense that she has a certain opinion on every issue to be voted on, thus defining her position in issue space. This position is referred to as the voter's *ideal point* and will be denoted by the vector $x_v \in \mathbb{R}^I$. We assume that a voter's position remains fixed for the duration of the campaign. Party platforms are seen as points in the issue space as well. Throughout this work, we consider models with two parties. The *utility* of party $p = 1, 2$ for a given voter v is measured by a function $u_v(y^p)$, which is assumed to be decreasing with the distance from the voter's ideal point and has its maximum at the voter's ideal point.

The probability that voter v votes for party p is described by the *response function* $P_0(z_v)$, where $z_v = u_v(y^p) - u_v(y^q)$ is the utility difference between party p and party q for voter v. It is assumed that a voter will most likely cast her vote for the party that yields the largest utility for her. If a voter is completely informed about the party positions, $P_0(z_v)$ will be a step function with value 0 for $z_v < 0$, i.e. if party q yields the larger utility, the probability of voting for party p will be zero; its value will be 1 for $z_v > 0$, i.e. the voter will certainly vote for party p if it yields the larger utility, and $P_0(z_v) = \frac{1}{2}$ if both parties yield the same utility for voter v which illustrates that the voter drops a coin in this case. We will assume here that in mass elections, the voters will generally not be completely informed and therefore $P_0(z_v)$ will be a sigmoidal function, independent of the individual voter v.

The slope $P_0'(0)$ of the response function at 0 is a measure of how critical the voters are. A steep slope indicates critical voters, who react strongly to small changes in platform positions, while a flat slope indicates uncritical voters whose behavior is more indifferent to small changes in platform positions.

In general, the voters are confronted with making a decision between participating in the election and thus voting for one of the competing parties, and abstaining from the election. In order to model voter abstention we assume that there are two different motives that cause voters to abstain from the election: For once, we will assume that the voter might loose her interest in casting a ballot if the party platforms are far away from her ideal point. We model this effect by the non-voting probability $\mu(z)$ that depends on the utility differences and satisfies the following conditions:

1 $\mu(z)$ is symmetric, i.e. $\mu(z) = \mu(-z)$. This reflects that the parties appear to be exchangeable to a person who does not vote at all.
2 $\mu(z)$ is decreasing for $z > 0$.
3 $\mu(z)$ shall be twice differentiable. This assumption is made for technical simplicity.

As an immediate consequence, $\mu'(0) = 0$ and $\mu''(0) \le 0$.

Furthermore, we also assume that a voter will abstain if the party platforms are far away from her ideal point. In this case, abstention depends explicitly on the utilities of the two parties for each voter. This effect can be modelled by a probability function $\Psi(u)$ that is increasing with the voter utility u. Since u is negative and decreasing with the distance from the voter's ideal point, $\Psi(u)$ is also decreasing with the distance from the voter's ideal point.

The two parties, or candidates, are the active players in our model. Their positions are described by the vectors $y^1 \in \mathbb{R}^I$ and $y^2 \in \mathbb{R}^I$. Parties are allowed to modify their positions in order to gain more votes. In this contribution, we want to model an interplay of both abstention motives. A party's payoff is the expected fraction of votes it gets. In a model with V voters the payoff for each one of the parties is given by:

$$\begin{aligned} E_1(y^1, y^2) &= \frac{1}{V} \sum_v P_0(z_v)(1 - \mu(z_v))\Psi(u_v(y^1) \\ E_2(y^1, y^2) &= \frac{1}{V} \sum_v P_0(-z_v)(1 - \mu(z_v))\Psi(u_v(y^2) \end{aligned} \tag{1}$$

Complete voter participation corresponds to setting $\mu(z) = 0$ and $\Psi(u_v(y^p)) = 1$ for $p = 1, 2$.

In order to gain more votes, each party makes small corrections to its platform along the gradient of the payoff function in its own coordinates.

Thus, the dynamics of platform adaptation is of the following form:

$$\begin{aligned} \dot{y}^1 &= \nabla_{y^1} \frac{1}{V} \sum_{v=1}^{V} P_0(z_v)(1 - \mu(z_v))\Psi(u_v(y^1)) \\ \dot{y}^2 &= \nabla_{y^2} \frac{1}{V} \sum_{v=1}^{V} P_0(-z_v)(1 - \mu(z_v))\Psi(u_v(y^2)) \end{aligned} \tag{2}$$

It is easy to see that $\mathcal{H}_2$, the manifold in which both party platforms are equal, is invariant under the dynamics since $\dot{y}^1 = \dot{y}^2$ holds.

We get the following Jacobian for a point on $\mathcal{H}_2$:

$$
\begin{aligned}
\frac{\partial \dot{y_j}^1}{\partial {y_k}^1}|_{z_v=0} &= -\frac{1}{2}\mu''(0)\frac{1}{V}\sum_{v=1}^{V}\Psi(u_v(y))\partial_k u_v(y)\partial_j u_v(y) \\
&+ 2(1-\mu(0))P_0'(0)\frac{1}{V}\sum_{v=1}^{V}\Psi'(u_v(y))\partial_k u_v(y)\partial_j u_v(y) \\
&+ \frac{1}{2}(1-\mu(0))\frac{1}{V}\sum_{v=1}^{V}\Psi''(u_v(y))\partial_k u_v(y)\partial_j u_v(y) \\
&+ (1-\mu(0))P_0'(0)\frac{1}{V}\sum_{v=1}^{V}\Psi(u_v(y))\partial_k\partial_j u_v(y) \\
&+ \frac{1}{2}(1-\mu(0))\frac{1}{V}\sum_{v=1}^{V}\Psi'(u_v(y))\partial_k\partial_j u_v(y) \qquad (3) \\
\frac{\partial \dot{y_j}^1}{\partial {y_k}^2}|_{z_v=0} &= \frac{1}{2}\mu''(0)\frac{1}{V}\sum_{v=1}^{V}\Psi(u_v(y))\partial_k u_v(y)\partial_j u_v(y) \\
&- (1-\mu(0))P_0'(0)\frac{1}{V}\sum_{v=1}^{V}\Psi'(u_v(y))\partial_k u_v(y)\partial_j u_v(y) \\
\frac{\partial \dot{y}^1}{\partial y^1}|z_v = 0 &= \frac{\partial \dot{y}^2}{\partial y^2}|z_v = 0 \\
\frac{\partial \dot{y}^1}{\partial y^2}|z_v = 0 &= \frac{\partial \dot{y}^2}{\partial y^1}|z_v = 0
\end{aligned}
$$

Note that $\partial_j u_v(y)$ denotes the j-th component of the vector $\nabla u_v(y)$.

The dynamics on $\mathcal{H}_2$ is given by the differential equation

$$
\dot{y} = (1-\mu(0))\frac{1}{V}\sum_{v=1}^{V}\left[P_0'(0)\Psi(u_v(y)) + \frac{1}{2}\Psi'(u_v(y))\right]\nabla u_v(y^p) \qquad (4)
$$

Little can be said about the general case. For discrete voter distributions, the expressions become rather unhandy.

We will therefore analyze two models using simple continuous voter distributions $\rho(x)$ in order to illustrate the dynamics of (2) and restrict ourselves to the one-issue-case.

We assume the Enelow-Hinich-type voter utility function

$$
u(y,x) = -(y-x)^2 \qquad (5)
$$

and an exponentially decaying voting probability of the form

$$\Psi(u_v(y)) := \exp(ku_v(y)) \tag{6}$$

For the moment, we will not specify the symmetric non-voting probability function μ.

At first, let us assume that the voters are normally distributed with mean 0 and variance σ^2.

The dynamics on $\mathcal{H}_2$ is given by

$$\dot{y} = \frac{1}{\sigma\sqrt{2\pi}}(1-\mu(0))\int_{-\infty}^{\infty}\left[P_0'(0)+\frac{1}{2}k\right]e^{-x^2/(2\sigma^2)-k(y-x)^2}(-2)(y-x)\,dx \tag{7}$$

With the abbreviations

$$a = \sqrt{\frac{2\sigma^2 k+1}{2\sigma^2}} \qquad b = \frac{-ky}{a} \qquad c = ky^2 - b^2 \tag{8}$$

we may write the exponent in the form $(ax+b)^2+c$. Setting $w = ax+b$, we obtain the following expression:

$$\dot{y} = -(1-\mu(0))\frac{(2P_0'(0)+k)(a^2-k)e^{b^2}}{a^3\sigma\sqrt{2}}y\exp(-ky^2) \tag{9}$$

Since $a^2 - k = 1/(2\sigma^2) > 0$, equ.(9) is of the form $\dot{y} = -Cy\exp(-ky^2)$ with a positive constant C. Thus, the mean voter point $y = 0$ is the only fixed point. It is stable within $\mathcal{H}_2$.

The Jacobian at $y = 0$ can be obtained from equ.(3) by evaluating similar integrals:

$$\begin{aligned}\frac{\partial\dot{y}^1}{\partial y^1} &= \frac{1}{\sqrt{2}a^3\sigma}\Bigg[-\mu''(0)+(1-\mu(0))4kP_0'(0)\\ &\quad+(1-\mu(0))k^2-(1-\mu(0))2a^2P_0'(0)-(1-\mu(0))ka^2\Bigg]\\ \frac{\partial\dot{y}^1}{\partial y^2} &= \frac{1}{\sqrt{2}a^3\sigma}\left[-\mu''(0)-(1-\mu(0))2kP_0'(0)\right]\end{aligned} \tag{10}$$

Rewriting equ.(10) in matrix form yields:

$$\mathbf{J}(y,y) = \frac{-\mu''(0)}{\sqrt{2}a^3\sigma}\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} + \frac{(1-\mu(0))2kP_0'(0)}{\sqrt{2}a^3\sigma}\begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}$$
$$+\frac{(1-\mu(0))}{\sqrt{2}a^3\sigma}[k^2 - 2a^2P_0'(0) - a^2k]\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \tag{11}$$

It is easy to see that the matrices $\mathbf{M} = \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}$ and $\mathbf{N} = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}$ have the same eigenvectors. The eigenvalues of $\mathbf{M}$ are 0 and 2, those of $\mathbf{N}$ are 1 and 3. Thus, the eigenvalues of the Jacobian are

$$\lambda_1 = \frac{(1-\mu(0))}{\sqrt{2}a^3\sigma}\left[(k-a^2)(2P_0'(0)+k)\right]$$
$$\lambda_2 = \frac{(1-\mu(0))}{\sqrt{2}a^3\sigma}\left[2P_0'(0)(3k-a^2) + k(k-a^2)\right] - \frac{2\mu''(0)}{\sqrt{2}a^3\sigma} \tag{12}$$

Taking into account that $k - a^2 = -1/2\sigma^2$, we see that the first eigenvalue, λ_1, is always negative. Substituting $3k - a^2 = (4k\sigma^2 - 1)/2\sigma^2$, the conditions for the second eigenvalue λ_2 to be negative can be rewritten in the form:

$$2P_0'(0)(1-\mu(0))(4k\sigma^2-1) < k(1-\mu(0)) + 4\sigma^2\mu''(0) \tag{13}$$

There are the following cases to distinguish:

Case I. $k < \dfrac{1}{4\sigma^2}$

In this case $\lambda_2 < 0$ is fulfilled for

$$P_0'(0) > \frac{1}{2(4k\sigma^2-1)}\left[k + \frac{4\sigma^2\mu''(0)}{1-\mu(0)}\right] = \mathbf{P}* \tag{14}$$

For $k > \frac{-4\sigma^2\mu''(0)}{1-\mu(0)}$ the term in the square brackets on the right hand side of equ.(14) is positive. Thus for $\frac{-4\sigma^2\mu''(0)}{1-\mu(0)} < k < \frac{1}{4\sigma^2}$ the second eigenvalue is always negative and the mean voter equilibrium is stable independent of $P_0'(0)$.

For $k < \frac{-4\sigma^2\mu''(0)}{1-\mu(0)}$ the term in the square brackets on the right hand side of (14) is negative. Thus we get a bifurcation at the mean voter equilibrium for $P_0'(0) = \mathbf{P}^*$. The mean voter equilibrium becomes thus stable for critical voters.

Case II. $k > \dfrac{1}{4\sigma^2}$.

In this case, $\lambda_2 < 0$ is fulfilled for $P_0'(0) < \mathbf{P}^*$.

If the term under the square brackets on the right hand side of equ.(14) is negative, then the condition $\lambda_2 < 0$ cannot be fulfilled, since $P_0'(0)$ has to be positive.

Thus, for $\frac{1}{4\sigma^2} < k < \frac{-4\sigma^2\mu''(0)}{1-\mu(0)}$ the mean voter equilibrium is always unstable independent of $P_0'(0)$.

If the term under the square brackets on the right hand side of (14) is positive, we get a bifurcation at the mean voter equilibrium for $P_0'(0) = \mathbf{P}^*$.

Thus, the mean voter equilibrium is stable for uncritical voters and becomes unstable for critical voters.

Let us now assume that the voters are uniformly distributed on the interval $[-1, 1]$.

The dynamics on $\mathcal{H}_2$ is given by the equation:

$$\dot{y} = \frac{1}{2}\left[P_0'(0) + \frac{k}{2}\right]\int_{-1}^{1} e^{-k(y-x)^2}(-2)(y-x)dx \tag{15}$$

With the substitution $w = y - x$, the above equation reads:

$$\dot{y} = -\left[P_0'(0) + \frac{k}{2}\right]\int_{y-1}^{y+1} e^{-kw^2} w dw \tag{16}$$

Evaluating the integral, we get:

$$\int_{y-1}^{y+1} e^{-kw^2} w dw = -\frac{1}{2k} e^{-k(y+1)^2}(1 - e^{4ky}) \tag{17}$$

Thus, the dynamics on $\mathcal{H}_2$ is given by:

$$\dot{y} = [P_0'(0) + \frac{k}{2}]\frac{1}{2k} e^{-k(y+1)^2(1-e^{4ky})} \tag{18}$$

It is easy to see that $\dot{y} = 0$ iff $e^{4ky} = 1$, i.e. $y = 0$. Thus, $y = 0$ is the only fixed point of equ.(15). The Jacobian at $y=0$ is given by:

$$\begin{aligned}
\frac{\partial \dot{y}^1}{\partial y^1} &= [-\mu''(0) + 4k(1-\mu(0))P_0'(0) + k^2(1-\mu(0))] \int_{-1}^{1} x^2 e^{-kx^2} dx \\
&\quad - (1-\mu(0))(P_0'(0) + \frac{k}{2}) \int_{-1}^{1} e^{-kx^2} dx \\
\frac{\partial \dot{y}^1}{\partial y^2} &= [\mu''(0) - 2k(1-\mu(0))P_0'(0)] \int_{-1}^{1} x^2 e^{-kx^2} dx
\end{aligned} \tag{19}$$

With the substitution $\mu x^2 = t^2$ and using the definition of the error function we find

$$\int_{-1}^{1} x^2 e^{-kx^2} dx = -\frac{e^{-k}}{k} + \frac{1}{2k}\sqrt{\frac{\pi}{k}}\mathrm{erf}(\sqrt{k}) \tag{20}$$

Therefore, we get the following Jacobian at $y=0$:

$$\begin{aligned}
\mathbf{J} &= \Big(2k(1-\mu(0))P_0'(0) - \mu''(0)\Big)\left(\frac{1}{2k}\sqrt{\frac{\pi}{k}}\mathrm{erf}(\sqrt{k}) - \frac{e^{-k}}{k}\right)\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} \\
&+ (1-\mu(0))\Bigg[(2kP_0'(0) + k^2)\left(\frac{1}{2k}\sqrt{\frac{\pi}{k}}\mathrm{erf}(\sqrt{k}) - \frac{e^{-k}}{k}\right) \\
&\qquad - (P_0'(0) + \frac{k}{2})\sqrt{\frac{\pi}{k}}\mathrm{erf}(\sqrt{k})\Bigg]\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}
\end{aligned} \tag{21}$$

The eigenvalues of the first matrix are 0 and 2. Thus, the eigenvalues of the Jacobian are:

$$\begin{aligned}
\lambda_1 &= -(1-\mu(0))(2P_0'(0) + k)e^{-k} \\
\lambda_2 &= (1-\mu(0))2P_0'(0)\left(-3e^{-k} + \sqrt{\frac{\pi}{k}}\mathrm{erf}(\sqrt{k})\right) \\
&\quad + \frac{\mu''(0)}{k}\left(2e^{-k} - \sqrt{\frac{\pi}{k}}\mathrm{erf}(\sqrt{k})\right) - k(1-\mu(0))e^{-k}
\end{aligned} \tag{22}$$

It is easy to see that the first eigenvalue, λ_1, is always negative, while λ_2 may change its sign.

In order to simplify the analysis we will choose the function $\mu(z) = \mu_0 e^{-z^2}$. Then $\mu''(0) = -2\mu_0$. In the system with both abstention parameters, the expression for the eigenvalue λ_2 is of the following form:

$$\begin{aligned}
\lambda_2 &= (1-\mu_0)2P_0'(0)(\sqrt{\frac{\pi}{k}}\mathrm{erf}(\sqrt{k}) - 3e^{-k}) \\
&\quad + \left(\frac{2}{k}\sqrt{\frac{\pi}{k}}\mathrm{erf}(\sqrt{k}) + \frac{k^2-4}{k}e^{-k}\right)\mu_0 - ke^{-k}
\end{aligned} \tag{23}$$

We have the following cases to distinguish:

Case I. $\sqrt{\frac{\pi}{k}}\mathrm{erf}(\sqrt{k}) - 3e^{-k} < 0$ which is equivalent to $k < 0.586109$.

Then the eigenvalue λ_2 is negative iff

$$P_0'(0) > \frac{-\mu_0\left(2\sqrt{\frac{\pi}{k}}\mathrm{erf}(\sqrt{k}) + (k^2-4)e^{-k}\right) + k^2e^{-k}}{2k(1-\mu_0)(\sqrt{\frac{\pi}{k}}\mathrm{erf}(\sqrt{k}) - 3e^{-k})} = \mathbf{P}^* \tag{24}$$

We observe

$$2\sqrt{\frac{\pi}{k}}\mathrm{erf}(\sqrt{k}) + (k^2-4)e^{-k} > 0\,. \tag{25}$$

Thus, the numerator of (24) is positive for

$$\mu_0 < \mu^*(k) = \frac{k^2e^{-k}}{2\sqrt{\frac{\pi}{k}}\mathrm{erf}(\sqrt{k}) + (k^2-4)e^{-k}} \tag{26}$$

In this case, the expression on the right hand side of (24) is negative. Since $P_0'(0) > 0$, λ_2 is always negative which means that the mean voter equilibrium is stable.

For $\mu_0 > \mu^*(k)$ we get a bifurcation at the mean voter equilibrium for $P_0'(0) = \mathbf{P}^*$, such that $\lambda_2 < 0$ holds for $P_0'(0) > \mathbf{P}^*$. Thus, the mean voter equilibrium is stable for critical voters and it becomes unstable if the voters are not critical.

Case II. $\sqrt{\frac{\pi}{k}}\mathrm{erf}(\sqrt{k}) > 3e^{-k}$ which is equivalent to $k > 0.586109$.

Then λ_2 is negative for $P_0'(0) < \mathbf{P}^*$. If $\mu_0 > \mu^*(k)$ then the numerator of $\mathbf{P}^*$ is negative. Since $P_0'(0) > 0$, the condition for λ_2 to be negative cannot be fulfilled. Thus, the mean voter fixed point is always stable independent of the value of $P_0'(0)$. If, on the other hand, $\mu_0 < \mu^*(k)$ then the numerator of $\mathbf{P}^*$ is positive. A bifurcation on the mean voter equilibrium occurs at $P_0'(0) = \mathbf{P}^*$. Thus, the mean voter equilibrium is unstable for critical voters and it is stable as long as the voters are not critical.

3 Conclusions

We have extended the models of two-party platform dynamics incorporating voter abstention described in [16] to a setup which corresponds to an interplay of both abstention mechanisms.

We assume that voting becomes uninteresting if either the utility differences between the parties are small or the party positions are far away from the voter's ideal point and so the probability of voting for a particular party depends both on the value of the utility difference between the two parties and on the distance between voter and party platform. We model the mechanism of voting for party p by the product of voting probabilities $P_0(z)(1-\mu(z))\Psi(u)$, where $P_0(z)$ is the probability of voting for p in the case of complete participation, $\mu(z)$ is a non-voting probability that depends on the utility differences, and $\Psi(u)$ is a voting probability decreasing with the distance of party p's platform from voter v's ideal point.

We find a stable mean voter equilibrium for a moderately decreasing function Ψ and a small non-voting probability μ. For large μ we find that if Ψ is moderately decreasing the mean voter equilibrium is stable for critical voters, while it becomes unstable if the voters are not critical. For small μ and strongly decreasing Ψ the mean voter equilibrium is stable for uncritical voters and it becomes unstable if the voters are critical. If both μ is large and Ψ is strongly decreasing, the mean voter equilibrium is always unstable.

Acknowledgements

Part of this work was performed during a visit to the Santa Fe Institute in Feb. 2000. Special thanks to J. Fernando Fontanari's group at the Instituto de Física São Carlos, where this contribution was completed.

References

[1] H. Hotelling. Stability in competition. *Econ. J.*, 39:41–57, 1929.

[2] A. Smithies. Optimum location in spatial competition. *J. Pol. Economy*, 49:423–439, 1941.

[3] A. Downs. *An Economic Theory of Democracy.* Harper & Row, New York, 1957.

[4] D. Black. *Theory of Committees and Elections.* Cambridge University Press, Cambridge, 1958.

[5] James M. Enelow and Melvin J. Hinich. *The Spatial Theory of Voting: An Introduction.* Cambridge University Press, Cambridge UK, 1984.

[6] James M. Enelow and Melvin J. Hinich, editors. *Advances in the Spacial Theory of Voting.* Cambridge University Press, Cambridge, UK, 1990.

[7] Serge Galam. Social paradoxes of majority rule voting and renormalization group. *J. Stat. Phys.*, 61:943–951, 1990.

[8] John H. Miller and Peter F. Stadler. The dynamics of adaptive parties under spatial voting. *J. Econ. Dyn. & Control*, 23:171–189, 1998.

[9] Ken Kollman, John H. Miller, and Scott E. Page. Adaptive parties in spatial elections. *Amer. Pol. Sci. Rev.*, 86:929–937, 1992.

[10] Ken Kollman, John H. Miller, and Scott E. Page. Policy position-taking in two-party elections. Technical report, Carnegie Mellon Univ., 1996. http://zia.hss.cmu.edu/econ/misc/kmp3.html.

[11] Bärbel M.R. Stadler. Adaptive platform dynamics in multi-party spatial voting. *Adv. Complex Syst.*, 2:101–116, 1999.

[12] O. Davis, M. DeGroot, and M. J. Hinich. Social preference orderings and majority rule. *Econometrica*, 40:147–157, 1972.

[13] G. Kramer. On a class of equilibrium conditions for majority rule. *Econometrica*, 41:285–297, 1973.

[14] R. McKelvey. General conditions for global intransitivities in formal voting models. *Econometrica*, 47:1085–1111, 1979.

[15] Charles R. Plott. A notion of equilibrium and its possibility under majority rule. *Amer. Econ. Rev.*, 57:787–806, 1967.

[16] Bärbel M.R. Stadler. Abstention causes bifurcations in two-party voting dynamics. *Appl. Math. Comput.*, 2000. in press, Santa Fe Institute Preprint 98-08-072.