Minimum Cycle Bases of Halin Graphs

Peter F. Stadler
INSTITUTE FOR THEORETICAL CHEMISTRY AND MOLECULAR STRUCTURAL BIOLOGY, UNIVERSITY OF VIENNA WÄHRINGERSTRASSE 17, A-1090 VIENNA, AUSTRIA, & THE SANTA FE INSTITUTE 1399 HYDE PARK RD, SANTA FE, NM 87501, USA

ABSTRACT

Halin graphs are planar 3-connected graphs that consist of a tree and a cycle connecting the end vertices of the tree. It is shown that all Halin graphs that are not "necklaces" have a unique minimum cycle basis. *Keywords: Minimum Cycle Basis, Halin Graphs*

1. INTRODUCTION

A Halin graph $H(V, T \cup C^*)$ [3] consists of a tree (V, T) with end vertices $V^e \subseteq V$ and a cycle (V^e, C^*) such that the following conditions are satisfied: (i) $H(V, T \cup C^*)$ is planar and C^* is the boundary of the exterior face, and (ii) All interior vertices of the tree (V, T) have degree at least three. Halin graphs are 3-connected by construction. This class of graphs has received considerable attention in the literature since a number of hard optimization problems can be solved easily when restricted to Halin graphs, see e.g. [10] and the references therein.

Let $G(V, E)$ be a simple, unweighted, undirected graph with vertex set V and edge set E. A *cycle* is a connected minimal subgraph such that every vertex in V_C has degree 2. The set $\mathcal{C}$ of Eulerian subgraphs (unions of edge-disjoint cycles) forms a vector space over GF(2) with vector addition $X \oplus Y := (X \cup Y) \setminus (X \cap Y)$ and scalar multiplication $1 \cdot X = X$, $0 \cdot X = \emptyset$, which is called the *cycle space* of G [1]. Here we represent cycles by their edge sets. The dimension of the cycle space is the *cyclomatic number* $\nu(G) = |E| - |V| + p$, where p is the number of connected components of G.

If G is planar then the set $\mathcal{F}$ of interior faces forms a cycle basis [8]. Sysło [12] showed that a simple planar 3-connected graph G is a Halin graph if and only if G has a planar basis in which every cycle has an exterior edge.

The length $|C|$ of a generalized cycle C is the number of its edges. The length $\ell(\mathcal{B})$ of a cycle basis $\mathcal{B}$ is the sum of the lengths of its generalized cycles: $\ell(\mathcal{B}) = \sum_{C\in\mathcal{B}} |C|$. A *minimum cycle basis* is a cycle basis with minimal length. Polynomial time algorithms for computing minimum cycle bases are known [6, 5].

If G is outerplanar, then its minimal cycle basis is unique [7]. In general, of course, this is not true. In this contribution we show that almost all Halin graphs have the planar basis $\mathcal{F}$ as the *unique* minimum cycle basis, and we determine all alternative minimum cycle bases of the exceptional cases.

2. PRELIMINARIES

A cycle C is *relevant* if it cannot be represented as an $\oplus$-sum of shorter cycles [9]. Equivalently, a cycle is relevant if and only if it is contained in a minimum cycle basis [13]. A cycle is called *essential* if it is contained in every minimum cycle basis of G [2].

The proof of the main result will make use of the following proposition which is a special case of a theorem by Stepanec and Zykov [11, 14]. For the sake of completeness a (short) proof is included below.

Proposition 1. Let e be an edge in G and suppose there is a unique shortest cycle C_e in G that contains e. Then C_e is essential.

Proof. Suppose there is a minimum cycle basis $\mathcal{M}$ that does not contain C_e. Since e is contained in a cycle of G, there is a subset $\mathcal{M}^e \subseteq \mathcal{M}$ of cycles that contain e. Since $\mathcal{M}$ is a cycle basis, there are sets $\mathcal{P} \subseteq \mathcal{M}^e$ and $\mathcal{Q} \subseteq \mathcal{M} \setminus \mathcal{M}^e$ such that

$$C_e = \bigoplus_{C'\in\mathcal{P}} C' \oplus \bigoplus_{C''\in\mathcal{Q}} C'' \,.$$

From $e \in C_e$ we conclude that $\mathcal{P} \neq \emptyset$. For each $C' \in \mathcal{P}$, $\mathcal{M}' = \mathcal{M} \cup \{C_e\} \setminus \{C'\}$ is a cycle basis with length $\ell(\mathcal{M}') = \ell(\mathcal{M}) + |C_e| - |C'|$. By assumption, $|C_e| < |C'|$ for all $C' \in \mathcal{M}^e$, and hence $\ell(\mathcal{M}') < \ell(\mathcal{M})$, contradicting the minimality of $\mathcal{M}$. ∎

The well known result that *any* shortest cycle that goes through a given edge e is relevant can be proved analogously. The graphs whose minimum cycle bases consist entirely of shortest cycles are characterized in [4].

We need to single out a special class of Halin graphs, which have been called *necklaces*, Ne_h, in [12]. Let P be a path of length $h+1$, with vertices labelled from 0 to $h+1$ along P. The comb Cb_h is the tree consisting of P together with vertices $1'$ through h' and edges $\{1,1'\}$ through $\{h,h'\}$. The necklace Ne_h is obtained from Cb_h by adding the edges $\{0,1'\}$, $\{1',2'\}$, …, $\{h-1',h'\}$, $\{h',h+1\}$, and $\{h+1,0\}$, see Figure 1. The vertices 0 and $h+1$ will be called the *ends* of the necklace.

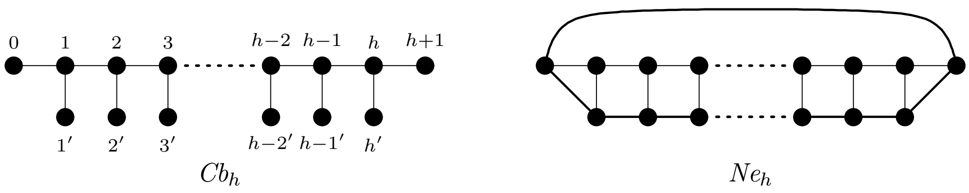

FIGURE 1. Comb Cb_h and Necklace Ne_h.

Note that the ends are uniquely determined in Ne_h if $h \geq 3$. We have $Ne_1 = K_4$ and Ne_2 is the trigonal prism. Sysło proved that a Halin graph that is not a necklace has a unique "H-feasible embedding", i.e., a planar embedding consisting of a cycle C^* as exterior boundary and a tree in the interior of C^*. Hence the partition of the edge set into the tree T and the cycle C^* is unique in this case. If the end points of Ne_h are unique, i.e., if $h \geq 3$, there are exactly two H-feasible embeddings. The trigonal prism, $h = 2$, and K_4, $h = 1$, have three and four such embeddings, respectively [12]. In the case of necklaces we fix a particular embedding so that the partition of the edge set into T and C^* is well defined.

3. RESULT

Theorem 1. Let $H = (V, T \cup C^*)$ be a Halin graph and let $\mathcal{F}$ denote the set of faces in the embedding in which C^* forms the boundary of the exterior face. Then $\mathcal{F}$ is a minimal cycle basis of H. Furthermore:
(i) $\mathcal{F}$ is the unique minimal cycle basis unless H is a necklace.
(ii) Suppose $H = Ne_h$ and $h \geq 3$ and let C_{xy} be the cycle consisting of the path P and the edge $\{0, h+1\}$ connecting the two ends of Ne_h. Then $|C^*| = |C_{xy}|$ and $\mathcal{F} \setminus \{C_{xy}\} \cup \{C^*\}$ is the only other minimum cycle basis.
(iii) If H is the trigonal prism graph then there are three minimal cycle bases consisting of the two essential triangles and two of the three quadrangles.
(iv) If $H = K_4$ then there are four minimum cycle basis consisting of any three of the four three-cycles of K_4.

Proof. Consider two adjacent vertices $x, y \in V_e$, $\{x, y\} \in C^*$. By definition, there is a unique path $\mathsf{w} = (x = w_0, w_1, \ldots, w_{k-1}, w_k = y)$ with edges $\{w_i, w_{i+1}\} \in T$, see Figure 2. We denote the edge set of w by W. We shall show below that w is a shortest path in $H'(V, T \cup C^* \setminus \{x, y\})$, the graph obtained from H by removing the edge $\{x, y\}$. It then follows immediately that $C_{xy} = W \cup \{x, y\}$ is a shortest cycle (in H) containing $\{x, y\}$. Furthermore, C_{xy} is the unique shortest cycle containing the edge $\{x, y\}$ if and only if w is the unique shortest path from x to y in H'. Claim (i) then follows from proposition 1.

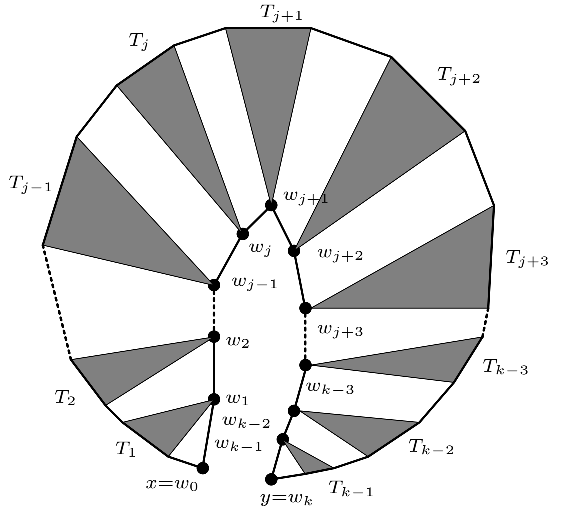

FIGURE 2. $\mathsf{w} = (x = w_0, w_1, \ldots, w_{k-1}, w_k = y)$ is a shortest path from x to y. The path along C has the same length if and only if each tree T_i, shown as shaded triangle, consists only of one vertex u_i and the edge $\{u_i, w_i\}$.

Since each internal vertex of T has degree at least 3, there is a subtree T_k extending from w_k to the cycle C^*. Any path in H' connecting x and y thus contains the edge $\{w_{i-1}, w_i\}$ or an edge $\{u_{i-1}, u_i\} \in C^*$ connecting an end point in T_{i-1} with an end point in T_i, $1 \le i \le k = |W|$. Any path that contains edges from both W and C^* must also contain at least one edge within the interior of a subtree T_i, $1 \le i < k$; it is therefore longer than $|W|$. Thus a shortest path from x to y in H' has length $|W|$, and hence a shortest cycle through $\{x, y\}$ has length $|W| + 1 \le |C^*|$. If this inequality is strict for every pair $\{x, y\} \in C^*$ then $\mathcal{F}$ is the unique minimal cycle basis by proposition 1.

It remains to deal with the case $|W| = |C^*| - 1$, in which the path along the "outside" of H' has the same length as W. In this case each of the subtrees T_i consists of a single vertex u_i and the edge $\{u_i, w_i\}$, $1 \le i < k$. Consequently, H is a necklace Ne_h and x and y must be ends of Ne_h. This completes the proof of claim (i).

Suppose $H = Ne_h$, $h \ge 3$. Then its ends x, y are unique. Each of the face cycles in $\mathcal{F} \setminus C_{xy}$ is therefore essential, and none of them contains the edge $\{x, y\}$. Any shortest cycle through $\{x, y\}$ therefore completes the minimum cycle basis. The only two possibilities are C_{xy} and C^* which indeed have equal length. This proves claim (ii).

If the end points of the necklace are not unique, then $h = 1$ or $h = 2$, i.e., $H = K_4$ or the trigonal prism. In both cases the minimal cycles bases are well known. ∎

Corollary. Every minimum cycle basis of a necklace consists of the interior faces of one of its H-feasible planar embeddings.

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