ADAPTIVE PLATFORM DYNAMICS IN SPATIAL VOTING MODELS

DISSERTATION

zur Erlangung des Akademischen Grades

Doctor rerum naturalium

eingereicht von

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an der

Formal- und Naturwissenschaftlichen Fakultät

der Universität Wien

Berndorf, im Mai 1998

To the Memory of

My Father and

My Grandmother

Zusammenfassung

In dieser Arbeit wird das dynamische Verhalten von konkurrierenden politischen Parteien im Rahmen eines räumlichen Wahl-Modells untersucht. Die Grundlage des spieldynamischen Modells ist die Annahme, dass politische Inhalte quantifiziert und deshalb als Punkte eines Euklidischen Raumes dargestellt werden können, dessen Koordinatenachsen den verschiedenen politischen Inhalten entsprechen. Wir nehmen an, dass die Wählerpositionen unveränderlich sind, während die Parteien die aktiven Spieler sind. Der Payoff einer Partei ist entweder der erwartete Anteil an Wählerstimmen oder der erwartete (relative) Vorsprung vor den anderen Parteien. Jede Partei adaptiert ihre Plattform-Position in kleinen Schritten, indem sie dem Gradienten ihrer Payoff-Funktion folgt. Das Ziel dieser Arbeit ist die Untersuchung des dynamischen Systems, das sich aus der Konkurrenz aller beteiligten Parteien ergibt.

Zunächst wird gezeigt, dass sich das dynamische System global "brav" verhält: Alle Bahnen erreichen in endlicher Zeit die Box, die von den extremen Wählerpositionen beschränkt wird, und alle (Hyper-) Ebenen im Phasenraum, in denen mindestens zwei Parteiplattformen übereinstimmen, sind invariant. Das System weist Permutationssymmetrie auf, falls die Wählerpräferenzen nur von den Plattformpositionen abhängen und unabhängig von den Parteinamen sind.

Modelle mit zwei Parteien zeigen ein überraschend einfaches dynamisches Verhalten: Unter einer Reihe von milden Voraussetzungen konvergieren alle Parteiplattformen zum Mittelwert der Wählerverteilung. Für konkave Wählerpräferenz-Funktionen ist dieser Mittelwert der global stabile Fixpunkt des Systems. Sogar bei Annahme von nicht-politischen Inhalten (welche eine Abhängigkeit der Wählerpräferenzen vom Parteinamen bedeuten), treten keine Bifurkationen auf.

Zwei verschiedene heuristische Modelle für Stimmenthaltung werden untersucht. Im ersten Fall wird angenommen, dass die Wahrscheinlichkeit, mit der ein Wahlberechtigter am Wahlgang teilnimmt, vom Unterschied der Präferenzen des Wählers für die beiden Parteien abhängt. In diesem Modell treten keine Bifurkationen auf. Wird hingegen angenommen, dass die Teilnahmewahrscheinlichkeit mit der Distanz zwischen Plattform und Wählerposition abnimmt, können je nach Wahl der Payoff-Funktion Bifurkationen auftreten.

Im Gegensatz zum Zweiparteienmodell wird der dem Mittelwert der Wählerverteilung entsprechende Fixpunkt in Modellen mit drei (und mehr) Parteien instabil, falls die Wähler hinreichend kritisch sind. Diese Beobachtung trifft sogar für den Fall von konkaven Wählerpräferenz-Funktionen zu. Das Auftreten von Bifurkationen hängt von der Steigung der multidimensionalen Payoff-Funktion ab, welche angibt, wie kritisch die Wähler sind. Bei numerischen Untersuchungen finden wir reichhaltige Bifurkationsdiagramme mit einer grossen Zahl von lokal stabilen Fixpunkten und Grenzzyklen. Viele der Resultate des Systems mit drei Parteien lassen sich auf Multiparteiensysteme verallgemeinern.

Abstract

We explore the dynamics of a model of competing political parties under spatial voting. Such models are based on the assumption that political issues can be quantified and hence party platforms as well as voter positions can be encoded as points in a Euclidean vector space the coordinates of which designate the different political issues. The active players are the parties; the payoff of a platform is determined as the expected fraction of votes that it receives, or as the expected fraction of votes that it is ahead of its competitor(s). The parties are allowed to *incrementally* adapt their platforms by following the voting gradient imposed by the preferences of the electorate and the platform of the opposition partie(s). The emphasis is on the dynamical system formed by these conditions.

The resulting dynamical system behaves reasonably at a global scale: all orbits are eventually bounded within a box that is spanned by the most extreme voter positions, and all planes in phase-space on which two or more platform positions coincide are invariant. If the voter utility functions are independent of the party labels the dynamical system has permutation symmetry.

Two-party models behave surprisingly simple: Under a wide variety of circumstances the platforms of all parties converge eventually to the mean voter fixed point. In particular, the mean voter point is globally stable for concave voter utility functions. For a much larger class of models we could at least ensure local stability of the mean voter fixed points. Even non-policy values (which introduce an explicit dependence of the voter utilities on the party labels) do not lead to bifurcations.

Abstention is modeled by two different heuristics: if the probability of participation depends only on the utility differences between two different platforms no bifurcations occur. If the participation probability decreases with distances between platforms and voter ideal points, the mean voter equilibrium may become unstable for some party payoff functions.

In contrast to the two-party case, in systems with three parties the mean voter equilibrium is unstable for sufficiently critical voters even in the case of concave voter utility functions. The bifurcation point is determined by the slope of the multidimensional sigmoidal response function that determines how critical the voters are. Numerically, we find elaborate bifurcation diagrams containing multiple locally stable fixed points and stable limit cycles. Many of the results for three parties can be generalized to multi-party models.

Preface

The idea of writing a thesis on spatial voting dynamics was born on a cold November evening in 1995 over a delicious sushi, when my husband Peter and I were visiting our friend John Miller at Carnegie Mellon University in Pittsburgh. I asked John, who was working on this subject, to lend me his copy of the book by Enelow and Hinich, which he has not received back since then. (It could have been worse, I could have borrowed his cool machine for peeling apples which at that time I liked even more.)

Back in Vienna, I asked Immanuel Bomze from the Department of Statistics, Operations Research, and Computer Methods at the Faculty of Social Sciences to supervise my thesis. He kindly agreed and it has been a pleasure to work with him and my second advisor Reinhard Bürger from the Department of Mathematics. The authorities of the Faculty of Natural Sciences approved of this slightly non-standard arrangement for a mathematics thesis without any bureaucratic complications. (Thanks to whom it may concern!)

Already during my masters thesis I had been using the computer resources of the Department of Theoretical Chemistry and it was very convenient for me that I was allowed to continue working as a non-resident alien in Peter Schuster's group. Thanks to the support of my friends there I became marginally computer-literate (I learned how to blame screw ups on the machine), and enjoyed the regular Tuesday-movies and a lot of great parties. Thanks everybody!

A substantial part of this work has been done at the Santa Fe Institute during our regular February and summer visits. Thanks for stimulating discussions, office space, and computer resources! John Miller's suggestions, who is an external faculty member at SFI, were very helpful for constructing my models. In February 1998 Paul Phillipson kindly agreed to proofread a first draft of this thesis during a visit to Boulder, Colorado. His efforts resulted in a substantial improvement of the presentation.

Vacations had a non-negligible impact on this work. The section on perturbations, for instance, originated at the beach of Loretto, a small town in Baja California Sur. The models for abstention are, as far as I recall, the eventual outcome of one Margarita too many in Guanajuato on another trip to Mexico.

This work, and indeed, all my studies, would not have been possible without the support from my family. My mother and my parents-in-law took care of our kids whenever I attended classes, worked in Vienna, or was on travel. My husband Peter always believed in me and always gave me hope when I was close to quitting. It was his enthusiasm for science that got me curious about this kind of work in the first place. My two sons Claus and Manuel were patient and understanding when I was working instead of playing with them. Last but not least I am grateful for all the help that I have received from my extended family over the years.

1. Introduction

1.1. Spatial Voting Theory

The importance of adaptive processes in economic theory has long been recognized (see, for example, Malthus [40]). In this thesis we shall employ the tools of evolutionary game dynamics to better understand the dynamic behavior of an adaptive system in which agents are constrained to *locally* adapt to their world.

Spatial voting theory describes two classes of agents: *voters* and *candidates* (parties) [22]. Spatial voting models are widely used in Economics and Political Science. Work on this subject can be traced back as far as to the 1920s in the papers of Hotelling, e.g. [28]. The core of the theory was developed in the classical works by Smithies [65], Downs [18], and Black [7], see section 1.9 for a brief overview.

Voters can vote directly over alternative policies in the case of a committee, or they can vote over alternative candidates in the case of an election. The key element in these types of models is that voters as well as candidates are seen as points in *issue space*. The issue space describes all the factors which are of concern to the voter. The voters are assumed to be able to evaluate the objects (i.e., policies or candidates) in terms of their own self-interest and to cast their votes on this basis. Each voter has a particular opinion on every issue to be voted on, and therefore a given stake or interest in the outcome of the vote which leads her to vote as she does. Furthermore, there are certain non-policy issues, such as a candidate's age, religion, and gender, on which parties have no influence and therefore these issues will remain fixed during the campaign.

A voter's position in issue space is referred to as the voter's ideal point. It is assumed that this position will does not change during the campaign. The number of different issues determines the dimension of the issue space. Party platforms are also regarded as points in issue space, but parties are allowed to modify their positions adaptively in order to gain more votes.

It is assumed that the voters are *sincere*[1], that means they know about their self-interest, evaluate alternative policies or candidates on the basis of which will best serve this interest and cast the vote for the policy or candidate most favorably evaluated. Candidates are sincere (in the sense that they do not lie about their true platform position) and opportunistic (in the sense that their only goal is to win the election). The theory of spatial voting does not explain the source or form that a voter's or a candidate's self-interest takes.

1.2. Voter Preferences

Spatial theory is an attractive framework to analyze choice because Euclidean geometry is easy to visualize and the language of politics itself is replete with

[1] We distinguish sincere voters from rational voters who might vote strategically for a party that is not closest to their ideal point in order to favor a particular coalition.

spatial references, e.g. consider the terms “left”,“right”, “moving left”, etc. Collective choice is interesting only when preferences are not perfectly homogeneous and conflict results by positing that different voters have different preferences over the alternatives. These preferences may be represented algebraically as well as geometrically.

Viewed in simplest spatial terms, the voter will cast her vote for the candidate “closest” to her. For our purposes, it is very useful to regard the issue space as an I-dimensional Euclidean space, where I denotes the number of different issues. We now want to introduce the analogy between preference and distance that is central to spatial voting models.

Let us first assume that all issues of the campaign are policy issues. The *weighed* Euclidean distance between voter v's ideal point x_v and party p's platform y^p is defined by

$$\|y^p - x_v\|_{\mathbf{S}_v}, \tag{1.1}$$

where $\mathbf{S}_v$ is a positive definite matrix of strength factors that determine how much voter v cares about each issue. Let us think of a four-issue model; the issues being money spent on education, money spent on defense, money spent on road construction, and voting for or against abortion. It may well be that a voter cares with equal strength about every issue and that her preferences are separated, i.e., that her position on any one of the issues does not depend on any position on the other issues. Then $\mathbf{S}$ is the 4×4 identity matrix. If the voter's feelings are more differentiated, for example, if she is very emotional about the abortion issue, less interested in education, her interest in defense and road construction is even less, but still the preferences are separated, then

$\mathbf{S}$ will be a diagonal matrix, the different entries representing the strengths given to different issues. Issues on which a voter puts more weight are called more *salient* to her. Other voters may have "ideal packages" of preferences, i.e., preferences will be correlated. If, for example, for a voter money spent on education is somehow correlated with money spent on road construction, then $\mathbf{S}$ will be a positive definite symmetric matrix, the off-diagonal entries determining correlation among the issues.

Example. Consider a two-issue model. Suppose (x_{b1}, x_{b2}) is voter b's preferred package of spending on both issues and preferences are correlated by

$$\mathbf{S} = \begin{pmatrix} s_{11} & s_{12} \\ s_{12} & s_{22} \end{pmatrix}.$$

It is required that $s_{11} > 0$, $s_{22} > 0$ and $s_{12}^2 < s_{11}s_{22}$. If spending on the first issue is fixed at $y_1 \neq x_{b1}$, then the voter's most preferred spending level on the second issue will be the value of x_2 that minimizes the function

$$f(x_2) = [s_{11}(y_1 - x_{b1})^2 + 2s_{12}(y_1 - x_{b1})(x_2 - x_{b2}) + s_{22}(x_2 - x_{b2})^2]^{\frac{1}{2}}.$$

Voter b's most preferred spending level on the second issue is thus given by

$$x_{b2}(y_1) = x_{b2} - \frac{s_{12}}{s_{22}}(y_1 - x_{b1}).$$

The ratio $\frac{s_{12}}{s_{22}}$ determines the size of the shift from his package ideal spending level for project 2, while the sign of s_{12} determines the direction of this shift.

1.3. Voter Utilities

A *utility function* $u_v(y)$ measures the utility of a particular candidate or political platform y for voter v. If all issues are policy issues, we define

$$u_v(y^p) = -(y^p - x_v)\mathbf{S}_v(y^p - x_v). \tag{1.2}$$

The voter's utility function has its maximum at the voter's ideal point, and it declines with the distance from this ideal point. In literature, often the term *dissatisfaction* function instead of *utility* function is used.

The geometric representation of preferences is in the form of indifference curves. By definition, all points on a voter's indifference curve have the property that the voter's ideal point associated with any point on that curve yields exactly the same utility. Indifference curves are analogous to contour lines in hiking maps, figure 1.

Each contour thus gives points of common utility (altitude). The most restrictive kind of preferences are *Euclidean preferences*, for which utility declines monotonically with distance. This can be represented algebraically by $\mathbf{S} = \mathbf{I}$, where $\mathbf{I}$ is the identity matrix. Geometrically, the indifference contours are circles in this case. A weaker characteristic of preferences is so-called separability [35]. Then $\mathbf{S}$ is a diagonal matrix of different strength entries and the indifference contours are ellipses in main position. In this case, points equidistant from a voter's ideal point lie on different indifference contours. In a two-dimensional choice space, if a one-step change in x_1-direction leads to a higher indifference contour than a one-step change in x_2-direction, the first issue is more salient to

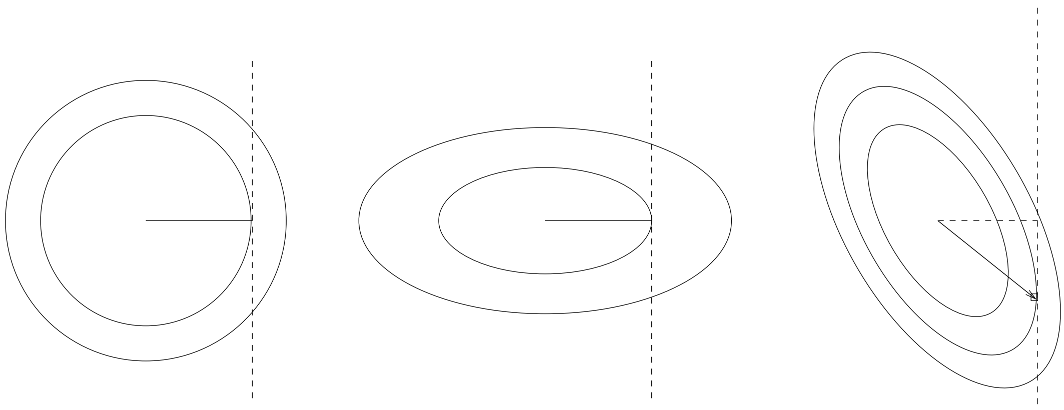

Figure 1: Geometric representation of voter preferences. The center point of each graph represents a voter's ideal point. In the case of separable preferences (left, middle) a voter's most preferred point at a given distance in x_1-direction stays at the voter's ideal x_2-level. Thus, Euclidean preferences are a special form of separable preferences. In the case of convex preferences, a voter's most preferred point at a given distance in x_1-direction is at an x_2-level that is different from its ideal x_2-level (right).

the voter. Convex preferences are the most general assumptions about preferences usually made in spatial voting theory. In this case, **S** corresponds to a symmetric matrix with correlation among several issues. Unlike in the case of separable preferences, the weaker assumption of convexity allows a voter's ideal point on a given dimension to be functionally related to policy on another dimension. Convex preferences hence lead to indifference curves that are general ellipses.

The model used most widely, originally proposed by Enelow and Hinich, is based on separable preferences, see [22]. We shall require a much less stringent condition on the voter utility functions in this work: we assume that $u_v : \mathbb{R}^I \to \mathbb{R}$ satisfies

$$(y_j - x_{vj})\frac{\partial u_v}{\partial y_j}(y) < 0 \tag{1.3}$$

for all $y \neq x_v$, i.e., u_v has a unique maximum at the voter's ideal position x_v, but $-u_v$ need not be convex.

Unless stated otherwise, we shall in general not require that the indifference curves are convex.

1.4. Strength Factors and Non-Policy Values

Davis, Hinich, and Ordeshook [16] broadened the scope of the spatial voting literature by including weights or strengths in a multidimensional model. Throughout their book [22], Enelow and Hinich use therefore a quadratic voter utility function of the form

$$u_v(y^p) \stackrel{\text{def}}{=\!=} m_{vp} - \sum_{i \in I} s_{vi}(y_i^p - x_{vi})^2 . \tag{1.4}$$

where s_{vi} is the *strength factor* measuring how strongly voter v feels about issue i, and m_{vp} is the so-called *non-policy value* of party p to voter v. Completely indifferent voters, for who $s_{vi} = 0$ for all issues, do not influence the expected outcome of elections and can therefore be neglected in our model. If u_v is not of the form (1.2) we may interpret the curvatures of the voter utility function at the ideal point as analogues of the strength factors.

Some voters base their decisions among candidates entirely on a single issue, such as abortion, civil rights, or foreign policy. Such a voter would attach weight 1 to a single issue and 0 to all other issues. Most voters probably distribute their strengths s_{vi} more evenly. The correlation between ideal points and strength may be used to characterize different types of voters. For instance, Kollman, Miller, and Page [32] consider *centrist* voters which place more weight on issues on which they have moderate views, *extremist* voters placing more weight on issues on which they have extreme views, and *uniform* voters with equal weights on every issue. In their model, strength is a function of ideal point:

$$\begin{array}{rlc} s_{vi} = & 1 - |x_{vi}| & \text{centrist} \\ s_{vi} = & |x_{vi}| & \text{extremist} \\ s_{vi} = & 1/2 & \text{uniform} \end{array} \tag{1.5}$$

Non-policy issues are relatively fixed characteristics of the candidate that are generally beyond his control, at least for the duration of the campaign. Age, religion, party, and gender are obvious examples of non-policy issues in, say, a presidential election. Similarly, the personal attributes of a party's politicians are non-policy issues in congressional elections. We may therefore assume that the non-policy values are in general independent of platform positions. The most general voter utility functions considered in this contribution are therefore of the form $\tilde{u}_v(y^p) = m_{vp} + u_v(y^p)$, where u_v depends on the policy issues only. Since only differences of the form $u_v(y^p) - u_v(y^q)$ will enter our models, we neglect the non-policy values without loosing generality if they do not explicitly depend on p. Throughout this thesis we shall assume $m_{vp} = 0$ unless stated otherwise.

1.5. Experimental Data for Spatial Voting Models

Extensive empirical analyses have been performed by Enelow, Hinich, and co-workers on U.S. elections, see e.g. [22, 20, 21, 19]. Earlier work on U.S. elections is reported in [69, 60, 55, 51]. Applications to congressional voting include [38, 39, 36, 37, 52, 50]. In these studies data from the American National Election Study [45], in particular so-called "thermometer scores" (for which a respondent is asked to gauge support or disapproval for a particular candidate on a numerical scale), are used to construct spatial maps of voter and candidate positions, see figure 2.

Empirical tests have generally supported spatial election theory but the estimation methods employed to produce the spatial representations of voters have raised serious statistical issues which have not been fully resolved. One of these issues is determining the number of dimensions, which is rather difficult because the number of estimated parameters increases with the number of dimensions [53].

1.6. Party Payoffs and Electoral Landscapes

Party platforms or candidates are described by their positions y^p in issue space $\mathbb{R}^I$, just as voters are defined by their ideal positions (and utility functions). The basic assumption of all spatial voting models is that each voter will vote for the party that yields the largest value of $u_v(y^p)$. However, the information

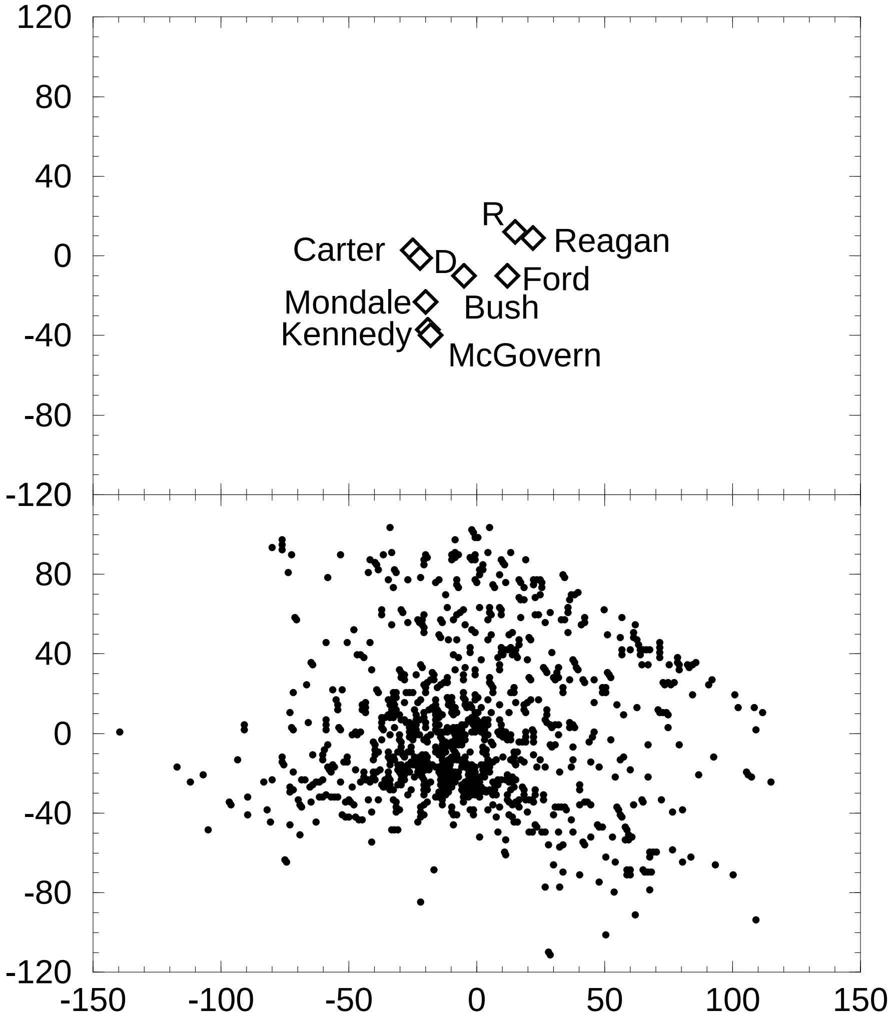

Figure 2: Candidate positions (above) and voter positions (below) for the 1980 US presidential election in a two-dimensional issue space.The data are taken from ref. [19]. The coordinate axes correspond to the first two principal components of the data set.

about a platform as well as the position of a voter in issue space will be known to the individual voter only with a certain accuracy [22, chap. 7]. We model this behavior following [44] by introducing the probability $\mathcal{P}$ that voter v chooses platform (party, candidate) 1 given the utility differences between the platform positions of all involved parties. We refer to $\mathcal{P}$ as the *response function* of the voters. If the voter's knowledge is infinitely accurate, then $\mathcal{P}_\infty = 1$ if party 1 has the largest utility for voter v. If m parties have the same largest utility as party 1, then $\mathcal{P}_\infty = 1/m$, since voter v will vote for each of these m parties with the same probability $1/m$; otherwise $\mathcal{P}_\infty = 0$.

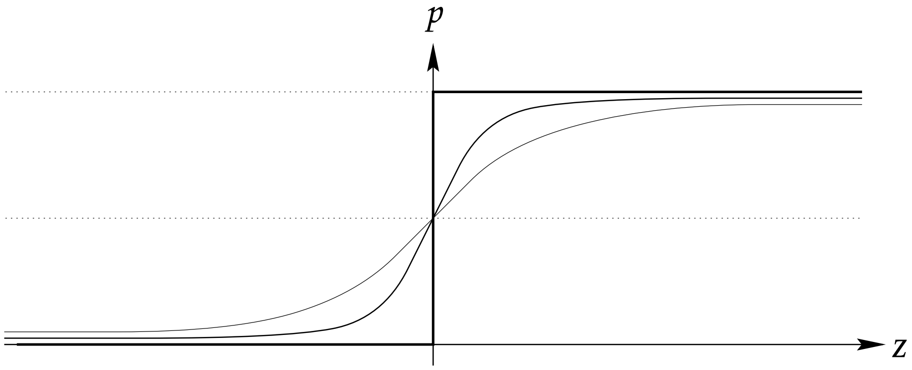

Figure 3: The sigmoidal response function $\mathcal{P}$ is used to model uncertainties in a voter's decision to vote for party 1. In the two-party case it depends only on the utility difference $z = u(y^1) - u(y^2)$. The uncertainty increases as the slope $\mathcal{P}'(0)$ decreases.

In the case of two parties $\mathcal{P}$ will be a sigmoidal function, see figure 3 for details. For the sake of mathematical tractability we shall assume throughout this work that the function $\mathcal{P} : \mathbb{R} \to [0,1]$ does *not* depend on the individual voter v. Almost the entire literature on spatial voting assumes complete knowledge and hence works with the discontinuous function $\mathcal{P}_\infty$. For our purposes it will be necessary to assume that $\mathcal{P}$ is a continuously differentiable approximation of $\mathcal{P}_\infty$.

The outcome of an election, that is the fraction of votes that each party receives, determines its *payoff.* More precisely, the payoff of (or utility for) party p is the

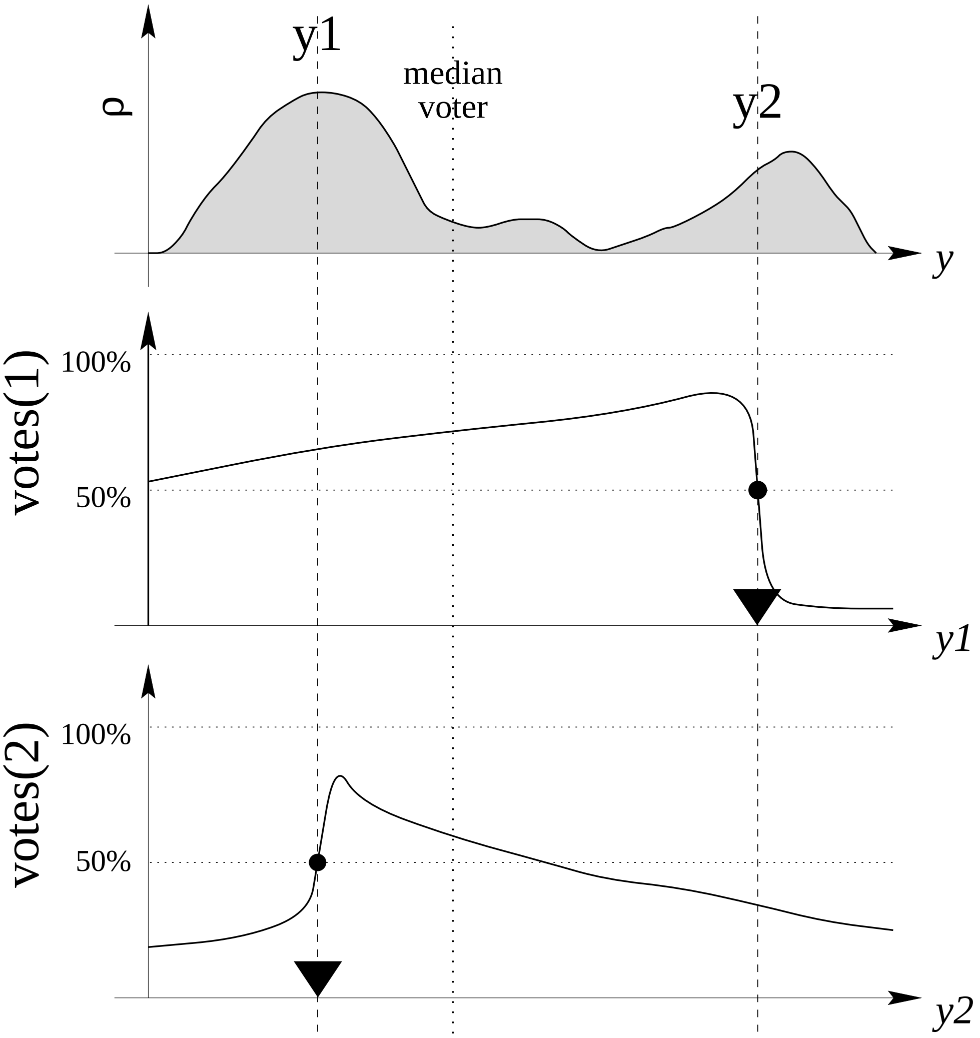

Figure 4: Voter distribution and electoral landscapes.
The upper part of the figure shows the voter distribution ρ in a 1-dimensional issue space. The two parties occupy at a given time the positions y^1 and y^2, respectively. The lower portion of the plot shows the expected number of votes, for parties 1 and 2 as a function of their own platforms, given that the other party stays at its current position. We call these curves the "electoral landscapes" perceived by the two parties. The shape of the electoral landscapes can be explained by the fact that party 1 receives the votes of all voters to the left of $(y_1 + y_2)/2$ (apart from the uncertainties introduced by $\mathcal{P}$). Hence Party 1 could receive more votes if it took a position closer to party 2, and conversely, party 2 could increase its share by moving closer to party 1.

expected fraction of votes

$$\begin{aligned}
&E_p(y^p|y^1,\ldots,y^{p-1},y^{p+1},\ldots,y^P)\\
&=\frac{1}{V}\sum_v \mathcal{P}(d_v(y^p,y^1),\ldots,d_v(y^p,y^{p-1}),d_v(y^p,y^{p+1}),\ldots,d_v(y^p,y^P),\\
&d_v(y^1,y^2),d_v(y^1,y^3),\ldots,d_v(y^1,y^P),\\
&d_v(y^2,y^3),\ldots,d_v(y^2,y^P),\ldots,d_v(y^{P-1},y^P))\,,
\end{aligned} \tag{1.6}$$

i.e., the sum over the probabilities that each voter v votes for party p. Here $d_v(y^k, y^l) = u_v(y^k) - u_v(y^l)$ is the utility difference of the platforms y^k and y^l for voter v. We shall refer to $E_p(y^p|\ldots)$, with the positions of all other parties fixed, as the *electoral landscape* [32, 33] of party p, see figure 4. The landscape metaphor is a common model in adaptive search. Applications to political science include [5] and [6]. In the context of biological evolution it goes back to Sewall Wright [71].

It is not hard to see that the vote landscapes in a two-party one-issue model are always unimodal. This is not necessarily true for more than one issue or more than two parties. There is a relationship between the distribution of voters' strengths and the slopes and positions of peaks on an electoral landscape [32]. The electoral landscapes defined in equ.(1.4) depend in addition on the slope of the sigmoidal function $\mathcal{P}$. A few examples of typical electoral landscapes arising from the same uniform voter distribution are collected in figure 5.

Instead of specifying the individual position of each voter v it may be more realistic and/or more convenient to introduce the density $\rho(x)$ of voters in issue space. It is normalized in the usual way: $\int_{\mathbb{R}^I} \rho(x)dx = 1$. Instead of $u_v(y)$ we now need to specify the utility function $u(y, x)$ of a platform y for a voter

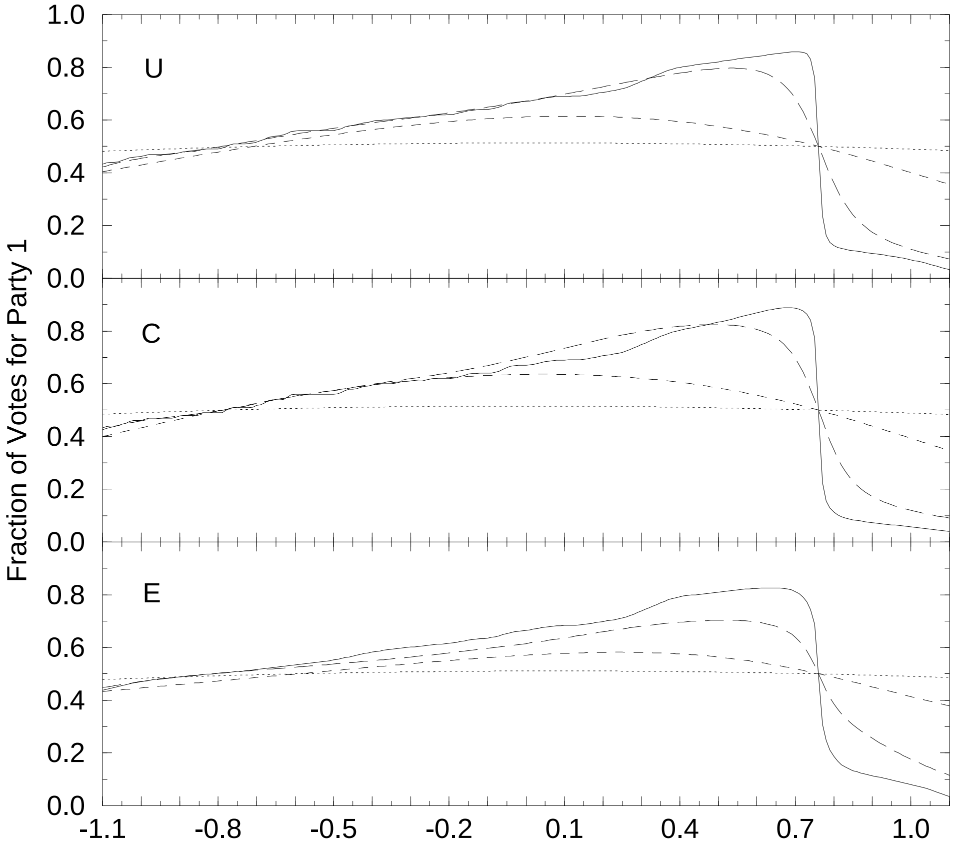

Figure 5: Examples of electoral landscapes with different strength functions $s(x)$ and sigmoidal functions $\mathcal{P}$ with different slopes $\mathcal{P}'(0)$. The letters U, C, E correspond to *uniform*, *centrist*, *extremist* voter, resp., as defined in equ.(1.5). For details see text.

with position x. Furthermore, we may assume that strength is a function of the ideal point, as defined in equ.(1.5). In analogy with the definition of d_v it is convenient to set $d^x(y^k, y^l) \stackrel{\text{def}}{=} u(y^k, x) - u(y^l, x)$, where $u(y, x)$ denotes the utility of party platform y for voter with ideal point x. For instance, we have

$$u(y, x) = -\sum_{i=1}^{I} s_i(x)(y_i - x_i)^2 \tag{1.7}$$

where the strength factor is a function of the platform position as in equ.(1.3). The expected outcome of an election is then

$$E_p(y^p|y^1,\dots,y^{p-1},y^{p+1},\dots y^P) = \int_{\mathbb{R}^I} \mathcal{P}\Big(d_x(y^p,y^1),\dots,d_x(y^p,y^{p-1}),d_x(y^p,y^{p+1}),\dots,d_x(y^p,y^P),\dots\Big)\,\rho(x)\,dx\,. \tag{1.8}$$

A more general model could be constructed by introducing a joint distribution $\rho(x,s)$ of voter positions x and strengths s.

Using the δ-distribution we may translate the "discrete" model (1.6) into this form by defining

$$\rho(x) = \frac{1}{V}\sum_v \delta(x - x_v)\,. \tag{1.9}$$

The relation of discrete and continuous voter distribution functions will be discussed in some more detail in chapter 5.

1.7. Platform Dynamics

The emphasis of our model is on an *adaptive dynamical system*, whereby global consequences emerge from locally adapting candidates. The inspiration for this model comes from the computational results of Kollman, Miller, and Page [30, 31]. They found that parties following simple locally adaptive rules rapidly converged toward common platforms. While the ideas of probabilistic voting (for a general review see [12]) and locally restricted strategy searches in such models [13, 63] have been widely discussed, here we assume that parties do not start at an identical status quo platform, and that their ability to maximize voter support is limited to climbing the local voting gradient.

The players in our models are the parties which may change their platforms in order to increase their share of votes. The expected fraction of votes E_p, that is, the electoral landscape for each party, defines the payoff of a strategy (platform). The basic assumption is that each party tries to increase its share of votes by small corrections to its platform y, see figure 6. Each party attempts to increase its own utility by means of steepest ascent in the variables under its own control, i.e., by locally optimizing its own platform under the assumption of a fixed position of the platforms of all other parties. The other players of course react to the changes in our party's platform and adjust their positions. Assuming that these platform adjustments are being conducted continuously, guided by, say, opinion polls, we argue that

$$\dot{y}^p = \nabla_{y^p} E_p(y^1, ..., y^P), \qquad p = 1, \ldots, P \tag{1.10}$$

is a plausible ansatz for the dynamics of platform adjustment. This dynamics corresponds to simultaneous hill-climbing of each party on its own electoral landscape. However, a party's vote landscape is constantly changing due to the movements of all other parties in issue space. This type of game dynamics was introduced in [44] for a two-party spatial voting model. Note, however, that the dynamics here depends on the fraction of votes, not on the number of votes. As a consequence, the velocity of adaptation does not depend on the size of the electorate.

The platform dynamics is a dynamical system, equ.(1.10), "living" on the *phase space* $\mathbb{R}^{PI}$. It will sometimes be necessary to directly refer to vectors in phase

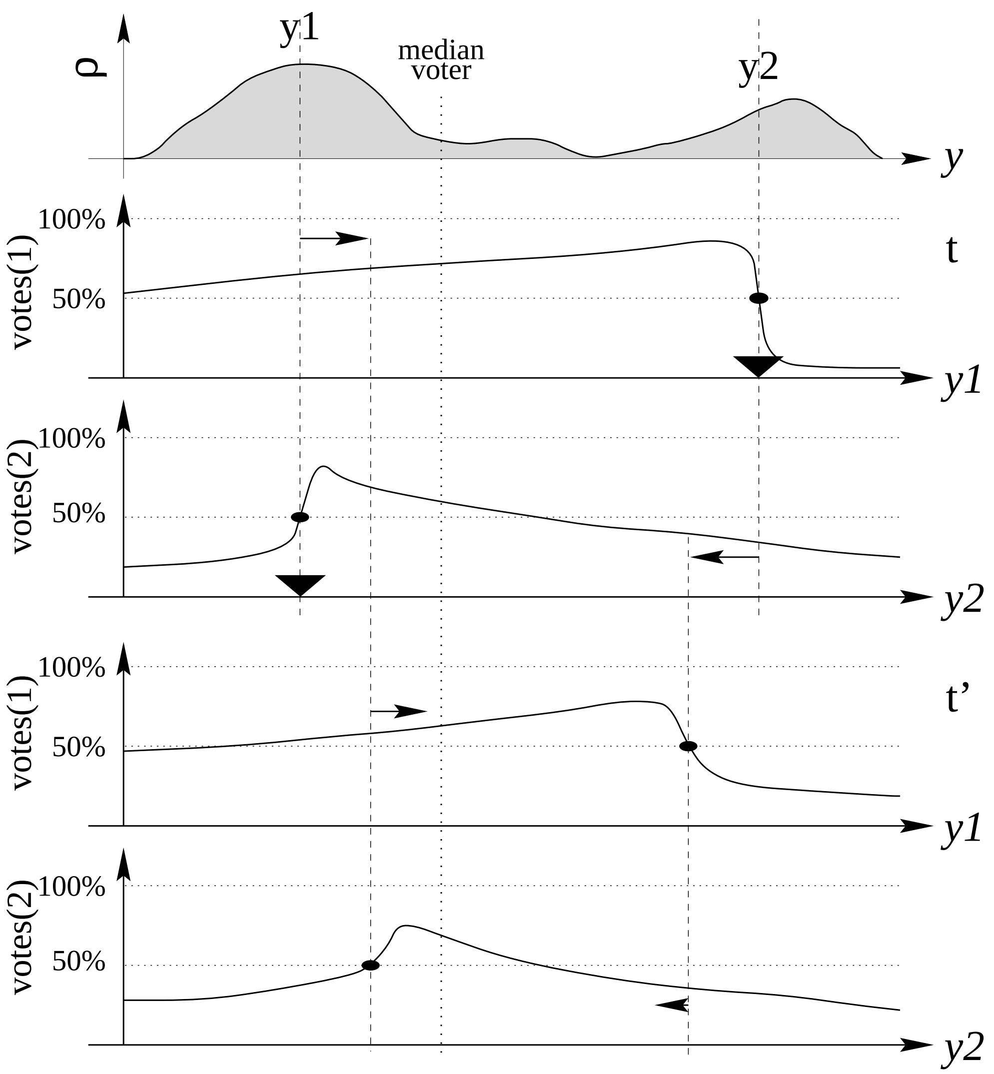

Figure 6: The topmost part of the figure shows again the voter distribution ρ in a 1-dimensional issue space. At time t, both parties will alter their positions following the gradient of their own expected election outcome in issue space. By changing its position y^2, however, party 2 changes the vote landscape for party 1 at time t', and *vice versa*. The "vote landscapes" thus change at the same time scale at which the parties try to hill-climb on them.

space. In the following, we will use the notation $x, y \in \mathbb{R}^I$ for positions (vectors) in issue space and $\vec{y} \in \mathbb{R}^{PI}$ for vectors in phase space.

A troubling result of voting theory is the general lack of stable equilibria once the assumption of a symmetric ideal point distribution is relaxed. Under simple preference-based voting it has been shown that in election theory, a pure strategy equilibrium for the candidates becomes quite rare when the election concerns two or more issues [49, 14, 34]. This discovery has its counterpart in the committee-voting theory when voting takes place over multidimensional policy alternatives. There exists rarely a policy alternative that cannot be defeated in a majority vote. An even more disappointing discovery has been made in the absence of majority rule equilibrium: the majority preference relation may engulf the entire outcome in one gigantic cycle [42, 41]. This discovery caused some theorists to despair of ever being able to predict candidate behavior or committee-voting outcomes [57]. In this work we shall see that the voting dynamics proposed above restores stable outcomes to a certain extent, at least in the long run.

1.8. Platform Dynamics Versus Other Types of Game Dynamics

It seems interesting to compare the dynamics of party platforms, equ.(1.10), with some more standard models of game dynamics.

Probably the most widely used type of game dynamics is the *replicator equation* [26]. One assumes a (usually finite) set of pure strategies (but see e.g. [10, 8, 9, 72] for the infinite case). Each strategy is played with a certain probability x_k. These probabilities are changed according to difference between the payoff $f_k(x)$ for strategy k in the current population and the average payoff of all strategies:

$$\dot{x}_k = x_k \big(f_k(x) - \sum_j x_j f_j(x) \big) \,. \tag{1.11}$$

The multi-population version [67, 68]

$$\dot{x}_k^p = x_k^p \big(f_k^p(x^1, \dots, x^P) - \sum_j x_j^p f_j^p(x^1, \dots x^P) \big) \,. \tag{1.12}$$

might be an alternative starting point for a model of platform adaptation. In this picture the platform of party p would be represented as a superposition of "pure" positions, the weights of which are given by the variables x_k^p. The payoff of such a pure strategy is $f_k^p(x^1, \dots, x^P)$ which of course depends on the position of all other parties. Such an approach, however, feels much less natural than the simple gradient dynamics. In particular, the choice of a set of pure strategies would be rather artificial in the context of spatial voting theory.

Hofbauer and Sigmund [27] consider the evolution of an essentially monomorphic population under the assumption that a small number of mutants y that are very similar to the consensus x test out alternatives. Let $E(y, x)$ be the fitness (payoff) of such a mutant in a monomorphic x-population. *Adaptive Dynamics* assumes that the whole population moves into the direction of the most promising mutant:

$$\dot{x} = \nabla_y E(y, x) \Big|_{y=x} \tag{1.13}$$

In contrast to replicator equations, this model is not restricted to the simplex. It describes, however, the time evolution of a single population rather than the coevolution of two populations. A multi-population version of equ.(1.13), however, might be useful in a more detailed model of the mechanism by which a party modifies its platform.

1.9. A Brief Overview of Spatial Voting Theory

Spatial voting theory can be traced back as far as to the 1920s in the papers of Hotelling [28]. Further research was done by Smithies in the forties [65]. In the early work of A. Downs [18] and D. Black [7] a large body of theory based on simple geometric representations of individual preferences was developed. It was used for modeling policy choice in legislatures and in the mass electorate. Black originally analyzed these two social choice problems with a single model. Nevertheless, it is now customary to divide the spatial theory of voting into the spatial theory of committees and the spatial theory of elections.

In the spatial theory of committees, the voters are the key actors, voting over different policy alternatives, each of which is usually represented as a point in a Euclidean space. In contrast, the spatial theory of elections treats the candidates as the key actors, with the voters playing a fixed role. Results in the spatial theory of elections have analogues in the spatial theory of committees. For example, the absence of a pure strategy equilibrium in two-candidate contests is equivalent to the absence of a policy alternative that is undefeated in pairwise committee voting. Still, one must proceed with caution when translating results obtained in election theory into results in committee theory.

Initially, given Black's formulation, candidates were interpreted as nothing more than policy alternatives. The voters perceived a candidate as a vector of positions on the policy issues of the campaign. Voters were assumed to vote strictly according to their preferences over policy alternatives. The resulting model thus bore no essential difference to that of a committee deciding which policy alternative to select. Downs [18] was the first to begin construction of a spatial theory explicitly designed for elections. Davis and Hinich [15] built the mathematical foundations for such a theory. From that point on, election theory and committee theory have taken a very different course. With the introduction of sophisticated voting, committee theory received a game-theoretic foundation, providing voters with a strategic theory of behavior appropriate for small groups [23].

Much effort was put on looking at solution concepts for n-person cooperative games that could predict the outcome of committee voting in the absence of an undominated outcome. Interesting ideas such as the "bargaining set" [4], the "main-simple V-set" [70], and the "competitive solution" [43] were explored, premised on weaker forms of outcome stability. On the non-cooperative side, Shepsle [64] originated a new approach to committee voting by emphasizing the role of structures and procedures in shaping outcomes. Romer and Rosenthal [59] introduced the field of positive models based on agenda theory. It became manifest that outcomes are affected by determining powers, such as the order in which policies are voted on, and the method of dividing the policies that are voted on.

Mixed results were obtained on the question of outcome stability in non-cooperative committee voting. In the case of one-issue-at-a-time voting, sincere voting, i.e., voting according to one's preferences, leads to an equilibrium majority outcome, but if voters are sophisticated, an equilibrium outcome may not exist [46]. It has become a challenge in committee voting to find out those equilibria that correspond to fully rational behavior on the part of the players. Sophisticated equilibrium, for example, meets the criterion of the term "subgame perfect equilibria". Keith Krehbiel [35] gives a review of recent developments in the field of legislative choice.

Election theory, on the other hand, having its focus on large electorates, was rather aimed at generalizing the model's key assumptions while retaining the basic framework of strategic candidates and non-strategic voters. Riker and Ordeshook [58] give an excellent description of spatial election theory. Besides a range of assumptions about voters and candidates a specification of the combinations of assumptions that imply the existence of a pure strategy equilibrium for the candidates is given. Furthermore, voters are permitted to have different-shaped utility functions. For instance they are allowed to abstain from alienation or indifference, i.e., ideal point distributions are no longer required to be unimodal or symmetric. Spatial election theory now has mushroomed to the point where it can be broken into subfields, such as agenda theory and probabilistic election theory.

2. Two Parties

2.1. Mathematical Model

We consider a system with V voters, $P = 2$ parties, and I issues. The *platform* of a party p is a point y^p in the I-dimensional *issue space* $\mathbb{R}^I$. Each voter is characterized by her preferred position $x_v = (x_{v1}, \ldots, x_{vI})$ in issue space and by the function $u_v : \mathbb{R}^I \to \mathbb{R}$ which is used to evaluate the utility of a party platform y for voter v. We shall assume as usual that equ.(1.3) holds, i.e., that u_v has a unique maximum at the voter's ideal position x_v. An example is the quadratic voter utility function equ.(1.4) introduced by Enelow and Hinich [22]. The expected outcome of an election, equ.(1.6), is simply

$$\begin{aligned} E_1(y^1, y^2) &= \frac{1}{V} \sum_v \mathcal{P}(u_v(y^1) - u_v(y^2)) \\ E_2(y^1, y^2) &= \frac{1}{V} \sum_v \mathcal{P}(u_v(y^2) - u_v(y^1)) \end{aligned} \tag{2.1}$$

in the case of two parties. It is convenient to use the notation

$$z_v \stackrel{\text{def}}{=\!=} u_v(y^1) - u_v(y^2) \tag{2.2}$$

for the utility difference (instead of d_v^{12}). Following [44], $\mathcal{P}(z_v)$ is the probability that voter v chooses platform 1 that has a difference z_v in utility against the opposing platform 2. For the sake of mathematical tractability we assume that the sigmoidal function $\mathcal{P} : \mathbb{R} \to [0, 1]$ does *not* depend on the individual voter

v. Furthermore, we assume throughout this section that $\mathcal{P}$ is twice continuously differentiable. Sigmoidal functions are discussed to some detail in appendix A.

In two-party models it is customary to assume that party p's utility, W_p, is given by the fraction of votes that it is ahead of its rival:

$$\begin{aligned} W_1(y^1, y^2) &= E_1(y^1, y^2) - E_2(y^1, y^2) \\ W_2(y^1, y^2) &= E_2(y^1, y^2) - E_1(y^1, y^2) \end{aligned} \tag{2.3}$$

The corresponding game dynamics is therefore $\dot{y^p} = \nabla_{y^p} W_p(y^1, y^2)$. This dynamical system was first described in [44]. Explicitly we have

$$\begin{aligned} \dot{y}^1 &= \nabla_{y^1} \frac{1}{V} \sum_v \left[\mathcal{P}(z_v) - \mathcal{P}(-z_v)\right] = \frac{1}{V} \sum_v [\nabla_{y^1} \mathcal{P}(z_v) - \nabla_{y^1} \mathcal{P}(-z_v)] \\ &= \frac{1}{V} \sum_v [\mathcal{P}'(z_v) + \mathcal{P}'(-z_v)] \nabla_{y^1} z_v = \frac{1}{V} \sum_v [\mathcal{P}'(z_v) + \mathcal{P}'(-z_v)] \nabla u_v(y^1) \\ \dot{y}^2 &= \frac{1}{V} \nabla_{y^2} \sum_v \left[\mathcal{P}(-z_v) - \mathcal{P}(z_v)\right] = \frac{1}{V} \sum_v [\nabla_{y^2} \mathcal{P}(-z_v) - \nabla_{y^2} \mathcal{P}(z_v)] \\ &= \frac{1}{V} \sum_v [\mathcal{P}'(z_v) + \mathcal{P}'(-z_v)] \nabla_{y^2}(-z_v) = \frac{1}{V} \sum_v [\mathcal{P}'(z_v) + \mathcal{P}'(-z_v)] \nabla u_v(y^2) \end{aligned} \tag{2.4}$$

In the case of continuous voter distributions we have instead

$$\dot{y^p} = \int_{\mathbb{R}^I} \Big[\mathcal{P}'\big(u(y^p, x) - u(y^q, x)\big) + \mathcal{P}'\big(u(y^q, x) - u(y^p, x)\big) \Big] \nabla u(y^p, x)\, \rho(x) \mathrm{d}x \tag{2.4$'$}$$

For sake of reference we note that equ.(2.4) reads in component-wise notation

$$\begin{aligned} \dot{y}^1_j &= \frac{1}{V} \sum_v \left[\mathcal{P}'(z_v) + \mathcal{P}'(-z_v)\right] \partial_j u_v(y^1) \\ \dot{y}^2_j &= \frac{1}{V} \sum_v \left[\mathcal{P}'(-z_v) + \mathcal{P}'(z_v)\right] \partial_j u_v(y^2) \end{aligned} \tag{2.5}$$

where $\partial_j u_v(y^p)$ denotes the j-th component of the vector $\nabla u_v(y^p)$. The manifold $\mathcal{H}_2 \overset{\text{def}}{=\!=} \left\{ \vec{y} \in \mathbb{R}^{2I} | y^1 = y^2 \right\}$ on which z_v vanishes is invariant under the game dynamics (2.4). It is worth noting that (2.4) has permutation symmetry, i.e., a permutation of the platform indices leaves the dynamical system unchanged. Consequently fixpoints outside $\mathcal{H}_2$ have to come in pairs: if $(\hat{a}, \hat{b})$ is a fixed point of (2.4), so is $(\hat{b}, \hat{a})$.

2.2. Boundedness of the Orbits

The extremal voter positions are $x_i^{\min} = \min_v x_{vi}$ and $x_i^{\max} = \max_v x_{vi}$. These points span the box

$$\mathcal{B} \stackrel{\text{def}}{=\!=} \left(\prod_{i=1}^{I} [x_i^{\min}, x_i^{\max}] \right)^2 \tag{2.6}$$

in the state space $\mathbb{R}^{2I}$. Miller and Stadler [44] showed that all orbits of a special case of the above model will eventually approach $\mathcal{B}$.

Theorem 1. The box $\mathcal{B}$ defined by the extremal voter positions is forward invariant under the dynamics (2.4); all orbits of the ODE (2.4) are eventually bounded in an arbitrary small neighborhood of $\mathcal{B}$.

Proof. By definition we have $\mathcal{P}'(z_v) + \mathcal{P}'(-z_v) > 0$ everywhere. Suppose $y_k^p > x_k^{\max}$. Then $\partial_k u_v(y^p) < 0$ for all voters v and hence $\dot{y}_k^p < 0$. In fact, it is even bounded away from 0 within any compact set fulfilling $y_k^p > x_k^{\max} + \delta$ with any arbitrarily small constant δ. The same argument can be made for $y_k^p < x_k^{\min}$. Thus all orbits will eventually wind up in a δ-neighborhood of $\mathcal{B}$. ∎

Remark. If $x_k^{\min} < x_k^{\max}$ for all issues k then $\mathcal{B}$ is even strictly forward invariant under the dynamics (2.4) and all orbits are eventually bounded within $\mathcal{B}$.

2.3. Trivial Fixed Points

We shall call a fixed point $(\hat{x}, \hat{x}) \in \mathcal{H}_2$ a *trivial fixed point* of the voting dynamics (2.4). Introducing the *average voter utility*

$$U(y) \stackrel{\text{def}}{=\!=} \frac{1}{V} \sum_v u_v(y) \tag{2.7}$$

we find that the dynamics (2.4) reduces to the gradient system

$$\dot{y} = 2\mathcal{P}'(0) \nabla U(y) \tag{2.8}$$

within the manifold $\mathcal{H}_2$. The trivial fixed points are thus the critical points of U. Using the identity $\dfrac{\partial^2}{\partial y_k^p \partial y_l^q} u_v(y^p) = 0$ for $p \neq q$ we obtain the general form of the Jacobian matrix of the dynamical system (2.4)

$$\begin{aligned}
\frac{\partial \dot{y}_j^1}{\partial y_k^1} &= \frac{1}{V} \sum_v \left[\mathcal{P}'(z_v) + \mathcal{P}'(-z_v)\right] \partial_k \partial_j u_v(y^1) \\
&\quad + \frac{1}{V} \sum_v \left[\mathcal{P}''(z_v) - \mathcal{P}''(-z_v)\right] \partial_k u_v(y^1) \partial_j u_v(y^1) \\
\frac{\partial \dot{y}_j^1}{\partial y_k^2} &= \frac{1}{V} \sum_v \left[-\mathcal{P}''(z_v) \partial_k u_v(y^2) + \mathcal{P}''(-z_v) \partial_k u_v(y^2)\right] \partial_j u_v(y^1) \\
\frac{\partial \dot{y}_j^2}{\partial y_k^1} &= \frac{1}{V} \sum_v \left[-\mathcal{P}''(-z_v) \partial_k u_v(y^1) + \mathcal{P}''(z_v) \partial_k u_v(y^1)\right] \partial_j u_v(y^2) \\
\frac{\partial \dot{y}_j^2}{\partial y_k^2} &= \frac{1}{V} \sum_v \left[\mathcal{P}''(-z_v) \partial_k u_v(y^2) + \mathcal{P}''(z_v)[-\partial_k u_v(y^2)]\right] \partial_j u_v(y^2) \\
&\quad + \frac{1}{V} \sum_v \left[\mathcal{P}'(-z_v) + \mathcal{P}'(z_v)\right] \partial_k \partial_j u_v(y^2)
\end{aligned} \tag{2.9}$$

For a point $(y, y) \in \mathcal{H}_2$ this simplifies to

$$\begin{aligned}
\left.\frac{\partial \dot{y}_j^1}{\partial y_k^1}\right|_{z_v=0} &= \left.\frac{\partial \dot{y}_j^2}{\partial y_k^2}\right|_{z_v=0} = 2\mathcal{P}'(0) \frac{1}{V} \sum_v \partial_k \partial_j u_v(y) = 2\mathcal{P}'(0) \mathbf{H}_{jk}(y) \\
\left.\frac{\partial \dot{y}_j^1}{\partial y_k^2}\right|_{z_v=0} &= \left.\frac{\partial \dot{y}_j^2}{\partial y_k^1}\right|_{z_v=0} = 0
\end{aligned} \tag{2.10}$$

The Jacobian of a trivial fixed point is thus

$$\mathbf{J}(y,y) = 2\,\mathcal{P}'(0)\begin{pmatrix}\mathbf{H}(y) & 0\\ 0 & \mathbf{H}(y)\end{pmatrix}, \tag{2.11}$$

where $\mathbf{H}(y)$ denotes the Hessian matrix of the average voter utility $U(y)$:

$$\mathbf{H}_{jk}(y) = \frac{\partial^2}{\partial y_j \partial y_k}U(y)\,. \tag{2.12}$$

The stability of a fixed point on $\mathcal{H}_2$ is determined by the curvature of the average voter dissatisfaction $U(y)$. The stable trivial fixed points are the maximizers of $U(y)$.

When, as usual, one assumes that $u_v(y)$ is a strictly concave function for every voter v, $U(y)$ is strictly concave, (i.e. there is a unique maximizer point of $U(y)$ which is a stable trivial fixed point). For the quadratic voter dissatisfaction function (1.2) there is a unique extremal point of U (which yields a maximum) defined by the *mean voter position*

$$\hat{y}_j = \frac{1}{\sum_v s_{vj}}\sum_v s_{vj}x_{vj}\,. \tag{2.13}$$

This corresponds to a stable equilibrium of the platform dynamics. The Jacobian at $\hat{y}$ is of the form:

$$\mathbf{J}(y,y) = 2\,\mathcal{P}'(0)\begin{pmatrix}-2\bar{s}_1 & 0 & 0 & \dots & 0\\ 0 & -2\bar{s}_2 & 0 & \dots & 0\\ \vdots & & & & \\ 0 & 0 & 0 & \dots & -2\bar{s}_I\end{pmatrix}\otimes\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix} \tag{2.14}$$

where $\bar{s}_i = \frac{1}{V}\sum_v s_{vi}$. If each voter's interest is equally strong concerning each issue, i.e. $s_{vi} = s_v$, then we get $\bar{s}_i = \bar{s}_j \stackrel{\text{def}}{=} \bar{s}$ for all issues i. Then the Jacobian is simply

$$\mathbf{J}(y,y) = -4\,\mathcal{P}'(0)\bar{s}\mathbf{I} \tag{2.15}$$

where $\mathbf{I}$ is the $2I \times 2I$ identity matrix.

A much stronger result is proved in [44].

Theorem 2. The mean voter equilibrium (2.13) is a globally stable fixed point of the platform dynamics (2.4) with a quadratic voter utility function of the form (1.4) in the absence of non-policy values.

The stability and uniqueness results for the trivial equilibrium require strictly concave utility functions (a typical assumption for similar dynamic game models). However, some non-concavity in such functions may be quite reasonable, for example, once platforms are at a sufficient distance from a voter's preferred position, changes that move the platform even further away are likely to have little impact.

The following example, which is taken from ref. [44], illustrates the behavior of our model in the cases where the concavity assumption on u is violated. Consider the voter utility function

$$u_v(y^j) = -\gamma^2 \sum_i s_{vi} \left[1 - \exp\left\{ -\frac{(y_i^j - x_{vi})^2}{\gamma^2} \right\} \right]. \tag{2.16}$$

Note that the Enelow-Hinich utility function (1.2) agrees with equ.(2.16) up to third order in $(y - x_v)$. While the utility diverges in (1.2), it levels off in (2.16), however. Gaussian voter utility functions were recently used in models of voter turnout [29]. The curvature of u_v changes sign at y_j if $|y_j - x_{vj}| = \gamma/\sqrt{2}$. The coordinates of non-trivial equilibria cannot be obtained explicitly even in very simple examples such as the one in figure 7, which exhibits eight stable

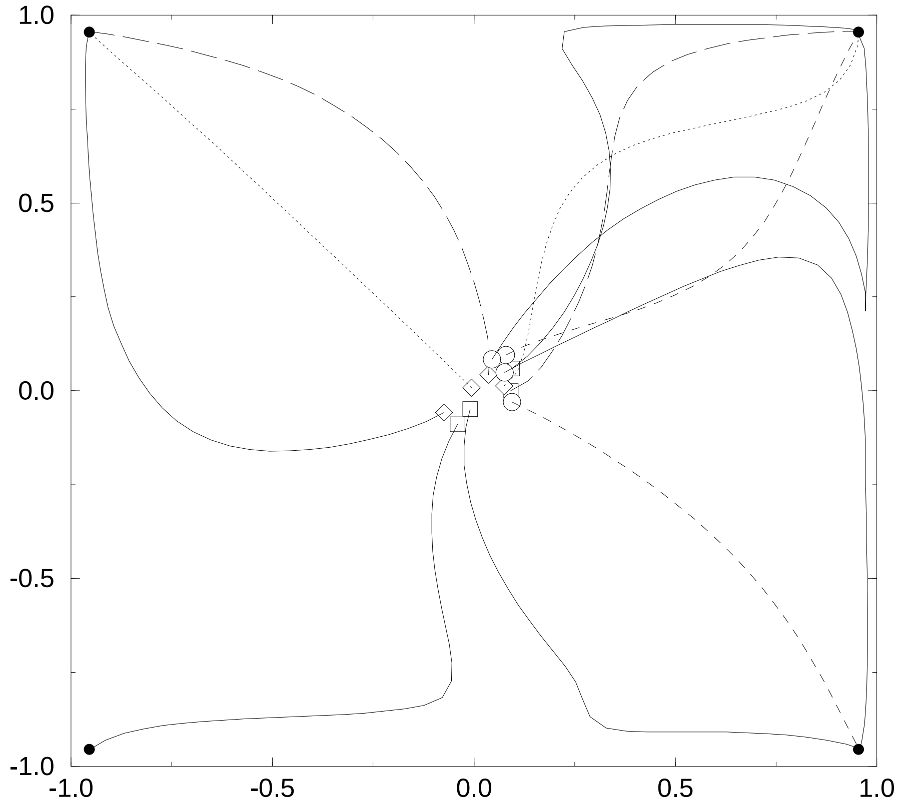

Figure 7: The platform dynamics for two parties are shown in issue space. Two voters with strengths equal to one on all issues and ideal points of $(-1, 1)$ and $(1, -1)$ were used. Voters have utility functions given by (2.16) with $\gamma = 1$. The mean voter equilibrium is the origin $(0, 0)$. It is unstable. Six different initial platform pairs were randomly generated about the origin (each pair has the same marker and line style). Similar markers end up at the same stable equilibrium, which are of the form

$$\begin{array}{rclcrcl} y^1 & = & (\xi, \xi) & \quad & y^2 & = & (\xi, -\xi) \\ y^1 & = & (\xi, \xi) & & y^2 & = & (-\xi, \xi) \\ y^1 & = & (-\xi, -\xi) & & y^2 & = & (\xi, -\xi) \\ y^1 & = & (-\xi, -\xi) & & y^2 & = & (-\xi, \xi) \end{array}$$

with $\xi \approx 0.955$ for $\gamma = 1$ (of course, solutions with y^1 and y^2 exchanged also exist). Which of these locally stable equilibria is actually reached depends on the initial conditions. Note that none of the equilibria are symmetric— the parties agree on one issue and dissent on the other.
This figure is reproduced from [44].

equilibria (due to its high symmetry) if $\gamma < \sqrt{2}$, while the trivial fixed point is globally stable for $\gamma > \sqrt{2}$.

2.4. An Example with Continuous Voter Distribution

We consider here a single issue model with a continuous distribution of voter preferences and voter utilities $u(y,x) = -s(y-x)^2$. We shall see that the trajectories of Equ.(2.4') are determined by a surprisingly simple differential equation, provided one succeeds in explicitly computing the integral over the voter distribution. For the sake of tractability we choose the normal distribution

$$\rho(x) = \frac{1}{\sqrt{2\pi}\sigma} \exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right) \tag{2.17}$$

with mean μ and variance σ^2. Furthermore we choose the sigmoidal response function $P_0(z) = (1 + \mathrm{erf}\,(az))/2$ derived from the error function $\mathrm{erf}\,(z) = \frac{2}{\sqrt{\pi}} \int_0^z \exp(-t^2)dt$. The utility functions for the two parties are therefore

$$U_1(z) = \int_{-\infty}^{\infty} \mathrm{erf}\,(az)\rho(x)dx \qquad \text{and} \qquad U_2(z) = \int_{-\infty}^{\infty} -\mathrm{erf}\,(az)\rho(x)dx\,. \tag{2.18}$$

Thus the dynamics of the continuous system (1.4') is given by:

$$\begin{aligned} \dot{y}_1 &= \frac{4as}{\sigma\pi\sqrt{2}} \int_{-\infty}^{\infty} (x-y_1)\exp(-(s^2a^2(y_2-y_1)^2(y_1+y_2-2x)^2 + 1/2(x-\mu)^2/\sigma^2))dx \\ \dot{y}_2 &= \frac{4as}{\sigma\pi\sqrt{2}} \int_{-\infty}^{\infty} (x-y_2)\exp(-(s^2a^2(y_2-y_1)^2(y_1+y_2-2x)^2 + 1/2(x-\mu)^2/\sigma^2))dx \end{aligned} \tag{2.19}$$

Using the abbreviations

$$
\begin{aligned}
A &= 8a^2s^2(y_2-y_1)^2\sigma^2+1 \\
B &= -(8a^2s^2(y_2-y_1)^2(y_1+y_2)\sigma^2+2\mu) \\
C &= 2a^2s^2(y_2-y_1)^2(y_1+y_2)^2\sigma^2+\mu^2
\end{aligned}
\tag{2.20}
$$

we may write the exponent in the form $(Ax^2+Bx+C)/(2\sigma^2)$, or more conveniently, in the form $(A\,(x-\gamma)^2+D)/(2\sigma^2)$ with

$$
\gamma(y_1,y_2) = -\frac{B}{2A} = \frac{4a^2s^2(y_2-y_1)^2(y_1+y_2)\sigma^2+\mu}{8a^2s^2(y^2-y^1)^2\sigma^2+1}
\tag{2.21}
$$

and $D(y_1,y_2) = C - A\gamma^2$. Using the change of variables $\xi \stackrel{\text{def}}{=\!=} (x-\gamma)$ and setting $A^* = A/(2\sigma^2)$ and $D^* = D/(2\sigma^2)$ we obtain

$$
\begin{aligned}
\dot{y}_1 = {} & \frac{4as}{\sigma\pi\sqrt{2}}\exp(-D^*(y_1,y_2))\int_{-\infty}^{\infty}\xi\exp(-A^*(y_1,y_2)\xi^2)d\xi \\
& + \frac{4as}{\sigma\pi\sqrt{2}}\exp(-D^*(y_1,y_2))(\gamma(y_1,y_2)-y_1)\int_{-\infty}^{\infty}\exp(-A^*(y_1,y_2)\xi^2)d\xi
\end{aligned}
\tag{2.22}
$$

and an analogous expression for $\dot{y}_2$. Since the first integrand is an odd function, the first integral vanishes. The second integral is a Gaussian integral yielding $\sqrt{\pi/A^*}$. Hence Equ.(2.22) simplifies to the following system of differential equations

$$
\begin{aligned}
\dot{y}_1 &= \frac{4as}{\sigma\pi\sqrt{2}}(\gamma(y_1,y_2)-y_1)\sqrt{\pi/A^*(y_1,y_2)}\exp(-D^*(y_1,y_2)) \\
\dot{y}_2 &= \frac{4as}{\sigma\pi\sqrt{2}}(\gamma(y_1,y_2)-y_2)\sqrt{\pi/A^*(y_1,y_2)}\exp(-D^*(y_1,y_2))
\end{aligned}
\tag{2.23}
$$

Let us define

$$
\psi(t) \stackrel{\text{def}}{=\!=} \frac{4as}{\sigma\pi\sqrt{2}}\exp(-D^*(y_1,y_2))\sqrt{\pi/A^*(y_1,y_2)}\frac{1}{A(y_1,y_2)}
\tag{2.24}
$$

It easy to verify that $\psi(t) > 0$ since each individual factor is positive and A is bounded away from 0. Hence we may write

$$\begin{aligned}
\dot{y}_1 &= \psi(t)\, A(y_1, y_2)\, (\gamma(y_1, y_2) - y_1) = \psi(t)\ [-B(y_1, y_2)/2 - y_1 A(y_1, y_2)] \\
\dot{y}_2 &= \psi(t)\, A(y_1, y_2)\, (\gamma(y_1, y_2) - y_2) = \psi(t)\ [-B(y_1, y_2)/2 - y_2 A(y_1, y_2)]
\end{aligned} \tag{2.25}$$

Since $\psi(t) > 0$ the trajectories of the above dynamical system are the same as those of the time-rescaled differential equations

$$\begin{aligned}
\dot{y}_1 &= 4a^2s^2(y_2 - y_1)^2(y_1 + y_2)\sigma^2 + \mu - 8a^2s^2(y_2 - y_1)^2 y_1\sigma^2 - y_1 \\
&= 4a^2s^2\sigma^2(y_2 - y_1)^3 + \mu - y_1 \\
\dot{y}_2 &= 4a^2s^2(y_2 - y_1)^2(y_1 + y_2)\sigma^2 + \mu - 8a^2s^2(y_2 - y_1)^2 y_2\sigma^2 - y_2 \\
&= 4a^2s^2\sigma^2(y_1 - y_2)^3 + \mu - y_2
\end{aligned} \tag{2.26}$$

Remark. Consider a differential equation

$$\dot{x} = \frac{\partial x}{\partial t} = \psi(t)F(x),$$

with a scalar function $\psi(t)$. Then the equation can be rewritten in the form:

$$\frac{\partial x}{\partial \tau}\frac{\partial \tau}{\partial t} = \psi(t)F(x)$$

Now let us set $\frac{\partial \tau}{\partial t} = \psi(t)$. If $\psi(t) > 0$ then this is a one-to-one relation of t and τ, and hence $\dot{x} = F(x)$ and $\dot{x} = \psi(t)F(x)$ have the same trajectories. Explicitly, the transformation of the time axis is

$$\tau(t) = \int_0^t \psi(t')dt'.$$

Set $\eta = y_1 + y_2$, $\zeta = y_1 - y_2$, and $w = 8a^2s^2\sigma^2$. With this transformation we obtain

$$\begin{aligned}
\dot{\eta} &= 2\mu - \eta \\
\dot{\zeta} &= -\zeta(w\zeta^2 + 1)
\end{aligned} \tag{2.27}$$

This decoupled system is easily solved explicitly.

$$\begin{aligned} \eta(t) &= 2\mu + (\eta(0) - 2\mu)\exp(-t) \\ \zeta(t) &= \zeta(0)\frac{\exp(-t)}{\sqrt{1 + wy(0)^2(1-\exp(-2t))}} \end{aligned} \tag{2.28}$$

As an immediate consequence we have the final result of this section:

Theorem 3. Equ.(2.19) has a unique, globally stable fixed point which coincides with the mean voter point (μ, μ).

Again, the mean voter position is the unique globally stable equilibrium. The voter distribution function has its maximum at μ. Thus, the equilibrium point in the system where both parties have the same platform coincides with the maximum of the voter distribution function in this case.

2.5. Non-Policy Values

In this section we briefly discuss the effect of non-policy contributions to the voter utility functions, i.e., we consider voter dissatisfaction functions of the form

$$\tilde{u}_v(y^p) = m_v^p + u_v(y^p)\,, \tag{2.29}$$

where $u_v(y^p)$ is the policy-dependent utility and m_v^p is a constant that depends on the voter and (the label of) the party, but not on the party's platform position. The difference in the dynamics stems from the fact that

$$\tilde{z}_v = \tilde{u}_v(y^1) - \tilde{u}_v(y^2) = m_v^1 - m_v^2 + u_v(y^1) - u_v(y^2) = \delta_v + z_v \tag{2.30}$$

replaces z_v in equ.(2.4). Note that $\nabla\tilde{u}_v(y) = \nabla u_v(y)$. On $\mathcal{H}_2$, we have $z_v = 0$, and thus

$$\dot{y}^p = \frac{1}{V}\sum_v [\mathcal{P}'(\delta_v) + \mathcal{P}'(-\delta_v)]\nabla u_v(y^p)\,, \tag{2.31}$$

and hence $\mathcal{H}_2$ is invariant. The stability of a trivial fixed point is determined by its Jacobian (which can be obtained from equ.(2.9) by subsituting $z_v + \delta_v$ for z_v and subsequently setting $z_v = 0$). One finds

$$\begin{aligned}
\frac{\partial \dot{y}^1_j}{\partial y^1_k} &= \frac{1}{V}\sum_v \left[\mathcal{P}'(\delta_v) + \mathcal{P}'(-\delta_v)\right]\partial_k\partial_j u_v(y) \\
&\qquad + \frac{1}{V}\sum_v \left[\mathcal{P}''(\delta_v) - \mathcal{P}''(-\delta_v)\right]\partial_k u_v(y)\partial_j u_v(y) \\
\frac{\partial \dot{y}^1_j}{\partial y^2_k} &= \frac{1}{V}\sum_v \left[-\mathcal{P}''(\delta_v) + \mathcal{P}''(-\delta_v)\right]\partial_k u_v(y)\partial_j u_v(y) \\
\frac{\partial \dot{y}^2_j}{\partial y^1_k} &= \frac{1}{V}\sum_v \left[-\mathcal{P}''(-\delta_v) + \mathcal{P}''(\delta_v)\right]\partial_k u_v(y)\partial_j u_v(y) \\
\frac{\partial \dot{y}^2_j}{\partial y^2_k} &= \frac{1}{V}\sum_v \left[\mathcal{P}''(-\delta_v) - \mathcal{P}''(\delta_v)\right]\partial_k u_v(y)\partial_j u_v(y) \\
&\qquad + \frac{1}{V}\sum_v \left[\mathcal{P}'(-\delta_v) + \mathcal{P}'(\delta_v)\right]\partial_k\partial_j u_v(y)
\end{aligned} \tag{2.32}$$

This can be recast in a much more convenient form by defining

$$\begin{aligned}
\mathbf{Q}_{kl}(y) &= \frac{1}{V}\sum_v \left[\mathcal{P}''(\delta_v) - \mathcal{P}''(-\delta_v)\right]\partial_k u_v(y)\partial_j u_v(y)\,, \\
\mathbf{H}_{kl}(y) &= \frac{1}{V}\sum_v \left[\mathcal{P}'(\delta_v) + \mathcal{P}'(-\delta_v)\right]\partial_k\partial_j u_v(y)\,.
\end{aligned} \tag{2.33}$$

We find then

$$\mathbf{J}(y,y) = \mathbf{Q}(y) \otimes \begin{pmatrix} 1 & -1 \\ 1 & -1 \end{pmatrix} + \mathbf{H}(y) \otimes \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\,. \tag{2.34}$$

The first term is nilpotent since $\begin{pmatrix} 1 & -1 \\ 1 & -1 \end{pmatrix}$ is nilpotent. The stability of a trivial equilibrium is therefore determined by the eigenvalue of $\mathbf{H}(y)$ alone. This matrix can be regarded as the Hessian of the potential function

$$\Psi(y) = \frac{1}{V} \sum_v \left[\mathcal{P}'(\delta_v) + \mathcal{P}'(-\delta_v)\right] u_v(y)\,, \tag{2.35}$$

which determines the dynamics on $\mathcal{H}_2$ since $\dot{y} = \nabla\Psi(y)$ on this invariant manifold.

3. Complete Participation and Abstention

3.1. Introduction

In general, the voters face a decision between participating in the election (i.e., voting for one of the competing parties) and abstention. The literature on voter turnout is extensive, see e.g. [2]. Formal modeling of voter turnout in the rational choice tradition has typically not been very successful in explaining many aspects of turnout behavior. Green and Shapiro [24, chap.4] summarize much of the work in this area. A less restrictive model was proposed recently by J. S. Irons [29]. It also assumes explicitly that voters weigh the costs and benefits of voting but allows that these costs and benefits depend on the policy alternatives in a more general manner.

In the context of the dynamical models presented here it will be more convenient to recast the turnout behavior in terms of the probabilities $\Theta_v(\vec{y})$ that voter v participates in the election as a function of the platform positions. For the sake of tractability we shall restrict ourselves to very simple heuristic examples of $\Theta_v(\vec{y})$ instead of incorporating a detailed model. It is then always possible to write the expected fraction of votes E_k in the form $\sum_v \Theta_v(\vec{y}) E_k^\circ(\vec{y})$, where $E_k^\circ(\vec{y})$ corresponds to a model with complete participation.

Complete participation of the voters in the election process implies additional symmetries in the response function $\mathcal{P}$. We shall use the notation P_0 instead of $\mathcal{P}$ if the following conditions are satisfied:

(i) $P_0(-z) = 1 - P_0(z)$ for all $z \in \mathbb{R}$.

(ii) $P_0(-\infty) = 0$ and $P_0(\infty) = 1$.

(iii) P_0 is monotonically increasing on $\mathbb{R}$.

P_0 is the voting probability given that the voter participates in the election.

In the original model of Enelow and Hinich, which corresponds to exact knowledge of both party platforms, a voter always votes for the party that offers the larger utility, and if both parties provide the same utility the voter is indifferent and flips a coin; thus we have

$$P_0(z) = \begin{cases} 0 & \text{if} \quad z < 0\,, \\ 0.5 & \text{if} \quad z = 0\,, \\ 1 & \text{if} \quad z > 0\,. \end{cases} \tag{3.1}$$

For reasons of mathematical tractability and reality (as discussed in section 1.6) we shall require instead:

(iii') $P_0(z)$ is *strictly* monotonically increasing on $\mathbb{R}$.

(iv) $P_0(z)$ is twice continuously differentiable.

Two party models with these properties are discussed in detail in [44]. Examples for valid response functions P_0 are

$$\begin{array}{ll} P_0(z) = & \frac{1}{2}(1+\tanh(az)) \quad = 1/(1+\exp(-2az)) \\ P_0(z) = & \frac{1}{2}(1+\mathrm{erf}(az)) \end{array} \tag{3.2}$$

Complete participation implies

$$\begin{aligned} W_1(y^1,y^2) &= E_1(y^1,y^2) - E_2(y^1,y^2) = 2E_1(y^1,y^2) - 1 \\ W_2(y^1,y^2) &= E_2(y^1,y^2) - E_1(y^1,y^2) = 1 - 2E_1(y^1,y^2) = -W_1(y^1,y^2) \end{aligned} \tag{3.3}$$

Since the dynamics depends only on $\nabla_{y^p} W_p$ the constant 1 has no influence. Rescaling the time axis by a factor of 2 leaves us with $\dot{y}^p = \nabla_{y^p} E_p(y^1,y^2)$, i.e., using E_p instead of W_p leads to the same orbits in the case of complete participation.

In the case of incomplete participation we obtain different dynamical systems depending on whether E_p or W_p is chosen. The resulting models may exhibit qualitatively different behavior; an example will be discussed in section 3.4.

3.2. Abstention Depending on Voter Utility Differences

In order to incorporate abstention into the 2-party model we first consider the case where the probability for *non-voting* is a (universal) function μ of the utility difference z_v. A special case, in which μ is a step function, is considered by Irons [29]. An alternative ansatz will be explored in the following section.

Voting becomes uninteresting if the utility $u_v(y^p)$ is very small for both parties $p = 1, 2$, and also the platform positions of the two parties are (almost) the same. It seems reasonable therefore to assume that the non-voting probability $\mu(z)$ is a non-increasing function of $|z|$. More precisely, we require:

(i) $\mu(z)$ is symmetric, i.e. $\mu(z) = \mu(-z)$. This means that the parties seem exchangeable to a person who does not vote at all.

(ii) $\mu(z)$ is decreasing for $z > 0$.

(iii) $\mu(z)$ is twice differentiable. (We make this assumption for technical simplicity.)

As an immediate consequence, $\mu'(0) = 0$ since

$$\begin{aligned} \mathcal{P}(z) &= (1 - \mu(z))P_0(z)\,, \\ \mathcal{P}(-z) &= (1 - \mu(z))P_0(-z)\,, \\ \mathcal{P}(z) + \mathcal{P}(-z) &= 1 - \mu(z)\,, \\ \mathcal{P}'(z) - \mathcal{P}'(-z) &= -\mu'(z)\,, \\ \mathcal{P}'(0) - \mathcal{P}'(0) &= -\mu'(0) = 0\,, \end{aligned} \tag{3.4}$$

and $\mu''(0) \leq 0$ by definition. Complete participation corresponds to setting $\mu(z) = 0$.

Using W as the party utility function, as in equ.(2.3), the dynamics can be written in the form

$$\begin{aligned} \dot{y}^1 &= \frac{1}{V}\nabla_{y^1}\sum_{v=1}^{V}(1 - \mu(z_v))(2P_0(z_v) - 1) \\ \dot{y}^2 &= \frac{1}{V}\nabla_{y^2}\sum_{v=1}^{V}(1 - \mu(z_v))(1 - 2P_0(z_v)) \end{aligned} \tag{3.5}$$

and the Jacobian at the trivial fixed point becomes

$$\mathbf{J} = 2{P_0}'(0)(1 - \mu(0))\begin{pmatrix} \mathbf{H}(y) & 0 \\ 0 & \mathbf{H}(y) \end{pmatrix} \tag{3.6}$$

As in the previous section, $\mathbf{H}(y)$ denotes the Hessian matrix of the average voter utility $U(y)$. The mean voter equilibrium remains stable also with incomplete participation, as long as $\mathbf{H}(y)$ is negative definite, since $1 - \mu(0) > 0$.

If we use E instead of W, the dynamics is given by

$$\begin{aligned} \dot{y}^1 &= \frac{1}{V}\nabla_{y^1}\sum_{v=1}^{V}(1 - \mu(z_v))P_0(z_v) \\ \dot{y}^2 &= \frac{1}{V}\nabla_{y^2}\sum_{v=1}^{V}(1 - \mu(-z_v))P_0(-z_v) \end{aligned} \tag{3.7}$$

The Jacobian at the trivial fixed point thus becomes

$$\mathbf{J} = \frac{1}{2}\mu''(0)\mathbf{C}(y) \otimes \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} + P_0'(0)(1-\mu(0)) \begin{pmatrix} \mathbf{H}(y) & 0 \\ 0 & \mathbf{H}(y) \end{pmatrix} \tag{3.8}$$

where we have used the abbreviation

$$\mathbf{C}_{kj}(y) \stackrel{\text{def}}{=} \frac{1}{V}\sum_{v=1}^{V} \partial_k u_v(y)\partial_j u_v(y)\,, \tag{3.9}$$

and $\mathbf{H}(y)$ is again the Hessian of the average voter satisfaction $U(y)$, equ.(2.12).

If (y, y) is a fixed point on $\mathcal{H}_2$ we have necessarily $(1/V)\sum_v \partial_k u_v(y) = \partial_k U(y) = 0$. Therefore we can write

$$\mathbf{C}_{kj}(y) = \frac{1}{V}\sum_{v=1}^{V} \big(\partial_k u_v(y) - \partial_k U(y)\big)\big(\partial_j u_v(y) - \partial_k U(y)\big)\,. \tag{3.10}$$

Thus the matrix $\mathbf{C}(y)$ is the covariance matrix of the components of the vectors $\nabla u_v(y)$ with the the average being taken over all voters v, and hence $\mathbf{C}(y)$ is non-negative definite for any fixed point $(y, y) \in \mathcal{H}_2$.

Thus the eigenvalues corresponding to the first part of $\mathbf{J}$ are 0 and $\mu''(0)\lambda_j$, resp., where λ_j is an eigenvalue of $\mathbf{C}$, and therefore positive[2]. In the case of strictly concave voter utility functions, all eigenvalues of the Hessian $\mathbf{H}$ are negative. Since the sum of two negative definite matrices is again negative definite, all eigenvalues of $\mathbf{J}$ are also negative and the mean voter equilibrium is stable.

[2]Strictly speaking covariance matrices are only positive semi-definite. However, eigenvalues 0 occur only in the degenerate cases where the voter positions are confined to a hyper-plane in issue space. We will neglect this complication in the following.

3.3. Abstention Depending on Voter Utilities

In a more realistic model, we assume that a voter will abstain if the platforms are far away from her ideal point. For simplicity we set:

$$\begin{aligned} E_1(y^1, y^2) &= \frac{1}{V}\sum_v P_0(z_v)\Psi(u_v(y^1)) \\ E_2(y^1, y^2) &= \frac{1}{V}\sum_v P_0(-z_v)\Psi(u_v(y^2)) \end{aligned} \tag{3.11}$$

where $\Psi(u)$ is an increasing function of u. Recall that the voter utility u is negative and decreasing with distance from the voter's ideal point. The voting dynamics becomes

$$\begin{aligned} \dot{y}^1 &= \frac{1}{V}\sum_v \left[P_0'(z_v)\Psi(u_v(y^1)) + P_0(z_v)\Psi'(u_v(y^1))\right] \nabla u_v(y^1) \\ \dot{y}^2 &= \frac{1}{V}\sum_v \left[P_0'(-z_v)\Psi(u_v(y^2)) + P_0(-z_v)\Psi'(u_v(y^2))\right] \nabla u_v(y^2) \end{aligned} \tag{3.12}$$

with a discrete voter distribution. The continuous case reads

$$\begin{aligned} \dot{y}^1 = \int_{-\infty}^{\infty} &\left[P_0'(u(y^1, x) - u(y^2, x))\Psi(u(y^1, x)) + P_0(u(y^1, x) - u(y^2, x))\Psi'(u(y^1, x))\right] \\ &\times\ \nabla u(y^1, x)\, \rho(x) dx \\ \dot{y}^2 = \int_{-\infty}^{\infty} &\left[P_0'(u(y^2, x) - u(y^1, x))\Psi(u(y^1, x)) + P_0(u(y^2, x) - u(y^1, x))\Psi'(u(y^2, x))\right] \\ &\times\ \nabla u(y^2, x)\, \rho(x) dx \end{aligned} \tag{3.12$'$}$$

Again this dynamical system remains unchanged under permutations of the platform indices. As in the other two-party models, $\mathcal{H}_2$,i.e., $z_v = 0$, is invariant. The Jacobian at a trivial fixed point is

$$
\begin{aligned}
\frac{\partial \dot{y}_j^1}{\partial y_k^1} &= \frac{1}{V}\sum_v \Big[[2P_0'(0)\Psi'(u_v(y)) + \frac{1}{2}\Psi''(u_v(y))]\partial_k u_v(y)\partial_j u_v(y) \\
&\quad + [P_0'(0)\Psi(u_v(y)) + \frac{1}{2}\Psi'(u_v(y))]\partial_k\partial_j u_v(y)\Big] \\
\frac{\partial \dot{y}_j^1}{\partial y_k^2} &= \frac{1}{V}\sum_v -\mathcal{P}'(0)\Psi'(u_v(y))\partial_k u_v(y)\partial_j u_v(y) \\
\frac{\partial \dot{y}_j^2}{\partial y_k^1} &= \frac{1}{V}\sum_v -\mathcal{P}'(0)\Psi'(u_v(y))\partial_k u_v(y)\partial_j u_v(y) \\
\frac{\partial \dot{y}_j^2}{\partial y_k^2} &= \frac{1}{V}\sum_v \Big[[2P_0'(0)\Psi'(u_v(y)) + \frac{1}{2}\Psi''(u_v(y))]\partial_k u_v(y)\partial_j u_v(y) \\
&\quad + [P_0'(0)\Psi(u_v(y)) + \frac{1}{2}\Psi'(u_v(y))]\partial_k\partial_j u_v(y)\Big]
\end{aligned}
\tag{3.13}
$$

The dynamics on $\mathcal{H}_2$ is given by the differential equation:

$$
\dot{y} = \frac{1}{V}\sum_v \left[P_0'(0)\Psi(u_v(y)) + \frac{1}{2}\Psi'(u_v(y))\right]\nabla u_v(y). \tag{3.14}
$$

In the following we analyze a few special cases with continuous voter distributions $\rho(x)$. We shall restrict ourselves to the one-issue case. Furthermore, we assume the Enelow-Hinich-type voter utility function $u(y,x) = -(y-x)^2$ and an exponentially decaying voting probability of the form $\Psi(u_v(y)) \stackrel{\text{def}}{=} \exp(\mu u_v(y))$ with $\mu > 0$.

As a first example suppose that the voters are normally distributed with mean 0 and variance σ^2, i.e., $\rho(x) = \frac{1}{\sigma\sqrt{2\pi}}\exp(\frac{-x^2}{2\sigma^2})$. Equ.(3.14) then becomes

$$
\dot{y} = \frac{1}{\sigma\sqrt{2\pi}}\int_{-\infty}^{\infty} [P_0'(0) + \frac{1}{2}\mu] \exp\left(-\frac{x^2}{2\sigma^2} - \mu(y-x)^2\right)(-2)(y-x)\,dx. \tag{3.15}
$$

Using the abbreviations

$$a = \sqrt{\frac{2\sigma^2\mu + 1}{2\sigma^2}} \qquad b = \frac{-\mu y}{a} \qquad c = \mu y^2 - b^2 \tag{3.16}$$

we may write the exponent in the form $(ax+b)^2 + c$. Setting $w = (ax+b)$, we obtain the following expression

$$\begin{aligned} \dot{y} &= -\frac{1}{\sigma\sqrt{2\pi}} e^{-c}[2P_0'(0)+\mu]\left(\frac{1}{a} - \frac{\mu}{a^3}\right) y \int_{-\infty}^{\infty} e^{-w^2} dw \\ &\quad + \frac{1}{a^2\sigma\sqrt{2\pi}} e^{-c}[2P_0'(0)+\mu] \int_{-\infty}^{\infty} w e^{-w^2} dw \\ &= -\frac{(2P_0'(0)+\mu)(a^2-\mu)e^{b^2}}{\sqrt{2}a^3\sigma} y \, \exp(-\mu y^2) \end{aligned} \tag{3.17}$$

The second part of the sum vanishes, since the second integrand is an odd function. Since $a^2 - \mu = 1/(2\sigma^2) > 0$, this is of the form $\dot{y} = -Cy\exp(-\mu y^2)$ with a positive constant C. Thus the mean voter point $y = 0$ is the only fixed point. It is obviously stable within $\mathcal{H}_2$.

The Jacobian at $y = 0$ can be obtained from equ.(3.13) by evaluating similar integrals:

$$\begin{aligned} \frac{\partial \dot{y}^1}{\partial y^1} &= \frac{8\mu P_0'(0) + 2\mu^2}{2\sqrt{2}\sigma a^3} - \frac{2P_0'(0)+\mu}{\sqrt{2}\sigma a} \\ \frac{\partial \dot{y}^1}{\partial y^2} &= \frac{-4\mu P_0'(0)}{2\sqrt{2}\sigma a^3} \\ \frac{\partial \dot{y}^2}{\partial y^1} &= \frac{-4\mu P_0'(0)}{2\sqrt{2}\sigma a^3} \\ \frac{\partial \dot{y}^2}{\partial y^2} &= \frac{8\mu P_0'(0) + 2\mu^2}{2\sqrt{2}\sigma a^3} - \frac{2P_0'(0)+\mu}{\sqrt{2}\sigma a} \end{aligned} \tag{3.18}$$

Rewriting equ.(3.18) in matrix form yields

$$\mathbf{J}(y,y) = \frac{4\mu P_0'(0)}{2\sqrt{2}\sigma a^3}\begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix} + \frac{2\mu^2}{2\sqrt{2}\sigma a^3}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - \frac{2P_0'(0)+\mu}{\sqrt{2}\sigma a}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \tag{3.19}$$

The eigenvalues of the first matrix are 3 and 1, resp. Therefore, the eigenvalues of the Jacobian are given by:

$$\begin{aligned}\lambda_1 &= \frac{1}{2\sqrt{2}\sigma a^3}\left[4P_0'(0)(3\mu - a^2) + 2\mu(\mu - a^2)\right]\\ \lambda_2 &= \frac{1}{2\sqrt{2}\sigma a^3}\left[(4P_0'(0) + 2\mu)(\mu - a^2)\right]\end{aligned} \tag{3.20}$$

The second eigenvalue, λ_2, is always negative since $\mu - a^2 = -1/(2\sigma^2)$. For the first eigenvalue, λ_1, a bifurcation occurs at $P_0'(0) = \frac{\mu(a^2-\mu)}{2(3\mu-a^2)}$. The mean voter fixed point is therefore stable if $\mu \leq 1/(4\sigma^2)$ independent of $P_0'(0)$. For $\mu > 1/(4\sigma^2)$, it is stable for small values of $P_0'(0)$. The mean voter equilibrium becomes unstable if

$$P_0'(0) > \frac{\mu}{8\sigma^2\mu - 2} \tag{3.21}$$

in this case.

Using W instead of E we obtain the seemingly more complicated dynamical system

$$\begin{aligned}\dot{y}^1 &= \int_{-\infty}^{\infty} \nabla_{y^1} P_0(u(y^1,x) - u(y^2,x))\Psi(u(y^1,x)) - P_0(u(y^2,x) - u(y^1,x))\Psi(u(y^2,x))\,\rho(x)dx\\ \dot{y}^2 &= \int_{-\infty}^{\infty} \nabla_{y^2} P_0(u(y^2,x) - u(y^1,x))\Psi(u(y^2,x)) - P_0(u(y^1,x) - u(y^2,x))\Psi(u(y^1,x))\,\rho(x)dx\end{aligned} \tag{3.22}$$

Of course $\mathcal{H}_2$ is again invariant. Choosing Ψ, the voter distribution ρ, and the voter utility u as above, yields the following analogue of equ.(3.17) on $\mathcal{H}_2$:

$$\dot{y} = -\frac{(4P'(0) + \mu)e^{b^2}(a^2 - \mu)}{\sqrt{2}a^3\sigma} y \exp(-\mu y^2) \tag{3.23}$$

Again, the mean voter equilibrium $y = 0$ is the only fixed point on $\mathcal{H}_2$. Since the first integral yields zero, it is easy to see that $\dot{y} = 0$ holds for $y = 0$. As above $a^2 - \mu > 0$ implies that this fixed point is stable within the invariant manifold $\mathcal{H}_2$. The Jacobian of the complete system can be computed easily. At the mean voter equilibrium we find

$$\begin{array}{rclrcl} \frac{\partial \dot{y}^1}{\partial y^1} & = & \frac{(4P'(0)+\mu)(\mu-a^2)}{a^3\sigma\sqrt{2}} & \frac{\partial \dot{y}^1}{\partial y^2} & = & 0 \\ \frac{\partial \dot{y}^2}{\partial y^1} & = & 0 & \frac{\partial \dot{y}^2}{\partial y^2} & = & \frac{(4P'(0)+\mu)(\mu-a^2)}{a^3\sigma\sqrt{2}} \end{array} \tag{3.24}$$

In this system, the mean voter equilibrium is always stable.

Now let us assume that the voters are uniformly distributed in the interval $[-1, 1]$. Again, we will consider both the expected number of votes E_p and the differences W_p as payoff functions. Explicitly we have

$$\begin{aligned} E_1(y^1, y^2) &= \frac{1}{2}\int_{-1}^{1} P_0(z)\exp\big(-\mu(y^1 - x)^2\big)dx \\ E_2(y^1, y^2) &= \frac{1}{2}\int_{-1}^{1} P_0(-z)\exp\big(-\mu(y^2 - x)^2\big)dx \end{aligned} \tag{3.25}$$

In the first case, the differential equations read

$$\begin{aligned} \dot{y}^1 &= \frac{1}{2}\int_{-1}^{1} \big\{P_0'(z)\exp[-\mu(y^1 - x)^2] + P_0(z)\mu\exp[-\mu(y^1 - x)^2]\big\}(-2)(y^1 - x)dx \\ \dot{y}^2 &= \frac{1}{2}\int_{-1}^{1} \big\{P_0'(-z)\exp[-\mu(y^2 - x)^2] + P_0(-z)\mu\exp[-\mu(y^2 - x)^2]\big\}(-2)(y^2 - x)dx \end{aligned} \tag{3.26}$$

Again, it is easy to see that $\mathcal{H}_2$, where $y^1 = y^2$, is invariant. On this plane, a fixed point must fulfill

$$\dot{y} = \frac{1}{2}\int_{-1}^{1} \big(P_0'(0) + \mu P_0(0)\big)\exp\big(-\mu(y - x)^2\big)(-2)(y - x)dx = 0 \tag{3.27}$$

If we make the substitution $w \stackrel{\text{def}}{=} y - x$, the dynamical equation is transformed to

$$\dot{y} = \big(P_0'(0) + \mu/2\big) \int_{y-1}^{y+1} w \exp(-\mu w^2) dw \tag{3.28}$$

Since $\int w \exp(-\mu w^2) = -\frac{1}{2\mu} \exp(-\mu w^2)$ a fixed point must satisfy $\exp\big(-\mu(y+1)^2\big)[1 - \exp(4\mu y)] = 0$. Clearly, the only solution is $\exp(4\mu y) = 1$, i.e., $y = 0$, corresponding to the mean voter fixed point.

The next step is the computation of the Jacobian of the full dynamical system at the mean voter fixed point. We find

$$\begin{aligned}
\frac{\partial \dot{y}^1}{\partial y^1} &= (4P_0'(0)\mu + \mu^2) \int_{-1}^{1} x^2 \exp(-\mu x^2) dx - (P_0'(0) + \frac{\mu}{2}) \int_{-1}^{1} \exp(-\mu x^2) dx \\
\frac{\partial \dot{y}^1}{\partial y^2} &= -2P_0'(0)\mu \int_{-1}^{1} x^2 \exp(-\mu x^2) dx \\
\frac{\partial \dot{y}^2}{\partial y^1} &= -2P_0'(0)\mu \int_{-1}^{1} x^2 \exp(-\mu x^2) dx \\
\frac{\partial \dot{y}^2}{\partial y^2} &= (4P_0'(0)\mu + \mu^2) \int_{-1}^{1} x^2 \exp(-\mu x^2) dx - (P_0'(0) + \frac{\mu}{2}) \int_{-1}^{1} \exp(-\mu x^2) dx
\end{aligned} \tag{3.29}$$

The Jacobian is therefore of the form

$$\begin{aligned}
\mathbf{J} = &\int_{-1}^{1} x^2 \exp(\mu x^2) dx \begin{pmatrix} 4P_0'(0)\mu + \mu^2 & -2P_0'(0)\mu \\ -2P_0'(0)\mu & 4P_0'(0)\mu + \mu^2 \end{pmatrix} \\
&+ \int_{-1}^{1} \exp(-\mu x^2) dx \begin{pmatrix} -P_0'(0) - \frac{\mu}{2} & 0 \\ 0 & -P_0'(0) - \frac{\mu}{2} \end{pmatrix}
\end{aligned} \tag{3.30}$$

The eigenvalues of the first matrix are

$$\lambda_1 = \mu(\mu + 6P_0'(0)) \qquad \text{and} \qquad \lambda_2 = \mu(\mu + 2P_0'(0))\,. \tag{3.31}$$

The integrals can be evaluated using the substitution $\mu x^2 \stackrel{\text{def}}{=} t^2$:

$$\begin{aligned}
\int_{-1}^{1} x^2 e^{-\mu x^2} dx &= \frac{-\exp(-\mu)}{\mu} + \frac{1}{2\mu}\sqrt{\frac{\pi}{\mu}} \operatorname{erf}(\sqrt{\mu}) \\
\int_{-1}^{1} e^{-\mu x^2} dx &= \sqrt{\frac{\pi}{\mu}} \operatorname{erf}(\sqrt{\mu})
\end{aligned} \tag{3.32}$$

The largest eigenvalue of the Jacobian therefore is

$$\lambda_{\max} = -\mu e^{-\mu} + 2P_0'(0)\left[\sqrt{\frac{\pi}{\mu}}\operatorname{erf}(\sqrt{\mu}) - 3e^{-\mu}\right] \tag{3.33}$$

The mean voter equilibrium becomes unstable for

$$P_0'(0) \geq \frac{\mu e^{-\mu}}{2\left[\sqrt{\frac{\pi}{\mu}}\operatorname{erf}(\sqrt{\mu}) - 3e^{-\mu}\right]} \tag{3.34}$$

A bifurcation can only occur if

$$\sqrt{\frac{\pi}{\mu}}\operatorname{erf}(\sqrt{\mu}) \geq 3e^{-\mu}\,, \quad \text{i.e., if} \qquad \mu > \mu^* \approx 0.5861855742\,. \tag{3.35}$$

The numerical value of μ^* was obtained using `octave`.

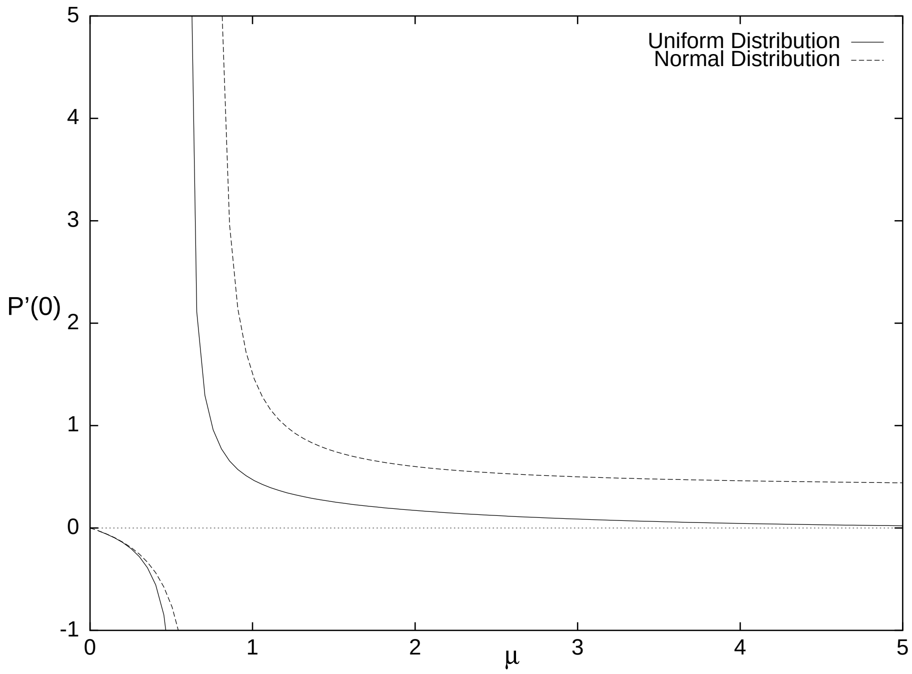

Figure 8: Critical values of $P_0'(0)$ as a function of μ for the two-party models using E as party utility Gaussian voter distribution (equ.(3.12'), dashed line) and uniform voter distribution (equ.(3.25), full line). We set $\sigma^2 = 1/3$ in equ.(3.21), so that both voter distributions have the same variance. There is no bifurcation for $\mu \leq \mu^*$. The critical values are $\mu^* \approx 0.58618557$ for uniform distribution and $\mu^* = 3/4$ in the Gaussian case.

Finally, let us use W instead of E. Then the dynamics is given by

$$
\begin{aligned}
\dot{y}^1 = \frac{1}{2}\int_{-1}^{1} \Big[& P_0'(z)\exp\big(-\mu(y^1-x)^2\big) + P_0(z)\mu\exp\big(-\mu(y^1-x)^2\big) + \\
& P_0'(-z)\exp\big(-\mu(y^2-x)^2\big)\Big](-2)(y^1-x)dx \\
\dot{y}^2 = \frac{1}{2}\int_{-1}^{1} \Big[& P_0'(-z)\exp\big(-\mu(y^2-x)^2\big) + P_0(-z)\mu\exp\big(-\mu(y^2-x)^2\big) + \\
& P_0'(z)\exp\big(-\mu(y^1-x)^2\big)\Big](-2)(y^2-x)dx
\end{aligned}
\tag{3.36}
$$

Table 1. Summary of Section 3.3: Bifurcation at the Trivial Equilibrium

Distribution	Party Utility Function $E(y^1, y^2)$ equ.(3.12')	 $W(y^1, y^2)$ equ.(3.22)
Gaussian	$P_0'(0) = \frac{\mu}{8\sigma^2\mu - 2}$	no bifurcation
Uniform	$P_0'(0) = \frac{\mu e^{-\mu}}{2\left[\sqrt{\frac{\pi}{\mu}}\operatorname{erf}(\sqrt{\mu}) - 3e^{-\mu}\right]}$	no bifurcation

It is easy to see that $\mathcal{H}_2$, i.e., $y^1 = y^2$, is invariant. A fixed point on this surface has to fulfill $\dot{y} = 0$.

$$
\dot{y} = (2P_0'(0) + \mu/2)\int_{-1}^{1}(y-x)\exp\big(-\mu(y-x)^2\big)dx \tag{3.37}
$$

We substitute $y - x \overset{\text{def}}{=} w$ and obtain

$$
\dot{y} = (2P_0'(0) + \mu/2)\int_{y-1}^{y+1} w\exp(-\mu w^2)dw\,. \tag{3.38}
$$

Thus, $\dot{y} = 0$ holds iff $1 - e^{4\mu y} = 0$, i.e., iff $y = 0$. The mean voter point is an equilibrium in this model as well. Let us now determine the entries of the Jacobian at this point:

$$\begin{aligned}\frac{\partial \dot{y}^1}{\partial y^1} &= (4P_0'(0)\mu + \mu^2)\int_{-1}^{1} x^2 e^{-\mu x^2}\,dx - (2P_0'(0) + \frac{\mu}{2})\int_{-1}^{1} e^{-\mu x^2}\,dx \\ \frac{\partial \dot{y}^2}{\partial y^2} &= (4P_0'(0)\mu + \mu^2)\int_{-1}^{1} x^2 e^{-\mu x^2}\,dx - (2P_0'(0) + \frac{\mu}{2})\int_{-1}^{1} e^{-\mu x^2}\,dx\end{aligned} \tag{3.39}$$

while the off-diagonal terms $\partial \dot{y}^1/\partial y^2 = \partial \dot{y}^2/\partial y^1 = 0$ vanish on $\mathcal{H}_2$. Substituting the explicit expressions for the integrals we find that the Jacobian has the twofold eigenvalue $\lambda = -(4P_0'(0) + \mu)\exp(-\mu) < 0$. The mean voter equilibrium in this system is stable, since λ is always negative.

3.4. Numerical Analysis

Not much can be said in general about the dynamics of equ.(3.12). We expect a globally stable mean voter equilibrium for small values of μ and $P'(0)$ by analogy with the continuous cases discussed above.

In order to study the dynamics outside the invariant manifold $\mathcal{H}_2$, numerical bifurcation diagrams have been computed, see fig. 9. The fixed point coordinates are calculated using Broyden's multidimensional secant method [11], as implemented in the *Numerical Recipes* [54] routine `brodyn()`. 500 to 5000 independent initial values, uniformly distributed in the box $\mathcal{B}$ have been used in order to ensure that all fixed points are found. Computations were performed on a `Pentium` PC running `Linux`.

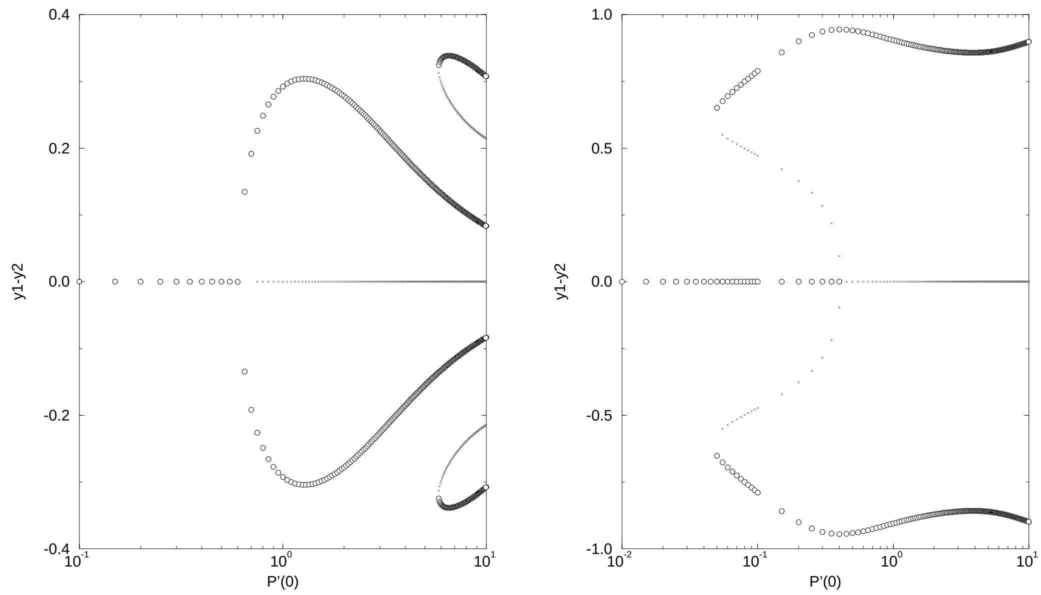

Figure 9: Bifurcation diagram for equ.(3.12). Parameters: 100 voters uniformly distributed in $[-1, 1]$, $u(y, x) = -(y - x)^2$, $\Psi(u) = \exp(-\mu u)$, $P(z) = (1 + \exp(az))^{-1}$. Stable fixed points are indicated by black circles, small gray dots denote saddle points. For small values of μ there is a supercritical pitchfork bifurcation (l.h.s., $\mu = 1$). For large values of μ one finds a saddle-node bifurcation (which is doubled due to the symmetry of the model) and a sub-critical pitchfork bifurcation (r.h.s., $\mu = 5$).

The eigenvalues of the Jacobian are computed by first reducing **J** to Hessenberg form (using `balanc()` and `elmhes()` and then applying the QR algorithm using `hqr()`). The reason for this approach is that the programs have been designed to deal with an arbitrary number of issues and parties, and thus a direct computation of the eigenvalues of **J** from the characteristic polynomial of **J** is infeasible.

Bifurcation diagrams, among them the examples shown in fig. 9, show that a variety of different bifurcations occur in equ.(3.12). Of course, the exact parameter values of the bifurcation points depend quite strongly on the details of

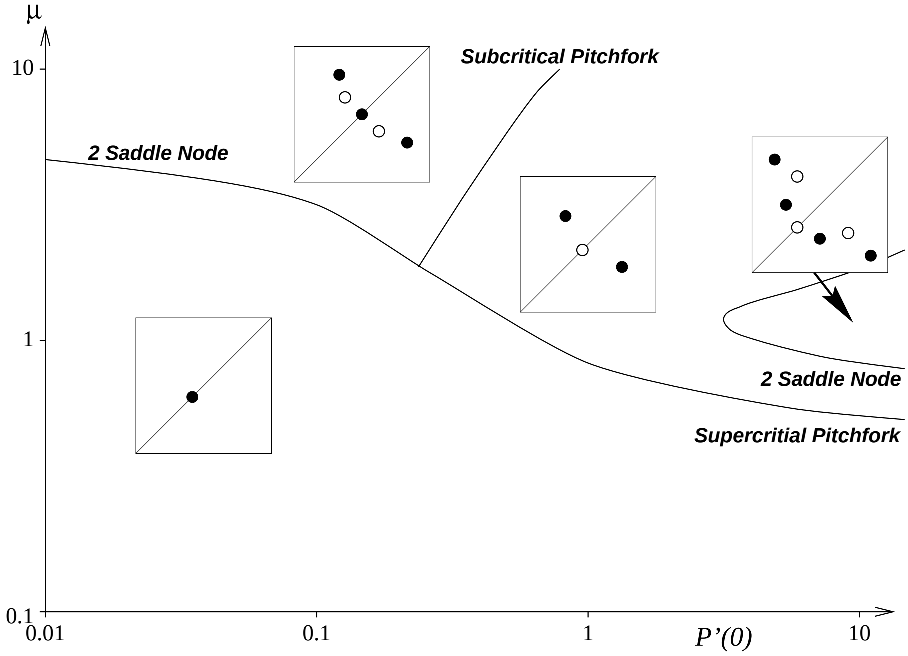

Figure 10: Schematic phase diagram for 2 party model with utility dependent abstention, equ.(3.12).
Parameters: $u(x,y) = -(y-x)^2$, $\Psi(u) = \exp(-\mu u)$, $P(z) = (1+\exp(az))^{-1}$, 100 voters uniformly distributed in $[-1,1]$.
The regions of the phase diagram are labeled by schematic phase portraits indicating the forward invariant box $\mathcal{B} = [-1,1]^2$, the invariant manifold $\mathcal{H}_2$ (diagonal line), stable fixed points (full circles) and saddle points (open circles). We find 4 different phase portraits with up to 7 equilibria. For small $P'(0)$ and small μ there is a single stable equilibrium on $\mathcal{H}_2$ close to the mean voter position. The exact locations of the bifurcations depend on the voter distribution.

the underlying voter distribution. Different (random) choices of the voter positions are found to lead to qualitatively the same bifurcation diagrams, though the numerical values are different.

The schematic phase diagram in fig. 10 was obtained from a series of 1-parameter bifurcation diagrams (with μ fixed and parameter $P'(0)$). Within the range of the numerical survey ($\mu, P'(0) \leq 10$) we encounter four different generic phase portraits. For small values of both μ and $P'(0)$ there is only a single equilibrium on $\mathcal{H}_2$ which is stable. An increase of μ gives rise to bifurcations. With μ large enough, for small values of $P'(0)$ we find two saddle node bifurcations which are related by symmetry. As $P'(0)$ increases, first a sub-critical and then a supercritical pitchfork bifurcation occur. As a consequence, up to 7 equilibria could be found. Within the parameter range of the numerical survey there is no Hopf bifurcation.

4. Three Parties

4.1. Generalization to Three Parties

Let us now consider an election with three parties whose platforms are denoted by y^1, y^2, y^3. As in the previous section we denote by $u_v(y^p)$ voter v's utility of party p's platform. Voter v's utility differences will be denoted by

$$d_{pq}^v \stackrel{\text{def}}{=\!=} u_v(y^p) - u_v(y^q)\,. \tag{4.1}$$

Note that $d_{pq}^v = -d_{qp}^v$.

The generalization of the probability function $\mathcal{P}$ is non-trivial. We shall assume that the probability of voting for party p depends only on the three utility differences d_{pq}^v, d_{pr}^v, and d_{qr}^v with $\{p, q, r\} = \{1, 2, 3\}$. Hence we seek functions $\mathcal{P} : \mathbb{R}^3 \to [0, 1]$ with the following properties:

(i) $Prob[v \text{ votes for } p] = \mathcal{P}(d_{pq}^v, d_{pr}^v, d_{qr}^v) = \mathcal{P}(d_{pr}^v, d_{pq}^v, d_{rq}^v)$, since the probability of voting for party p cannot depend on the labeling of the other two parties.

(ii) $\dfrac{\partial}{\partial d_{pq}^v}\mathcal{P}(d_{pq}^v, d_{pr}^v, d_{qr}^v) > 0$ and $\dfrac{\partial}{\partial d_{pr}^v}\mathcal{P}(d_{pq}^v, d_{pr}^v, d_{qr}^v) > 0$ holds everywhere on $\mathbb{R}^3$, i.e., the probability of voting for p always increases with the utility differences d_{pq} and d_{pr}.

In the case of complete participation we have in addition

(iii) $\mathcal{P}(d_{pq}^v, d_{pr}^v, d_{qr}^v) + \mathcal{P}(d_{qp}^v, d_{qr}^v, d_{pr}^v) + \mathcal{P}(d_{rp}^v, d_{rq}^v, d_{pq}^v) = 1$.

The general three-party dynamics assumes that the expected fraction of votes for each party is

$$E_p = \frac{1}{V}\sum_{v=1}^{V}\mathcal{P}(d^v_{pq}, d^v_{pr}, d^v_{qr}) \tag{4.2}$$

and that each party tries to locally maximize its share of votes, i.e.,

$$\dot{y}^p = \nabla_{y^p} E_p = \nabla_{y^p}\frac{1}{V}\sum_{v=1}^{V}\mathcal{P}(d^v_{pq}, d^v_{pr}, d^v_{qr}) \tag{4.3}$$

where $\{p, q, r\} = \{1, 2, 3\}$ denotes the three parties.

Note that every permutation of the three parties leads to the same set of differential equations, i.e., any permutation of the three parties is a symmetry of the dynamical system (4.3). As a consequence, if there is a fixed point (a, b, c) of equ.(4.3), then (a, c, b), (b, a, c), (b, c, a), (c, a, b), and (c, b, a) are also equilibrium points of (4.3).

We may recast equ.(4.3) somewhat more explicitly in the form

$$\dot{y}^p = \frac{1}{V}\sum_{v=1}^{V}\left\{\frac{\partial}{\partial d^v_{pq}}\mathcal{P}(d^v_{pq}, d^v_{pr}, d^v_{qr}) + \frac{\partial}{\partial d^v_{pr}}\mathcal{P}(d^v_{pq}, d^v_{pr}, d^v_{qr})\right\}\nabla u_v(y^p) \tag{4.3$'$}$$

It will be convenient to use the notation $\partial_1\mathcal{P}$, etc., in order to denote the partial derivatives of $\mathcal{P}$ with respect to its first, second, and third argument, respectively. Equ.(4.3) reads in this notation

$$\dot{y}^p = \frac{1}{V}\sum_{v=1}^{V}\left\{\partial_1\mathcal{P}(d^v_{pq}, d^v_{pr}, d^v_{qr}) + \partial_2\mathcal{P}(d^v_{pq}, d^v_{pr}, d^v_{qr})\right\}\nabla u_v(y^p)\,. \tag{4.4}$$

4.2. Boundedness of the Orbits

The extremal voter positions span the box

$$\mathcal{B} \stackrel{\text{def}}{=\!=} \left(\prod_{i=1}^{I} [x_i^{\min}, x_i^{\max}] \right)^3 \tag{4.5}$$

in the state space $\mathbb{R}^{3I}$. Intuitively, a platform outside $\mathcal{B}$ can always be improved by approaching $\mathcal{B}$, since the platform comes "closer" to all voters. The following theorem shows that this is in fact true under the dynamics (4.3).

Theorem 1. (i) The box $\mathcal{B}$ defined by the extremal voter positions is forward invariant; all orbits are eventually bounded in an arbitrary small neighborhood of the box $\mathcal{B}$.
(ii) If $\mathcal{B}$ has non-zero volume, i.e., if $x_k^{\min} < x_k^{\max}$ for all issues k then $\mathcal{B}$ is even strictly forward invariant under the dynamics (4.3) and all orbits are eventually bounded in the interior of $\mathcal{B}$.

Proof. (i) The dynamics can be recast in the form

$$\dot{y}_k^p = \frac{1}{V} \sum_{v=1}^{V} \psi_v^p(y^1, y^2, y^3) \partial_k u_v(y^p)$$

where ψ_v^p serves as an abbreviation for the sum of the two partial derivatives of $\mathcal{P}$ in the curly parentheses of equ.(4.4). As an immediate consequence of the axioms of the probability function $\mathcal{P}$ we know that $\psi_v^p > 0$ in $\mathbb{R}^{3I}$.
The voter dissatisfaction functions fulfill of course

$$y_k > x_k^{\max} \implies \partial_k u_v(y) < 0 \qquad \text{and} \qquad y_k < x_k^{\min} \implies \partial_k u_v(y) > 0$$

for all voters v. Consequently, $\dot{y}^p_k < 0$ whenever $y^p_k > x^{\max}_k$ and $\dot{y}^p_k > 0$ whenever $y^p_k < x^{min}_k$. Thus there are no ω-limits outside $\mathcal{B}$, and all orbits are eventually contained in an arbitrary small neighborhood of $\mathcal{B}$.
(ii) If $\mathcal{B}$ has non-zero volume then for each issue k there are voters v and w such that

$$y_k \geq x^{\max}_k \implies \partial_k u_v(y) < 0 \qquad \text{and} \qquad y_k \leq x^{\min}_k \implies \partial_k u_w(y) > 0\,.$$

Thus $\dot{y}^p_k < 0$ whenever $y^p_k \geq x^{\max}_k$ and $\dot{y}^p_k > 0$ whenever $y^p_k \leq x^{min}_k$. Thus $\mathcal{B}$ is strictly forward invariant. ∎

4.3. Invariant Manifolds

Lemma 1. The plane $\mathcal{H}^{(pq)}_2 \stackrel{\text{def}}{=\!=} \{\vec{y} = (y^1, y^2, y^3) \in \mathbb{R}^{3I} \mid y^p = y^q\}$, is invariant under the three-party voting dynamics.

Proof. It is sufficient to show that $\mathcal{H}^{(23)}_2$ is invariant; the invariance of the other two planes follows then from the symmetries of the voting dynamics. Observing that $d^v_{23} = 0$ and $z_v \stackrel{\text{def}}{=\!=} d^v_{12} = d^v_{13}$ on $\mathcal{H}^{(23)}_2$ we find that

$$\begin{aligned}
\dot{y}^2 &= \frac{1}{V}\sum_{v=1}^{V}\left[\frac{\partial}{\partial d^v_{21}}\mathcal{P}(d^v_{21}, d^v_{23}, d^v_{13}) + \frac{\partial}{\partial d^v_{23}}\mathcal{P}(d^v_{21}, d^v_{23}, d^v_{13})\right] \nabla u_v(y^2) \\
&= \frac{1}{V}\sum_{v=1}^{V}\left[\partial_1\mathcal{P}(-z_v, 0, z_v) + \partial_2\mathcal{P}(-z_v, 0, z_v)\right] \nabla u_v(y^2) \\
\dot{y}^3 &= \frac{1}{V}\sum_{v=1}^{V}\left[\frac{\partial}{\partial d^v_{31}}\mathcal{P}(d^v_{31}, d^v_{32}, d^v_{12}) + \frac{\partial}{\partial d^v_{32}}\mathcal{P}(d^v_{31}, d^v_{32}, d^v_{12})\right] \nabla u_v(y^3) \\
&= \frac{1}{V}\sum_{v=1}^{V}\left[\partial_1\mathcal{P}(-z_v, 0, z_v) + \partial_2\mathcal{P}(-z_v, 0, z_v)\right] \nabla u_v(y^3)\,.
\end{aligned}$$

Thus, $y^2 = y^3$ implies $\dot{y}^2 = \dot{y}^3 = 0$, and $\mathcal{H}_2^{(23)}$ is in fact invariant. ∎

Note that the three-party dynamics does *not* reduce to the two party dynamics, equ.(2.4), when two of the three parties have identical platforms. In particular, a "coalition" of two parties receives two-thirds instead of half of the votes at a trivial equilibrium, since each party individually gets one third of the votes.

Corollary. The manifold $\mathcal{H}_3 \stackrel{\text{def}}{=} \{\vec{y} \in \mathbb{R}^{3I} \mid y^1 = y^2 = y^3\}$ is invariant under the three-party voting model (4.4).

Proof. Follows from $\mathcal{H}_3 = \mathcal{H}_2^{(12)} \cap \mathcal{H}_2^{(23)}$. ∎

Fixed points on $\mathcal{H}_3$ will be referred to as *trivial*. The first and second partial derivatives of the response function $\mathcal{P}$ will be important in determining their stability.

Lemma 2. On $\mathcal{H}_3$ we have

$$\begin{aligned} \partial_1\mathcal{P}(0,0,0) = \partial_2\mathcal{P}(0,0,0) &\stackrel{\text{def}}{=} \mathcal{P}' \\ \partial_3\mathcal{P}(0,0,0) &= 0 \\ \partial_1\partial_3\mathcal{P}(0,0,0) + \partial_2\partial_3\mathcal{P}(0,0,0) &= 0 \\ \partial_1^2\mathcal{P}(0,0,0) = \partial_2^2\mathcal{P}(0,0,0) &\stackrel{\text{def}}{=} \mathcal{P}'' \\ \partial_1\partial_2\mathcal{P}(0,0,0) &\stackrel{\text{def}}{=} \hat{\mathcal{P}} \end{aligned} \tag{4.6}$$

Proof. The equations between partial derivatives are immediate consequences of the symmetry property (i) of $\mathcal{P}$. ∎

On $\mathcal{H}_3$, where the platforms of all three parties coincide, we find again a gradient system:

$$\dot{y}^p = \frac{1}{V}\sum_v [\partial_1 \mathcal{P}(0,0,0) + \partial_2 \mathcal{P}(0,0,0)] \nabla u_v(y^p) = 2\mathcal{P}' \, \nabla U(y^p)\,. \tag{4.7}$$

The equilibria on this manifold are thus again exactly the critical points of the average voter utility $U(y)$.

4.4. Stability of Trivial Fixed Points

Let us now determine the Jacobian of our dynamical system for a fixed point on $\mathcal{H}_3$. By virtue of the symmetries of our model it is sufficient to compute the entries $\partial \dot{y}^1_j / \partial y^1_k$ and $\partial \dot{y}^1_j / \partial y^2_k$. We find explicitly

$$\begin{aligned}
\frac{\partial \dot{y}^1_j}{\partial y^1_k} &= \frac{1}{V}\sum_v \big\{ \partial_1^2 \mathcal{P}(d^v_{12}, d^v_{13}, d^v_{23}) + \partial_2^2 \mathcal{P}(d^v_{12}, d^v_{13}, d^v_{23}) + \\
&\qquad 2\partial_1\partial_2 \mathcal{P}(d^v_{12}, d^v_{13}, d^v_{23}) \big\} \partial_j u_v(y^1) \partial_k u_v(y^1) \\
&\qquad + \frac{1}{V}\sum_v \big\{ \partial_1 \mathcal{P}(d^v_{12}, d^v_{13}, d^v_{23}) + \partial_2 \mathcal{P}(d^v_{12}, d^v_{13}, d^v_{23}) \big\} \partial_k \partial_j u_v(y^1) \qquad (4.8) \\
\frac{\partial \dot{y}^1_j}{\partial y^2_k} &= \frac{1}{V}\sum_v \big\{ - \partial_1^2 \mathcal{P}(d^v_{12}, d^v_{13}, d^v_{23}) - \partial_1\partial_2 \mathcal{P}(d^v_{12}, d^v_{13}, d^v_{23}) \\
&\qquad + \partial_1\partial_3 \mathcal{P}(d^v_{12}, d^v_{13}, d^v_{23}) + \partial_2\partial_3 \mathcal{P}(d^v_{12}, d^v_{13}, d^v_{23}) \big\} \partial_j u_v(y^1) \partial_k u_v(y^2)
\end{aligned}$$

Substituting the values of the partial derivatives of $\mathcal{P}$ at the point $(0,0,0)$ as described in the previous subsection then yields

$$\begin{aligned}
\left.\frac{\partial \dot{y}^1_j}{\partial y^1_k}\right|_{\mathcal{H}_3} &= 2(\mathcal{P}'' + \hat{\mathcal{P}}) \frac{1}{V}\sum_v \partial_j u_v(y) \partial_k u_v(y) + 2\mathcal{P}' \frac{1}{V}\sum_v \partial_k \partial_j u_v(y) \\
\left.\frac{\partial \dot{y}^1_j}{\partial y^2_k}\right|_{\mathcal{H}_3} &= -(\mathcal{P}'' + \hat{\mathcal{P}}) \frac{1}{V}\sum_v \partial_j u_v(y) \partial_k u_v(y)
\end{aligned} \tag{4.9}$$

The Jacobian at point $(y,y,y) \in \mathcal{H}_3$ can thus be written in the form

$$\mathbf{J}(y,y,y) = (\hat{\mathcal{P}} + \mathcal{P}'')\mathbf{C}(y) \otimes \begin{pmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{pmatrix} + 2\mathcal{P}'\mathbf{H}(y) \otimes \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \tag{4.10}$$

where we have used the abbreviation

$$\mathbf{C}_{kj}(y) \stackrel{\text{def}}{=\!=} \frac{1}{V}\sum_{v=1}^{V} \partial_k u_v(y) \partial_j u_v(y) \,. \tag{4.11}$$

$\mathbf{C}$ is the covariance matrix of the components of the vectors $\nabla u_v(y)$ at a fixed point of (4.4) on $\mathcal{H}_3$ by the same argument as in equ.(3.11). Again, $\mathbf{H}(y)$ is the Hessian of the average voter utility.

We observe that the 3×3 matrix occurring in the first term is singular. It has the simple eigenvalue 0 and the double eigenvalue 3. The first term is therefore positive semi-definite provided $\mathcal{P}'' + \hat{\mathcal{P}} > 0$. The second term is negative definite for a maximum of the average voter dissatisfaction, e.g., for the unique fixed point when the Enelow-Hinich satisfaction function (2.1) is used. Bifurcations may therefore occur at the trivial fixed point depending on the ratio of $\mathcal{P}'' + \hat{\mathcal{P}}$ and $\mathcal{P}'$ even if U is a quadratic function. The three party model differs qualitatively from the two party case in this respect.

4.5. An Explicit Example for $\mathcal{P}$

The complicated form of the platform dynamics suggest that a detailed analysis by analytical techniques will not be possible. Supplementary numerical studies are necessarily restricted to varying a small number of parameters. We therefore choose the response function

$$\mathcal{P}(d^v_{ij}, d^v_{ik}, d^v_{jk}) \stackrel{\text{def}}{=} \frac{P_0(d^v_{ij})P_0(d^v_{ik})}{P_0(d^v_{12})P_0(d^v_{13}) + P_0(d^v_{21})P_0(d^v_{23}) + P_0(d^v_{31})P_0(d^v_{32})}\,, \tag{4.12}$$

which is a generalization of the sigmoidal function P_0 introduced in section 3.1. The denominator in the above expression,

$$A_v = P_0(d^v_{12})P_0(d^v_{13}) + P_0(d^v_{21})P_0(d^v_{23}) + P_0(d^v_{31})P_0(d^v_{32}) \tag{4.13}$$

is chosen such that condition (iii) from section 3.1., i.e., complete participation, is satisfied.

Lemma 3. Equ.(4.12) fulfills the properties (i), (ii), and (iii) for a valid response function $\mathcal{P}$ with complete participation.

Proof. We shall first simplify the notation by setting $a = d^v_{12}$, $b = d^v_{13}$, $c = d^v_{23}$. We denote the denominator by A_v. It is easy to see that (i) is fulfilled, since

$$\begin{aligned}\mathcal{P}(a,b,c) &\stackrel{\text{def}}{=} \frac{P_0(a)P_0(b)}{P_0(a)P_0(b) + P_0(-a)P_0(c) + P_0(-b)P_0(-c)} = \\ &= \frac{P_0(b)P_0(a)}{P_0(b)P_0(a) + P_0(-b)P_0(-c) + P_0(-a)P_0(c)} = \\ &= \mathcal{P}(b,a,-c)\,.\end{aligned}$$

Incorporating the fact that

$$\frac{\partial A}{\partial a} = P_0(a)[P_0(b) - P_0(c)],$$

since $P_0'(a) = P_0'(-a)$, we will show that condition (ii) is fulfilled since

$$\begin{aligned}\frac{\partial \mathcal{P}}{\partial a} &= \frac{1}{A^2}(P_0'(a)P_0(b)A - \frac{\partial A}{\partial a}) \\ &= P_0'(a)\frac{P_0(b)P_0(-a)P_0(c) + P_0(b)P_0(-b)P_0(-c) + P_0(a)P_0(b)P_0(c)}{A^2} > 0\end{aligned}$$

Symmetry of the system guarantees $\dfrac{\partial \mathcal{P}(a,b,c)}{\partial a} = \dfrac{\partial \mathcal{P}(b,a,-c)}{\partial a} = \dfrac{\partial \mathcal{P}(a,b,c)}{\partial b}$. ∎

Introducing the abbreviation $A_v^{(j)} \stackrel{\text{def}}{=} \partial A_v / \partial u_v(y^j)$ we obtain the following, rather complicated differential equation describing the platform dynamics of three parties:

$$
\begin{aligned}
\dot{y}_j^1 &= \frac{1}{V}\sum_{v=1}^{V} A_v^{-1}[P_0'(d_{12}^v)P_0(d_{13}^v) + P_0(d_{12}^v)P_0'(d_{13}^v)]\partial_j u_v(y^1) \\
&\quad - \frac{1}{V}\sum_{v=1}^{V} P_0(d_{12}^v)P_0(d_{13}^v)A_v^{-2}A_v^{(1)}\partial_j u_v(y^1)\,, \\
\dot{y}_j^2 &= \frac{1}{V}\sum_{v=1}^{V} A_v^{-1}[P_0'(d_{21}^v)P_0(d_{23}^v) + P_0(d_{21}^v)P_0'(d_{23}^v)]\partial_j u_v(y^2) \\
&\quad - \frac{1}{V}\sum_{v=1}^{V} P_0(d_{21}^v)P_0(d_{23}^v)A_v^{-2}A_v^{(2)}\partial_j u_v(y^2)\,, \\
\dot{y}_j^3 &= \frac{1}{V}\sum_{v=1}^{V} A_v^{-1}[P_0'(d_{31}^v)P_0(d_{32}^v) + P_0(d_{31}^v)P_0'(d_{32}^v)]\partial_j u_v(y^3) \\
&\quad - \frac{1}{V}\sum_{v=1}^{V} P_0(d_{31}^v)P_0(d_{32}^v)A_v^{-2}A_v^{(3)}\partial_j u_v(y^3)\,,
\end{aligned}
\tag{4.14}
$$

This form is used explicitly for numerical investigations, in particular for the calculation of bifurcation diagrams in section 4.11.

In order to determine the stability of the trivial equilibria we need the parameters $\mathcal{P}'$, $\mathcal{P}''$, and $\hat{\mathcal{P}}$. A short calculation yields

$$
\mathcal{P}' = \frac{2}{3}P_0'(0)\,, \qquad \mathcal{P}'' = 0\,, \qquad \text{and} \qquad \hat{\mathcal{P}} = \frac{8}{9}P_0'(0)^2\,. \tag{4.15}
$$

The Jacobian on $\mathcal{H}_3$ is of the form (4.10)

$$
\mathbf{J} = \frac{4}{3}P_0'(0)\mathbf{H}\otimes\begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix} + \frac{8}{9}P_0'(0)^2\mathbf{C}\otimes\begin{pmatrix}2&-1&-1\\-1&2&-1\\-1&-1&2\end{pmatrix}. \tag{4.16}
$$

The slope $P_0'(0)$ of the sigmoidal function P_0 thus plays a crucial role for the stability of the trivial equilibria. A trivial fixed point (y, y, y) that corresponds to a maximum of U will be stable for small values of $P_0'(0)$ and unstable for large values of the slope. Recall that a steep slope of the function P_0 at 0 indicates a strong division of voters for either party whenever the platforms are different. We can also say that $P_0'(0)$ expresses how critically voters react in case there are alternatives to choose between. For a more detailed analysis we shall restrict ourselves to quadratic voter dissatisfaction functions in the following subsection.

4.6. Three-Party Enelow-Hinich Model

We shall determine the *critical value* p^* of $P_0'(0)$ explicitly for the quadratic Enelow-Hinich model with equal strengths $s_{vi} = s$ for all v and i. It is easy to verify that the co-ordinates of the trivial fixed point are given by the average voter's position

$$y_i^1 = y_i^2 = y_i^3 = \bar{x} \overset{\text{def}}{=\!=} \frac{1}{V} \sum_v x_{vi} \,, \tag{4.17}$$

as in the two-party model discussed in the previous section. The matrix $\mathbf{C}$ reduces in this special case to

$$\mathbf{C}_{ij} = \frac{4s^2}{V} \sum_v (x_{vi} - \bar{x}_i)(x_{vj} - \bar{x}_j) \overset{\text{def}}{=\!=} 4s^2 \mathbf{V}_{ij} \tag{4.18}$$

i.e., it is a multiple of the co-variance matrix $\mathbf{V}$ of the distribution of voters in the issue space. Let ρ denote the spectral radius (maximum eigenvalue) of the covariance matrix

$$\mathbf{V}_{ij} \stackrel{\text{def}}{=} \frac{1}{V} \sum_v (x_{vi} - \bar{x}_i)(x_{vj} - \bar{x}_j) \tag{4.19}$$

of the distribution of voters in issue space. The largest eigenvalue of $\mathbf{C}$ is then $4s^2\rho$. In the Enelow-Hinich model the Hessian $\mathbf{H}$ is diagonal. With constant strength s we have explicitly $\mathbf{H} = -2s\mathbf{I}$. Therefore, each eigenvector of $\mathbf{C}$ is an eigenvector of $\mathbf{H}$. The largest eigenvalue of $\mathbf{J}$ is therefore

$$\begin{aligned} \lambda_{\max} &= \frac{4P_0'(0)}{3} \cdot (-2s) \cdot 1 + \frac{8P_0'(0)^2}{9} \cdot 4s^2\rho \cdot 3 \\ &= \frac{8sP_0'(0)}{3} \left(4s\rho P_0'(0) - 1 \right). \end{aligned} \tag{4.20}$$

The average voter equilibrium therefore becomes unstable when $P_0'(0)$ exceeds the critical value $p^* = \frac{1}{4s\rho}$. In the 1-issue case this reduces to

$$p^* = \frac{1}{4s\mathrm{var}(x)}, \tag{4.21}$$

where $\mathrm{var}(x)$ is the variance of the one-dimensional voter distribution.

Remark. The more general model with arbitrary strength cannot be solved explicitly since generally, $\mathbf{H}$ and $\mathbf{J}$ do not have the same eigenvectors and therefore the computation of the eigenvalues will not be obvious.

4.7. Incomplete Participation

In this subsection we consider a simple model with incomplete participation. Consider the response function

$$\mathcal{P}(a,b,c) = P_0(a)P_0(b)\,. \tag{4.22}$$

Lemma 4. The response function $\mathcal{P}$ defined above represents a probability, i.e., $\mathcal{P}$ is non-negative and

$$Q = \mathcal{P}(a,b,c) + \mathcal{P}(-a,c,b) + \mathcal{P}(-b,-c,a) < 1\,.$$

Proof. We have

$$\begin{aligned}\mathcal{P}(a,b,c) &= P_0(a)P_0(b)\\ \mathcal{P}(-a,c,b) &= (1-P_0(a))P_0(c)\\ \mathcal{P}(-b,-c,a) &= 1-P_0(b)-P_0(c)+P_0(b)P_0(c)\end{aligned}$$

We have to distinguish two cases:

case 1: $P_0(b) > P_0(c)$.

$$\begin{aligned}Q = &P_0(a)[P_0(b)-P_0(c)] + P_0(c) + 1 - P_0(b) - P_0(c) + P_0(b)P_0(c)\\ <&P_0(b)-P_0(c)+P_0(c)+1-P_0(b)-P_0(c)+P_0(c) < 1\end{aligned}$$

case 2: $P_0(b) \leq P_0(c)$. $\quad Q \leq P_0(c) + 1 - P_0(b) - P_0(c) + P_0(b) = 1.$ ∎

Now we have to compute the three parameters $\mathcal{P}''$, $\hat{\mathcal{P}}$, and $\mathcal{P}'$. A short calculation yields

$$\mathcal{P}'' = 0\,, \quad \hat{\mathcal{P}} = P_0'(0)^2\,, \quad \mathcal{P}' = P_0'(0)/2\,. \tag{4.23}$$

Therefore $\lambda_{\max} = 2sP_0'(0)[6s\rho P_0'(0) - 1]$ with Enelow-Hinich voter utilities with equal strength where $\mathbf{H} = -2s\mathbf{I}$. The critical value of $P_0'(0)$ is in this case $p^* = (6s\rho)^{-1}$ and $p^* = (6s\mathrm{var}(x))^{-1}$ in the 1-issue case.

4.8. Three Party Model with Normal Distribution of Voters

The dynamical system in (4.4) with normal voter distribution and a single issue explicitly reads

$$\begin{aligned}
\dot{y}^1 &= \frac{-2}{\sigma\sqrt{2\pi}} \int_{-\infty}^{\infty} \big(\partial_1 \mathcal{P}(d_{12}, d_{13}, d_{23}) + \partial_2 \mathcal{P}(d_{12}, d_{13}, d_{23})\big)(y^1 - x)\exp(\frac{-x^2}{2\sigma^2})dx \\
\dot{y}^2 &= \frac{-2}{\sigma\sqrt{2\pi}} \int_{-\infty}^{\infty} \big(\partial_1 \mathcal{P}(d_{21}, d_{23}, d_{13}) + \partial_2 \mathcal{P}(d_{21}, d_{23}, d_{13})\big)(y^2 - x)\exp(\frac{-x^2}{2\sigma^2})dx \\
\dot{y}^3 &= \frac{-2}{\sigma\sqrt{2\pi}} \int_{-\infty}^{\infty} \big(\partial_1 \mathcal{P}(d_{31}, d_{32}, d_{12}) + \partial_2 \mathcal{P}(d_{31}, d_{32}, d_{12})\big)(y^3 - x)\exp(\frac{-x^2}{2\sigma^2})dx
\end{aligned} \tag{4.24}$$

It is easy to see that $\mathcal{H}_3$ is invariant. The fixed points on this line are given by the equation

$$\begin{aligned}
\dot{y} &= \frac{-2}{\sigma\sqrt{2\pi}} \int_{-\infty}^{\infty} \big(\partial_1 \mathcal{P}(0,0,0) + \partial_2 \mathcal{P}(0,0,0)\big)(y - x)\exp(\frac{-x^2}{2\sigma^2})dx \\
&= -\frac{4}{\sigma\sqrt{2\pi}}\mathcal{P}'\, y \int_{-\infty}^{\infty} \exp(\frac{-x^2}{2\sigma^2})dx = 0
\end{aligned} \tag{4.25}$$

This is fulfilled for $y = 0$, which is in fact the mean of the voter distribution. Substituting $\frac{x^2}{2\sigma^2} \stackrel{\mathrm{def}}{=} t^2$ we get

$$\int_{-\infty}^{\infty} \exp(\frac{-x^2}{2\sigma^2})dx = \sigma\sqrt{2\pi} \tag{4.26}$$

and hence $\dot{y} = -4\mathcal{P}'\,y$. Let us first consider the simple model with incomplete participation. We simply set $\mathcal{P}(d_{ij}, d_{ik}, d_{jk}) \stackrel{\text{def}}{=} \mathcal{P}(d_{ij})\mathcal{P}(d_{ik})$. Now we compute the Jacobian at the trivial fixed point:

$$\mathbf{J} = \begin{pmatrix} 8(\mathcal{P}'' + \hat{\mathcal{P}})\sigma^2 - 4\mathcal{P}' & -4(\mathcal{P}'' + \hat{\mathcal{P}})\sigma^2 & -4(\mathcal{P}'' + \hat{\mathcal{P}})\sigma^2 \\ -4(\mathcal{P}'' + \hat{\mathcal{P}})\sigma^2 & 8(\mathcal{P}'' + \hat{\mathcal{P}})\sigma^2 - 4\mathcal{P}' & -4(\mathcal{P}'' + \hat{\mathcal{P}})\sigma^2 \\ -4(\mathcal{P}'' + \hat{\mathcal{P}})\sigma^2 & -4(\mathcal{P}'' + \hat{\mathcal{P}})\sigma^2 & 8(\mathcal{P}'' + \hat{\mathcal{P}})\sigma^2 - 4\mathcal{P}' \end{pmatrix} \tag{4.27}$$

where $\sigma^2 = \mathrm{var}(x)$. The expressions $\mathcal{P}''$, $\hat{\mathcal{P}}$, and $\mathcal{P}'$ are defined as in section 4.3. At the trivial fixed point, $\mathcal{P}'' = 0$, $\mathcal{P}' = \frac{P_0'(0)}{2}$, and $\hat{\mathcal{P}} = P_0'(0)^2$. The Jacobian is therefore of the following form

$$\mathbf{J} = 4P_0'(0)^2\sigma^2 \begin{pmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{pmatrix} - 2P_0'(0) \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{4.28}$$

The first matrix has 0 as a simple eigenvalue and 3 as an eigenvalue with multiplicity 2. A bifurcation in this system occurs at $P_0'(0) = (6\sigma^2)^{-1}$. Not surprisingly, this is the same result as for the discrete voter distribution discussed in the previous section.

In the case of complete participation, we have $\mathcal{P}'' = 0$, $\mathcal{P}' = \frac{2}{3}P_0'(0)$, and $\hat{\mathcal{P}} = \frac{8}{9}P_0'(0)^2$. The Jacobian thus reads:

$$\mathbf{J} = \frac{32}{9}P_0'(0)^2\sigma^2 \begin{pmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{pmatrix} - \frac{8}{3}P_0'(0) \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{4.29}$$

A bifurcation occurs at $P_0'(0) = (4\sigma^2)^{-1}$. This result agrees with that obtained for the discrete voter distribution discussed in the previous section.

4.9. Three Party Model with Uniform Distribution of Voters

Let us assume that the voters are distributed uniformly on the interval $[-1,1]$.

Then the dynamics in three dimensions is of the following form:

$$\begin{aligned}
\dot{y}^1 &= (-1)\int_{-1}^{1}\big(\partial_1\mathcal{P}(d_{12},d_{13},d_{23})+\partial_2\mathcal{P}(d_{12},d_{13},d_{23})\big)(y^1-x)dx \\
\dot{y}^2 &= (-1)\int_{-1}^{1}\big(\partial_1\mathcal{P}(d_{21},d_{23},d_{13})+\partial_2\mathcal{P}(d_{21},d_{23},d_{13})\big)(y^2-x)dx \\
\dot{y}^3 &= (-1)\int_{-1}^{1}\big(\partial_1\mathcal{P}(d_{31},d_{32},d_{12})+\partial_2\mathcal{P}(d_{31},d_{32},d_{12})\big)(y^3-x)dx
\end{aligned} \tag{4.30}$$

On $\mathcal{H}_3$, the dynamics reduces to

$$\dot{y} = -4\mathcal{P}'\,y \tag{4.31}$$

Thus, the mean voter $y=0$ is again a fixed point. Let us compute the Jacobian at the trivial fixed point for the case of incomplete participation:

$$\mathbf{J} = \begin{pmatrix} \frac{8}{3}(\mathcal{P}''+\hat{\mathcal{P}})-4\mathcal{P}' & -\frac{4}{3}(\mathcal{P}''+\hat{\mathcal{P}}) & -\frac{4}{3}(\mathcal{P}''+\hat{\mathcal{P}}) \\ -\frac{4}{3}(\mathcal{P}''+\hat{\mathcal{P}}) & \frac{8}{3}(\mathcal{P}''+\hat{\mathcal{P}})-4\mathcal{P}' & -\frac{4}{3}(\mathcal{P}''+\hat{\mathcal{P}}) \\ -\frac{4}{3}(\mathcal{P}''+\hat{\mathcal{P}}) & -\frac{4}{3}(\mathcal{P}''+\hat{\mathcal{P}} & \frac{8}{3}(\mathcal{P}''+\hat{\mathcal{P}})-4\mathcal{P}' \end{pmatrix} \tag{4.32}$$

where $\mathcal{P}''$, $\mathcal{P}'$, and $\hat{\mathcal{P}}$ are again defined as in section 4.3. We may write the Jacobian at the trivial fixed point as follows:

$$\mathbf{J} = \frac{4}{3}P_0'(0)^2\begin{pmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{pmatrix} - 2P_0'(0)\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{4.33}$$

A bifurcation occurs if the largest eigenvalue of the Jacobian (which in this case has multiplicity 2) becomes zero. This happens at $P_0'(0) = \frac{1}{2}$. Since the variance of the uniform distribution on [-1,1] is 1/3, the location of the bifurcation is consistent with the result $p^* = (6s\mathrm{var}(x))^{-1}$ from the previous two sections.

The case of complete participation yields the following Jacobian:

$$\mathbf{J} = \frac{32}{27}P_0'(0)^2 \begin{pmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{pmatrix} - \frac{8}{3}P_0'(0) \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{4.34}$$

The bifurcation at the trivial fixed point occurs at $P_0'(0) = \frac{3}{4}$, which is again consistent with the result $p^* = (4s\mathrm{var}(x))^{-1}$.

4.10. General Continuous Voter Distributions

The agreement between the results of the previous two sections and the discrete case considered in section 4.4 strongly suggests that equ.(4.10) can be generalized to continuous voter distributions. Assuming a continuous voter distribution, equ.(4.3) becomes

$$\dot{y}_k^p = \int_{\mathbb{R}^I} \left[\partial_1 \mathcal{P}(d_{pq}, d_{pr}, d_{qr}) + \partial_2 \mathcal{P}(d_{pq}, d_{pr}, d_{qr})\right] \partial_k u(y^p, x)\, \rho(x) dx\,. \tag{4.35}$$

We see immediately that on $\mathcal{H}_3$ this reduces to

$$\dot{y}_k = 2\mathcal{P}' \int_{\mathbb{R}^I} \partial_k u(y, x)\, \rho(x) dx \tag{4.36}$$

Setting $\dot{y}_k = 0$ in (4.36) defines the coordinates of the trivial fixed point(s). Similarly, we find for the Jacobian matrix at a point $\vec{y} = (y, y, y) \in \mathcal{H}_3$:

$$\begin{aligned} \frac{\partial \dot{y}_k^1}{\partial y_l^1} &= 2(\mathcal{P}'' + \hat{\mathcal{P}}) \int_{\mathbb{R}^I} \partial_l u(y, x) \partial_k u(y, x)\, \rho(x) dx \\ &\quad + 2\mathcal{P}' \int_{\mathbb{R}^I} \partial_k \partial_l u(y, x)\, \rho(x) dx \\ \frac{\partial \dot{y}_k^1}{\partial y_l^2} &= -(\mathcal{P}'' + \hat{\mathcal{P}}) \int_{\mathbb{R}^I} \partial_l u(y, x) \partial_k u(y, x)\, \rho(x) dx \end{aligned} \tag{4.37}$$

In analogy with the discrete case we introduce the average voter utility

$$U(y) = \int_{\mathbb{R}^I} u(y,x)\,\rho(x)dx\,. \tag{4.38}$$

Clearly, $\mathbf{H}_{kl}(y) = \int_{\mathbb{R}^I} \partial_k\partial_l u(y,x)\,\rho(x)dx$ is the Hessian of U. Using equ.(4.36) with $\dot{y} = 0$ we may interpret

$$\mathbf{C}_{kl}(y) = \int_{\mathbb{R}^I} \partial_l u(y,x)\partial_k u(y,x)\,\rho(x)dx \tag{4.39}$$

as the covariance of $\partial_k U$ and $\partial_l U$. With this notation we recover equ.(4.10) also in the case of general continuous voter distributions. In section 5 we shall discuss the relation of continuous and discrete voter distributions in some more detail.

Finally, consider the Enelow-Hinich type voter utilities (1.7) for a single issue, $u(y,x) = -s(x)(y-x)^2$. For simplicity we assume that $\rho(x)$ and $s(x)$ are symmetric around 0. Obviously, $\hat{y} = 0$ is the unique mean voter fixed point in this case. We consider the three example of position dependent strength functions introduced in equ.(1.5).

For *uniform* voters, $s(x) = 1/2$, we find $\mathbf{C} = \mathrm{var}(x)$ and $\mathbf{H} = -1$.

For *extremist* voters, $s(x) = |x|$, we find $\mathbf{C} = 4\mathrm{curt}(x) = 4\int_{-\infty}^{+\infty} x^4\rho(x)dx$ and $\mathbf{H} = -2\int_{-\infty}^{+\infty} |x|\rho(x)dx$

For *centrist* voters, $s(x) = \max[1-|x|, 0]$, we have to assume that the support of $\rho(x)$ is contained in $[-1, 1]$ in order to obtain "pretty" equations. With this additional assumption we find $\mathbf{H} = \int_{-\infty}^{+\infty} |x|\rho(x)dx - 2$ and a rather complicated expression for $\mathbf{C}$.

Table 2. Position Dependent Strength Factors.

The bifurcation at the trivial fixed point of a 3-party Enelow-Hinich model with position dependent strength factors depends quite strongly on the model for the strength factors, equ.(1.5). We assume a uniform voter distribution on $[-\alpha, \alpha]$ with $\alpha \leq 1$ and use the response function $\mathcal{P}$ defined in equ.(4.12).

	uniform	extremist	centrist
H	-1	$-\alpha$	$\alpha - 2$
C	$\alpha^2/3$	$(4/5)\alpha^4$	$(4/3)\alpha^2 - 2\alpha^3 + (4/5)\alpha^4$
p^*	$(3/2)\alpha^{-2}$	$(5/8)\alpha^{-3}$	$-\mathbf{H}/(2\mathbf{C})$

Suppose $\rho(x) = \begin{cases} 1/(2\alpha) & |x| \leq \alpha \\ 0 & |x| > \alpha \end{cases}$, i.e., the uniform distribution on $[-\alpha, \alpha]$. The value of $\mathbf{H}$, and $\mathbf{C}$ for the three models of the strength functions are compiled in table 2.

Using the response function (4.12) we obtain

$$\lambda_{max} = \frac{4}{3} P_0'(0)\mathbf{H} + \frac{8}{3} P_0'(0)^2 \mathbf{C} \tag{4.40}$$

from (4.10). The bifurcation points $p^* = -\mathbf{H}/(2\mathbf{C})$ are given in table 2. Note the strong dependence of the bifurcation points on the model for the strength factors.

4.11. Numerical Analysis

In this section we report a few numerical results obtained for a three party model with a single issue. Two bifurcation diagrams are shown in figure 11.

The procedure for obtaining the bifurcation diagrams is explained in section 3.4. The phase space is projected onto the single coordinate

$$x^* = \frac{y^1 + 2y^2 + 5y^3}{8} \tag{4.41}$$

in order to break the symmetry of the model and make all fixed points visible. The response function is $P_0(z) = (1 + \tanh(az))/2$.

A close inspection of the figures shows that the search for rest points does not work perfectly. For instance, the mean voter equilibrium is not found for all values of $P'(0)$. This problem could be overcome by increasing the number of initial guesses for the search algorithm, at the expense of a further increase in CPU requirements (The computation of figures 11a and b already took several weeks on fast workstations because the evaluation of the party utilities are computationally rather demanding).

There is a large number of bifurcation points in the three party model. For small and moderate values of $P'(0)$ we find the same sequence of bifurcations with different voter distributions. For large values of $P'(0)$ numerical difficulties make the interpretation of the bifurcation diagrams doubtful. We have identified a sequence of 8 bifurcations in both examples shown in figure 11 and in a number examples not shown here. These bifurcations are listed in table 2. An enlargement of the first three bifurcations is shown in figure 13.

The first bifurcation is a saddle node bifurcation with the invariant planes $\mathcal{H}_{pq}$. Hence it occurs with multiplicity 3. For slightly larger values of $P'(0)$ we find a degenerate transcritical bifurcation in which 3 unstable fixed points (that are located in the in the planes $\mathcal{H}_{pq}$ and have a single unstable manifold) “collide”

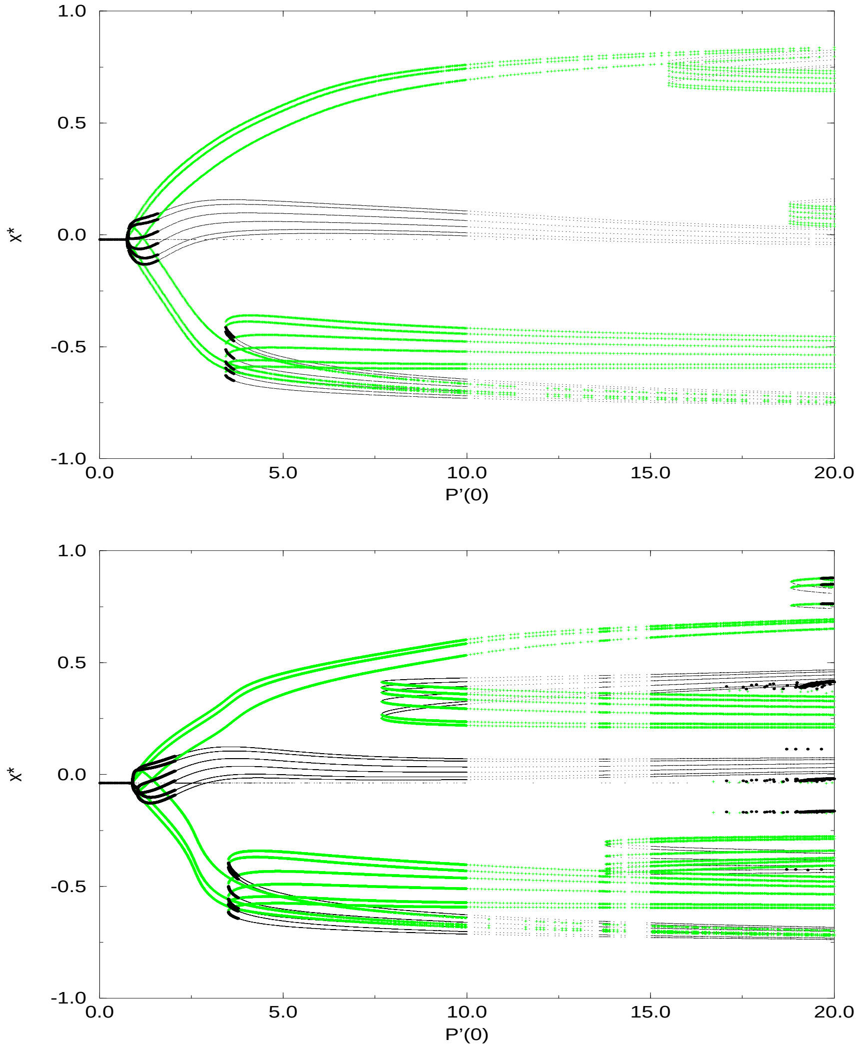

Figure 11: Numerical bifurcation diagrams of three-party one-issue Enelow-Hinich models. Stable fixed points appear as bold black lines. Two types of saddle points were found: Saddles with one unstable direction appear in bold grey, while those with two unstable manifolds correspond to thin black lines (or small dots).
top: 100 voters with ideal points uniformly distributed in $[-1, 1]$.
below: 20 voters with ideal points uniformly distributed in $[-1, 1]$. Beyond $P'(0) \approx 17$ the program seems to produce spurious solutions in regions of the phase space where the gradients become very small.
Note that the two diagrams differ only by the voter distribution. Hence the dynamics is very sensitive to the details of the voter distribution at least for large values of $P'(0)$.

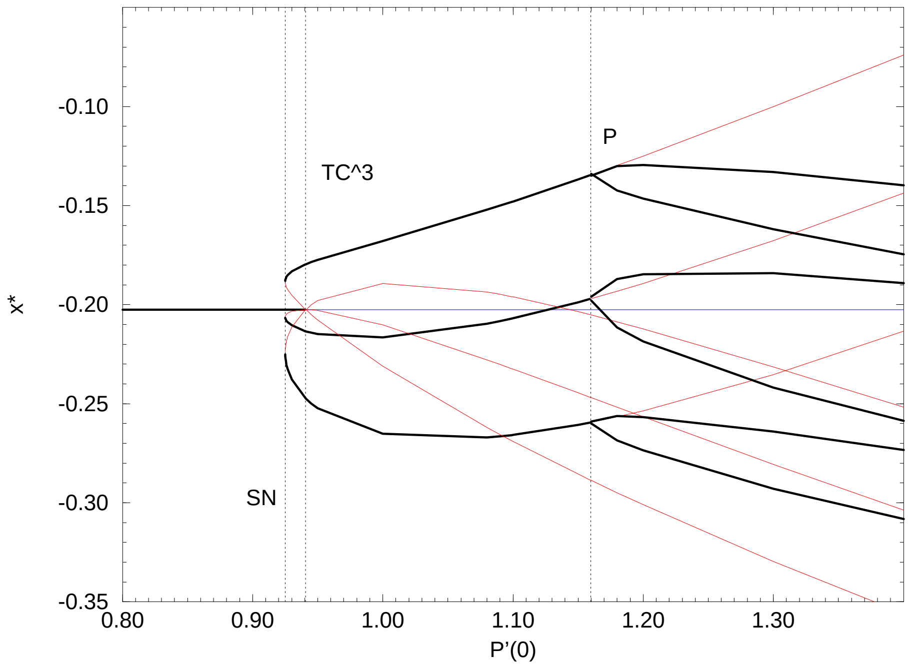

Figure 12: Detail of a bifurcation diagram for the 3-party 1-issue Enelow-Hinich model with 20 voters. Different random numbers were used to generate a voter distribution that is different from the one in figure 11.

Table 3. Bifurcations in the Three-Party Model.

Bifurcation points from the two cases shown in figure 11 are listed here.

Bifurcation Type	Multi-plicity	100 voters fig. 12 top	20 voters fig. 12 bottom
Saddle-Node	3	0.76225	0.905
deg. Transcritical	1	0.7627	0.910
Pitchfork, supercrit.	3	0.7785	1.05
Hopf, supercrit.	6	1.59	2.06
Saddle-Node	6	3.43	3.51
Hopf, supercrit.	6	3.69	3.78
Saddle-Node	6	15.45	7.685
Saddle-Node	6	18.75	14.0

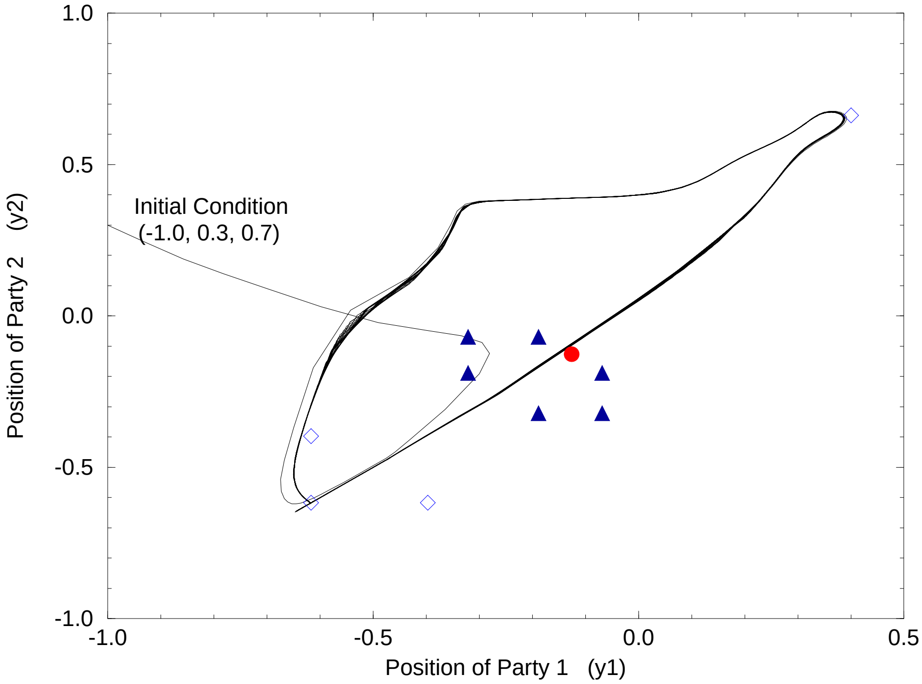

Figure 13: A large limit cycle for $P'(0) = 5$, $V = 20$ voters, and a single issue. Fixed points are indicated by symbols: $\diamond$ indicates saddle points with two equal coordinates (there are 2 sets, any two of them are projected onto the same point in the y^1/y^2 plane). A full circle indicates the mean voter point, $\triangle$ denotes saddle points with S_3-symmetry, i.e., not on any plane $\mathcal{H}_{pq}$.

with the (stable) mean voter fixed point. After the bifurcation the mean voter is unstable as well (with two unstable manifolds).

At higher value of $P'(0)$ the stable equilibria within $\mathcal{H}_{pq}$ undergo pitchfork bifurcations producing 6 stable fixed points and three saddle points, one within each of the three planes $\mathcal{H}_{pq}$. The coordinates of the 6 stable equilibria are related by permutation symmetry. As $P'(0)$ increases further these sinks undergo supercritical Hopf bifurcations, thereby giving rise to stable limit cycles.

At even larger values of the bifurcation parameter we find a sixfold saddle node bifurcation followed by a supercritical Hopf bifurcation. At large values of $P'(0)$ the system does not contain stable fixed points. A number of saddle node bifurcations produce additional saddle points as $P'(0)$ increases. In this regime we also find large stable limit cycles, such as the example in figure 13.

5. Perturbations

5.1. Perturbed Vector Fields

The question of how small perturbations affect the dynamics has been the object to extensive investigations. In the following, the most important facts on perturbation theory are summarized. We follow the presentation in reference [25, chap. 16]. Then we will consider two applications of perturbation theory: First we briefly discuss an extension of the spatial voting model to ideological parties. Then we show that discrete voter distributions may be viewed as perturbations of continuous ones and *vice versa.*

Consider a normed vector space E and let W be an open set in E. Let $V(W)$ be the set of all C^1 vector fields on W. For any vector field $\mathcal{G} \in V(W)$ we define the norm

$$\|\mathcal{G}\|_* \stackrel{\text{def}}{=\!=} \sup_{y \in W} \{ |\mathcal{G}(y)|, \, |\partial \mathcal{G}_i(y)/\partial y_j| \} \tag{5.1}$$

A neighborhood of $\mathcal{F} \in V(W)$ is therefore a subset $A \in V(W)$ containing a set of the form

$$\{ \mathcal{G} \in V(W) \big| \|\mathcal{G} - \mathcal{F}\|_* < \epsilon \} \tag{5.2}$$

We call $\mathcal{G}$ a regular ϵ-perturbation of $\mathcal{F}$ if it satisfies (5.2).

Proposition 1. Let $\mathcal{F} : W \to E$ be a C^1 vector field and $\hat{x} \in W$ an equilibrium of $\dot{x} = \mathcal{F}(x)$ such that the Jacobian of $\mathcal{F}$ at $\hat{x}$ is invertible. Then there exists a neighborhood $U \subset W$ of $\hat{x}$ and a neighborhood $A \subset V(W)$ of $\mathcal{F}$ such that for any $\mathcal{G} \in A$ there is a unique equilibrium $\hat{y} \in U$ of $\dot{y} = \mathcal{G}(y)$. For any $\delta > 0$ we can choose the neighborhood A of $\mathcal{F}$ so that $|\hat{y} - \hat{x}| < \delta$.

Proposition 1 applies to the special case where $\hat{x}$ is a hyperbolic equilibrium, i.e. where all the eigenvalues of the Jacobian at $\hat{x}$ have nonzero real parts and therefore the Jacobian is invertible. The *index* $\operatorname{ind}(\hat{x})$ of $\hat{x}$ is the number of eigenvalues (counting multiplicities) of the Jacobian having negative real parts. If $\dim E = n$, then $\operatorname{ind}(\hat{x}) = n$ means $\hat{x}$ is a sink, while $\operatorname{ind}(\hat{x}) = 0$ means it is a source. If the Jacobian of $\mathcal{F}$ in $\hat{x}$ is hyperbolic, then we may choose A in Proposition 1 such that $\hat{y}$ has the same index as $\hat{x}$. In many cases we encounter a family of vector fields depending on a parameter ξ. We shall write $\mathcal{F}_\xi$ to incorporate this fact.

Proposition 2. Suppose $\mathcal{F}_\xi$ is a family of functions that is continuous in ξ (i.e., the map $q : \mathbb{R} \to V(W) : \xi \mapsto \mathcal{F}_\xi$ is continuous) such that $\mathcal{F}_0 = \mathcal{F}$ and $\hat{y}$ is a hyperbolic fixed point of $\dot{y} = \mathcal{F}(y)$. Then there is $\xi_0 > 0$ and a neighborhood U of $\hat{y}$ such that for all ξ in the interval $0 < \xi < \xi_0$ there is a unique fixed point $\hat{y}(\xi) \in U$ of $\mathcal{F}_\xi$. The parameter ξ_0 can be chosen such that $\hat{y}(\xi)$ is hyperbolic and has the same index as $\hat{y}$. In addition, $\hat{y}(\xi)$ is a continuous function of the perturbation parameter ξ for a given fixed point $\hat{y}(0) = \hat{y}$ of the unperturbed vector field.

A similar result holds for periodic orbits as well [25, sect. 16.2]. If there is a unique closed orbit in the unperturbed system, one will also find closed orbits in the perturbed system if the perturbation is sufficiently small, but the uniqueness is in generally not guaranteed. However, there is one special case where the uniqueness of the closed orbit of the perturbation is certain: If γ is a periodic attractor (resp. repellor) under $\mathcal{F}$ and $\mathcal{G}$ is sufficiently close to $\mathcal{F}$, then there will be a unique closed orbit β in the perturbed system $\mathcal{G}$.

The flow $\dot{y} = \mathcal{F}(y)$ and the flow $\dot{y} = \mathcal{G}(y)$ are topologically equivalent if there is a diffeomorphism between the vector fields (resp. a homeomorphism between the solutions) that carries each trajectory of the original flow onto a trajectory of the perturbation.

If for some ϵ all ϵ-perturbations of $\mathcal{F}$ are topologically equivalent to $\mathcal{F}$, then $\mathcal{F}$ is called structurally stable. On $\mathbb{R}^2$ it is relatively easy to characterize stable flows. For instance, based on the Pontryagin-Andronov theorem [3], Peixoto [48] showed that a flow on a forward invariant disk D^2 is structurally stable if and only if

(i) the equilibria in D^2 are hyperbolic,

(ii) each closed orbit in D^2 is either a periodic attractor or a periodic repellor,

(iii) no trajectory in D^2 goes from saddle to saddle.

Furthermore, Peixoto's theorem guarantees that almost all flows on a forward invariant disk D^2 are structurally stable (on orientable manifolds).

The situation is more complicated in higher dimensions. An important result is the generalization of Peixoto's theorem that states that almost all gradient fields are structurally stable [47]. This result serves as a motivation to introduce perturbation in our model. We may expect that at least the gradient-like flow within the invariant planes $\mathcal{H}_p$, where all party platforms are equal, will be structurally stable for a wide range of parameters, excluding bifurcation points (which, of course, are non-hyperbolic).

5.2. Policy Dependent Platform Utilities

Let us suppose that, in addition to maximizing its share of voters a party also cares about achieving an "ideal" platform, designated by Y^p [44, sect. 4.6]. A party's utility then depends on both the number of expected votes E_p and the distance, $D(y^p, Y^p)$, between its actual and ideal platforms. We thus replace party p's utility by $(1-\xi)E_p(y^1, \dots, y^P) - \xi D(y^p, Y^p)$, where ξ parameterizes the relative importance of the election outcome versus the ideological component. The voting dynamics is then

$$\dot{y}^p = \Psi^p(\vec{y}) - \xi \left[\Psi^p(\vec{y}) + \Gamma^p(y^p)\right], \tag{5.3}$$

where $\Gamma^p(y^p) = \nabla_{y^p} D(y^p, Y^p)$ and $\Psi^p(\vec{y}) = \nabla_{y^p} E_p(y^1, \dots, y^P)$.

Restricting attention to the box $\mathcal{B}$, we may assume that Γ^p and its partial derivatives with respect to ${y_j}^p$ are bounded (as D is assumed to be continuously differentiable on $\mathbb{R}^I \times \mathbb{R}^I$), by some constant M. Without loss of generality (i.e., without changing the dynamics) we will assume that Γ^p and ξ are scaled such that $M = 1$ and ξ measures the size of the ideological payoff component. Thus $\xi[\Psi^p(\vec{y}) + \Gamma^p(y^p)]$ forms a regular perturbation of the vector field $\Psi(\vec{y})$.

Introducing an ideological payoff component will in general break the symmetry of the vector field and, hence, also the symmetry of the mean voter equilibrium. Thus, in general, we will find $\hat{y}^1(\xi) \neq \hat{y}^2(\xi)$ even for small values of $\xi > 0$. Using a perturbation approach that was also extensively applied to selection-mutation equations in theoretical biology [66] it is possible to estimate the location of the perturbed mean voter equilibrium. Taking into account the knowledge that $\Psi(\hat{y}) = 0$, with this approach it is possible to estimate the location of a perturbed mean-voter equilibrium:

$$\hat{y}(\xi) = \hat{y} + \xi \mathbf{J}^{-1}(\hat{y}) \begin{pmatrix} \Gamma^1(\hat{y}^1) \\ \vdots \\ \Gamma^P(\hat{y}^P) \end{pmatrix} + \mathcal{O}(\xi^2)\,. \tag{5.4}$$

Thus, the distance of the equilibrium positions of the P parties is proportional to the parameter ξ.

In the extreme case, when $\xi \approx 1$, and a party's behavior is mostly determined by its ideology, we may consider the voter-dependent term Ψ as a perturbation. In this case each party will settle down to an equilibrium close to Y^p, since in the limiting case where $\Psi = 0$ the vector field is of the form

$$\dot{y}^p = -\nabla_{y^p} D(y^p, Y^p) \tag{5.5}$$

This is a gradient system in which each party converges to a minimum of $D(y^p, Y^p)$ independent of the other parties.

5.3. Continuous Versus Discrete Voter Distributions

In this section we shall see that the dynamics of platform adaptation is essentially the same whether we assume a continuous or a discrete voter distribution. More precisely, we show that to each continuous voter distribution we can construct a discrete one leading to essentially the same behavior, and *vice versa*.

It will be convenient to abbreviate the vector field of a spatial voting model with voter distribution ρ simply by $\mathcal{F}(y, \rho)$. Our ODE is thus $\dot{y} = \mathcal{F}(y, \rho)$. Let

Q be a compact set in $\mathbb{R}^{3I}$. In practice we choose Q such that all, or at least the overwhelming majority of voter ideal positions are located within Q. In the following we shall be concerned only with the platform dynamics within Q in order to be able to apply the perturbation theory outlined in section 5.1.

Theorem 1. Let $Q \subset \mathbb{R}^{3I}$ be compact. Let ρ be a continuous voter distribution that is bounded above by $a_1|x|^{-\alpha}$ and suppose $|\partial_k u(y^p, x)|$, $|\partial_i \partial_j u(y^p, x)|$, and $|\partial_j u(y^p, x) \partial_j u(y^q, x)|$ are bounded by $A + B|x|^\beta$ uniformly for all $\vec{y} \in Q$ and all $x \in \mathbb{R}^I$. If $\alpha > 2(\beta + I)$ the following is true:
For each $\delta > 0$ there is a discrete voter distribution $\hat{\rho}$ with $V = V(\delta)$ voters, such that

$$\|\mathcal{F}(y, \rho) - \mathcal{F}(y, \hat{\rho})\|_* < \delta \,. \tag{5.6}$$

In other words, if $\rho(x)$ decreases sufficiently fast and/or if the absolute values of the derivatives of the voter dissatisfaction function do not increase too fast, then we may replace a continuous voter distribution by a discrete sample. The changes in the dynamics constitute a regular perturbation, the size of which is determined by the sample size, i.e., the number of voters in the discrete case. This result is of particular interest for numerical investigations, since the models with continuous voter distributions lead to integro-differential equations that are not easy to implement in practice.

Proof of Theorem 1.

For the proof of the theorem we shall need a number of technical lemmas. The first step looks rather far-fetched. It will be convenient, however, to restrict most of the work to a compact subset of $\mathbb{R}^I$:

Lemma 1. Let $F(x) > 0$ and suppose ρ is a continuous distribution and let $\Phi(r)$ and $\psi(r)$ be non-negative functions such that

$$\Phi(r) \geq \max_{|x|\leq r} |F(x)| \qquad \psi(r) \geq \max_{|x|\geq r} |\rho(x)| \,.$$

Suppose there is a positive function $R(\delta)$ such that for all $\delta > 0$ the following is true:

(i) There is a probability density $\tilde{\rho}$ such that $|\rho(x) - \tilde{\rho}(x)| < (1 + 1/\mathrm{vol}(1))\delta$ if $|x| \leq R(\delta)$ and $\tilde{\rho}(x) = 0$ for $|x| > R(\delta)$.

(ii) $\mathrm{surf}(1) \displaystyle\int_{R(\delta)}^{\infty} \Phi(r)\psi(r)r^{I-1}dr < \delta$.

Then: $\lim\limits_{\delta\to 0} \delta R(\delta)^I \Phi(R(\delta)) = 0$ implies $\lim\limits_{\delta\to 0} \displaystyle\int_{\mathbb{R}^I} F(x)(\rho(x) - \tilde{\rho}(x))dx = 0$.

Proof. $\Phi(R(\delta))$ is an upper bound for $|F(x)|$ on the compact sphere with radius $R(\delta))$ and $\Phi(r)\psi(r)$ is a radial symmetrical upper bound for $|F(x)|\rho(x)$. Using polar coordinates we obtain

$$\begin{aligned}
\left| \int_{\mathbb{R}^I} F(x)[\rho(x) - \tilde{\rho}(x)]dx \right| &\leq \int_{\mathbb{R}^I} \big|F(x)[\rho(x) - \tilde{\rho}(x)]\big|dx \\
&\leq \int_{|x|\leq R(\delta)} |F(x)|(1 + 1/\mathrm{vol}(1))\delta dx + \int_{|x|\geq R(\delta)} |F(x)|\rho(x)dx \\
&\leq (1 + 1/\mathrm{vol}(1))\Phi(R(\delta))\delta \mathrm{vol}(1)R(\delta)^I + \mathrm{surf}(1)\int_{R(\delta)}^{\infty} \Phi(r)\psi(r)r^{I-1}dr
\end{aligned}$$

where vol(1) is the volume and surf(1) is the surface of the n-dimensional hypersphere with radius 1, respectively. Since the second part of the above sum is smaller than δ by assumption the proof is complete. ∎

Next we show that we can satisfy the requirement of lemma 1 with reasonable upper bounds $\Phi(r)$ and $\psi(r)$:

Lemma 2. Set $\Phi(r) \stackrel{\text{def}}{=} c_1 + c_2 r^\beta$ and $\psi(r) \stackrel{\text{def}}{=} a_1 r^{-\alpha}$ with $c_1, c_2, a_1, \alpha, \beta > 0$ and $\alpha > \beta + I$. Furthermore let

$$R(\delta) \stackrel{\text{def}}{=} (\delta/B)^{1/(I+\beta-\alpha)} \quad \text{with} \quad B = \frac{a_1(c_1 + c_1)}{\alpha - I - \beta} \quad \text{for} \quad \delta < B$$

and $R(\delta) \stackrel{\text{def}}{=} 1$ for $\delta \geq B$. Then $\displaystyle\int_{R(\delta)}^\infty \Phi(r)\psi(r)r^{I-1}dr \leq \delta$.

Proof. <u>Case (i) $\delta < B$</u>: We find
$\int_{R(\delta)}^\infty \Phi(r)\psi(r)r^{I-1}dr = \frac{c_1 a_1}{\alpha - I}\left(\frac{\delta}{B}\right)^{\frac{\alpha-I}{\alpha-I-\beta}} + \frac{c_2 a_1}{\alpha-(\beta+I)}\,\frac{\delta}{B} < \frac{a_1(c_1+c_2)}{\alpha-(\beta+I)}\,\frac{\delta}{B} = \delta$
since $\delta/B < 1$ and $(\alpha - I)/(\alpha - I - \beta) > 1$.
<u>Case (ii) $\delta > B$</u>: Then $R(\delta) = 1$ and the integral simplifies to $c_1 a_1/(\alpha - I) + c_2 a_1/(\alpha - \beta - I) \leq B < \delta$. ∎

Lemma 3. Let Φ, ψ as in lemma 2 and suppose $\alpha > 2(I + \beta)$. Then

$$\lim_{\delta \to 0} \delta \Phi(R(\delta)) R(\delta)^I = 0\,.$$

Proof. Of course it is sufficient to consider $\delta < B$. In this case we find

$$\delta\Phi(R(\delta))R(\delta)^I = c_1\left(\frac{\delta}{B}\right)^{\frac{\alpha-2I-\beta}{\alpha-I-\beta}} + c_2\left(\frac{\delta}{B}\right)^{\frac{\alpha-2(I+\beta)}{\alpha-I-\beta}} < (c_1 + c_2)\left(\frac{\delta}{B}\right)^{\frac{\alpha-2(I+\beta)}{\alpha-(I+\beta)}},$$

which converges to 0 since the exponent of δ is positive. ∎

Next we construct a step-wise continuous distribution $\tilde\rho$ satisfying the assumption of lemma 1. To this end we divide $\mathcal{K}(\delta)$ into $N = N(\delta)$ small compact sets $\mathcal{K}_i$ with non-zero measure that intersect each other at most with their boundaries. We define:

$$\tilde\rho(x) \stackrel{\text{def}}{=} \frac{1}{\operatorname{vol}(\mathcal{K}_i)} \int_{\mathcal{K}_i} \rho(x)dx + \frac{1}{\operatorname{vol}(\mathcal{K}(\delta))} \int_{\mathbb{R}^I \setminus \mathcal{K}(\delta)} \rho(x)dx$$

in the interior of $\mathcal{K}_i$. The mean value theorem implies that there is an $x_i \in \mathcal{K}_i$ such that

$$\tilde{\rho}(x) = \rho(x_i) + L(\delta)$$

Furthermore we set $\tilde{\rho}(x) = 0$ on $\mathbb{R}^I \setminus \mathcal{K}(\delta)$ and adopt the convention that $\tilde{\rho}(x)$ is the (arithmetic) mean of $\rho(x_i)$ for all $\mathcal{K}_i$ that contain x. It is easy to verify that $\tilde{\rho}(x)$ is indeed a probability density:

$$\begin{aligned}\int_{\mathbb{R}^I} \tilde{\rho}(x)dx &= \int_{\mathcal{K}(\delta)} \tilde{\rho}(x)dx = \sum_i \mathrm{vol}(\mathcal{K}_i)\rho(x_i) + \mathrm{vol}(\mathcal{K}(\delta))L(\delta) \\ &= \int_{\mathcal{K}(\delta)} \rho(x)dx + \int_{\mathbb{R}^I \setminus \mathcal{K}(\delta)} \rho(x)dx = \int_{\mathbb{R}^I} \rho(x)dx = 1\,.\end{aligned}$$

As in Lemma 1 we assume $\int_{\mathbb{R}^I \setminus \mathcal{K}(\delta)} \rho(x)dx < \delta$, and the radius of $\mathcal{K}(\delta)$ is at least 1. Therefore $0 < L(\delta) \leq \delta/\mathrm{vol}(1)$. Since ρ is continuous and $\mathcal{K}(\delta)$ is compact we may choose the covering $\{\mathcal{K}_i\}$ such that $|\rho(x) - \rho(x_i)| < \delta$ on $\mathcal{K}_i$ for all i, and hence we have everywhere:

$$|\rho(x) - \tilde{\rho}(x)| < \delta + L(\delta) \leq \delta(1 + 1/\mathrm{vol}(1))\,.$$

Thus there is indeed a probability density satisfying the conditions posed in lemma 1.

In the next step, we construct a discrete voter probability distribution

$$\hat{\rho}(x) = \frac{1}{V}\sum_{i=1}^{N} p(x_i)\delta(x - x_i),$$

where $p(x_i)$ is the number of voters with ideal point x_i. Therefore, the fraction of voters having x_i as ideal point x_i is

$$\int_{\mathcal{K}_i} \hat{\rho}(x)dx = p(x_i)/V\,,$$

We define

$$p(x_i) = \lfloor V \,\mathrm{vol}(\mathcal{K}_i)\tilde{\rho}(x)\rfloor.$$

Then $|p(x_i) - V\mathrm{vol}(\mathcal{K}_i)\tilde{\rho}(x)| < 1$, which is equivalent to

$$|p(x_i)/V - \mathrm{vol}(\mathcal{K}_i)\tilde{\rho}(x)| = \left|\int_{\mathcal{K}_i} \hat{\rho}(x)dx - \int_{\mathcal{K}_i} \tilde{\rho}(x)dx\right| < 1/V\,.$$

It remains to link the function $F(\,.\,)$ in the above integrals to the spatial voting model. Consider the general case with P parties. The response function for each party p can be written in the following way:

$$\begin{aligned}&\mathcal{P}(d_{p,1}, d_{p,2}, \ldots, d_{p,P}, d_{1,2}, \ldots, d_{1,P}, \ldots, d_{P-1,P})\\ &= T_p(u(y^1, x), u(y^2, x), \ldots, u(y^P, x))\end{aligned}$$

For the j-th component of the differential equation for platform p we have

$$F(x) = \frac{\partial}{\partial y_j^p}(T_p(u(y^1,x), u(y^2,x), \ldots, u(y^P,x))) = \partial_p T_p \partial_j u(y^p, x)$$

Since T_p is a sum of continuously differentiable multi-dimensional sigmoidal functions, $\partial_p T_p$ is bounded by assumption. For simplicity we write $u_p = u(y^p, x)$ in the following. In case of the partial derivatives of the vector field we have

$$F(x) = \frac{\partial}{\partial y_k^q}(\partial_p T_p \partial_j u_p) = \partial_q\partial_p T_p \partial_k u_q \partial_j u_p + \partial_p T_p \partial_k{}^q \partial_j{}^p \delta_{p,q} u_p.$$

Again, $\partial_q\partial_p T_p$ is bounded by assumption. The behavior of $F(\,.\,)$ therefore depends only on the form of the voter dissatisfaction function (and its derivatives).

In summary, we have

$$|F(x)| \leq C \max_{k,j,p,q,y\in Q} \{|\partial_j u(y^p,x)|\,, |\partial_k\partial_j u(y^p,x)|\,, |\partial_j u(y^p,x)\partial_k u(y^q,x)|\}$$

with some constant $C > 0$, and theorem 1 follows. ∎

Corollary 1. For the Enelow-Hinich model the conclusion of theorem holds if $\rho(x) \le a_1/|x|^{2(I+2)}$. For the Gaussian voter dissatisfaction functions in the example (2.16) is suffices to require $\rho(x) \le a_1/|x|^{2I}$.

Proof. One easily verifies, that the Enelow-Hinich model satisfies $\partial_k u(y^p, x) \le A + B|x|$ while the second derivatives are constant. Thus $\beta = 2$. Both the first and second derivatives are bounded in the example (2.16), i.e., $\beta = 0$. ∎

Corollary 2. The conclusion of the theorem is true if the first and second derivatives of u are bounded by a polynomial in $|x|$ and ρ decreases exponentially with $|x|$. It is trivially satisfied if ρ has compact support (as in the case of the uniform distribution) for arbitrary C^2 functions u.

Remark. The converse is rather trivial. Given a discrete voter distribution of the form

$$\rho(x) = \frac{1}{V} \sum_{v=1}^{V} \delta(x - x_v)\,, \tag{5.7}$$

we may replace the δ-distribution by a continuously differentiable approximation such that the difference between the vector fields is arbitrarily small. For instance, if $|u(x,y)| \le f(y) e^{K|x|^2}$ for an arbitrary function f depending only on y, we may use a Gaussian distribution with sufficiently small variance to approximate the δ-distribution.

6. Discrete Time Dynamics

6.1. Discrete versus Continuous Time Models

Elections usually are held at regular time intervals. One may therefore view the platform dynamics as a discrete process mapping one election outcome onto the next one. This amounts to dropping the assumption that opinion polls are conducted continuously, focusing instead on the platform changes due to the pre-election campaign. Instead of a continuous time model of the form $\dot{y} = f(y)$ we have to consider its discrete time analogue $y' = y + \tau f(y)$ where the time scale τ measures the platform mobility from one election to the next.

In the following, let f be a continuously differentiable vector field on $\mathbb{R}^n$ and τ a positive constant.

Lemma 1. $\dot{y} = f(y)$ and $y' = y + \tau f(y)$ have the same fixed points.

Proof. $\dot{y} = 0 \iff f(y) = 0$ and $y' = y \iff f(y) = 0$. ∎

Lemma 2. Let $\mathbf{J}$ denote the Jacobian of f, i.e. $\mathbf{J}_{kl}(y) = \partial f_k / \partial y_l$. Then the Jacobian of the discrete time system is $\tilde{\mathbf{J}} = \mathbf{I} + \tau \mathbf{J}$, where $\mathbf{I}$ denotes the identity matrix.

Proof. In componentwise notation, the entries of $\tilde{\mathbf{J}}$ read $\frac{\partial {y_k}'}{\partial y_l} = \delta_{kl} + \tau \frac{\partial f_k(y)}{\partial y_l}$, i.e. $\tilde{\mathbf{J}}_{kl} = \delta_{kl} + \tau \mathbf{J}_{kl}$. ∎

Lemma 3. If λ is an eigenvalue of $\mathbf{J}$ then $\tilde{\lambda} \stackrel{\text{def}}{=} 1 + \tau\lambda$ is an eigenvalue of $\tilde{\mathbf{J}}$ that belongs to the same eigenvector.

Proof. Let x be an eigenvector of $\mathbf{J}$ for the eigenvalue λ. Then
$\tilde{\mathbf{J}}x = \mathbf{I}x + \tau\lambda x = (1 + \tau\lambda)x = \tilde{\lambda}x.$ ∎

We know that for a stable fixed point in the continuous time system, all eigenvalues of $\mathbf{J}$ are negative. A fixed point is stable in the discrete time model if $|\tilde{\lambda}| < 1$ for all eigenvalues.

Lemma 4. If a fixed point is stable in the discrete time model it is also stable in the continuous time model. If a fixed point is unstable in the continuous time model it is also unstable in the discrete time model.

Proof. For the sake of generality, we will write down all the eigenvalues with complex entries, i.e.

$$\tilde{\lambda}_1 \stackrel{\text{def}}{=} \Re(\tilde{\lambda}) \qquad \tilde{\lambda}_2 \stackrel{\text{def}}{=} \Im(\tilde{\lambda}) \qquad \lambda_1 \stackrel{\text{def}}{=} \Re(\lambda) \qquad \lambda_2 \stackrel{\text{def}}{=} \Im(\lambda),.$$

Then the relations between the real parts of the eigenvectors are

$$\tilde{\lambda}_1 = 1 + \tau\lambda_1 \qquad \tilde{\lambda}_2 = \tau\lambda_2 \qquad \lambda_1 = (\tilde{\lambda}_1 - 1)/\tau \qquad \lambda_2 = \tilde{\lambda}_2/\tau,.$$

If the fixed point is stable in the discrete time model, then

$$|\tilde{\lambda}|^2 = 1 + 2\tau\lambda_1 + (\tau)^2|\lambda|^2 < 1$$

for all eigenvalues which is equivalent to $\lambda_1 < -\tau|\lambda|^2/2 < 0$ for all eigenvalues. If the fixed point is unstable in the continuous time model, i.e., $\lambda_{\max;1} > 0$, then $\tilde{\lambda}_{\max;1} > 1$ follows, and therefore $|\tilde{\lambda}_{\max}| = \sqrt{\tilde{\lambda}^2_{\max;1} + \tilde{\lambda}^2_{max;2}} > 1.$ ∎

For our purposes it will be useful to consider the case of real eigenvalues (since the Jacobian at the trivial fixed point of the voting model is symmetric).

Lemma 5. Suppose $\mathbf{J}$ has only real eigenvalues. Then a stable fixed point in the continuous time model is also stable in the discrete time model provided $\tau < 2/r(\mathbf{J})$, where $r(\mathbf{J})$ is the spectral radius of $\mathbf{J}$.

Proof. First we observe that $r(\mathbf{J}) = -\lambda_{\min}$ since all eigenvalues of $\mathbf{J}$ are non-positive. $-1 < 1 + \tau\lambda_{\min}$ and $1 + \tau\lambda_{\max} < 1$ have to be fulfilled for a stable fixed point in the discrete time version. The second condition is fulfilled for every value of τ. The first condition holds iff $\tau < -2/\lambda_{\min} = 2/r(\mathbf{J})$. ∎

Remark. A bifurcation occurs at $\tau^* = -2/\lambda_{\min} = 2/r(\mathbf{J})$.

6.2. Discrete-Time Enelow-Hinich Models

6.2.1. Two-Party Model

Lemma 1. In the discrete time two party model a bifurcation occurs at the trivial fixed point for $\tau^* = [2\mathcal{P}'(0)\max_i \bar{s}_i]^{-1}$.

Proof. The eigenvalues of the Jacobian at the trivial fixed point are

$$\lambda_i = -4\mathcal{P}'(0)\bar{s}_i,$$

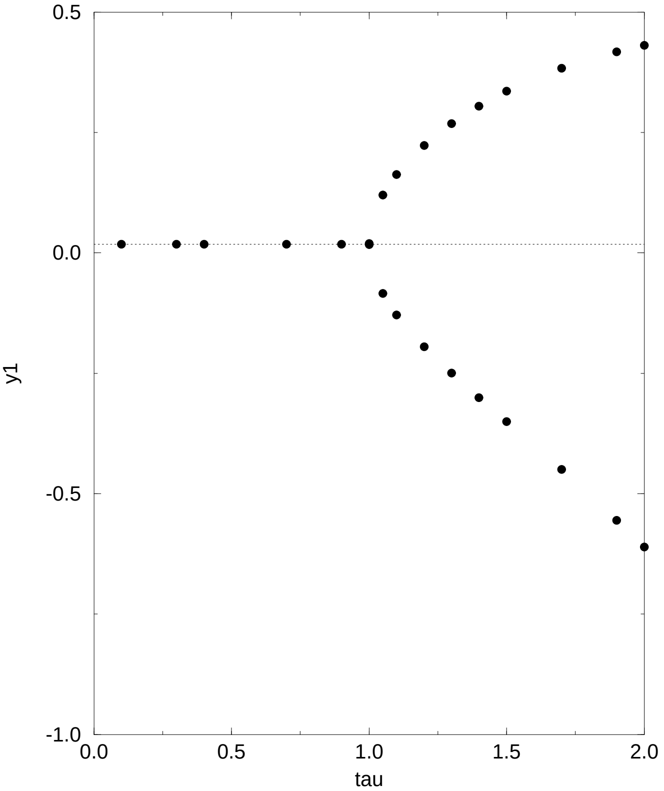

Figure 14: Bifurcation in the discrete time 2-party model. $P'(0) = 1/2$. Back dots indicate the stable equilibria for different values of τ. The dotted line marks the mean voter equilibrium.

where $i = 1, \ldots, I$. The fixed point is stable in the discrete time system iff $|\tilde{\lambda}_i| = |1 + \lambda_i| < 1$ for all i. This inequality can be written as

$$-1 < 1 + \tau \min_i \lambda_i \leq 1 + \tau \max_i \lambda_i < 1.$$

The second part of the above inequality is of course fulfilled since all eigenvalues λ are negative. The first part of the inequality is fulfilled only if

$$\tau < \frac{-2}{\min_i \lambda_i}.$$

Since $\min_i(-4\mathcal{P}'(0)\bar{s}_i) = -4\mathcal{P}'(0)\max_i \bar{s}_i$, the first part of the inequality is fulfilled only if $\tau < [2\mathcal{P}'(0)\max_i \bar{s}_i]^{-1}$. ∎

Remark. If we assume equal strength of voters' interests in all the issues, i.e. $\bar{s}_i = \bar{s}$ for all i, then $\tau^* = \dfrac{1}{2\mathcal{P}'(0)\bar{s}}$.

6.2.2. 3-Party Enelow-Hinich Model with Complete Participation

Lemma 2. A bifurcation at the trivial fixed point in the discrete time system occurs at $\tau^* = \frac{3}{4\mathcal{P}_0{}'(0)\bar{s}}$.

Proof. Let us assume that the trivial fixed point is stable in the continuous time model and that $\bar{s}_i = \bar{s}$ for all i. The Jacobian at $(y, y, y) \in \mathcal{H}_3$ can be written in the form

$$\mathbf{J}(y,y,y) = (\hat{\mathcal{P}} + \mathcal{P}'')\mathbf{C}(y) \otimes \begin{pmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{pmatrix} + 2\mathcal{P}'\mathbf{H}(y) \otimes \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Since the 3×3 matrix occurring in the first term has the simple eigenvalue 0 and the eigenvalue 3 with algebraic multiplicity 2, the largest eigenvalue of $\mathbf{J}$ is given by

$$\begin{aligned} \lambda_{\max} &= \frac{4P_0{}'(0)}{3} \cdot -2\bar{s} \cdot 1 + \frac{8P_0{}'(0)^2}{9} \cdot 4(\bar{s})^2\rho \cdot 3 \\ &= \frac{8sP_0{}'(0)}{3}\left(4s\rho P_0{}'(0) - 1\right). \end{aligned}$$

Since the trivial fixed point is stable in the continuous time model, $1 + \lambda_{\max} < 1$ holds. The minimal eigenvalue is

$$\lambda_{\min} = \frac{4P_0{}'(0)}{3} \cdot (-2\bar{s}) \cdot 1 \,.$$

The critical value of τ^* in the discrete time model is therefore $\tau^* = \dfrac{-2}{\lambda_{\min}} = \dfrac{3}{4\mathcal{P}_0{}'(0)\bar{s}}$. ∎

As a consequence of the above considerations, we get the following result for the discrete time three party model with incomplete participation:

Lemma 3. The bifurcation at the trivial fixed point occurs for $\tau^* = \dfrac{1}{\mathcal{P}_0{}'(0)\bar{s}}$.

7. Multi-Party Systems

7.1. Generalization to P parties

In this section, we will consider an election with P parties, the respective platforms being denoted by $y^1, \ldots, y^P$. Voter v's utility of party p's platform will be denoted by $u_v(y^p)$, as in previous sections. The utility differences for voter v will be denoted by

$$d_{pq}^v \stackrel{\text{def}}{=} u_v(y^p) - u_v(y^q) \tag{7.1}$$

and hence $d_{pq}^v = -d_{qp}^v$.

The probability function $\mathcal{P} : \mathbb{R}^{\binom{P}{2}} \mapsto [0,1]$ depends on all $\binom{P}{2}$ pairwise voter utility differences. We use the notation

$$\text{Prob}[v \text{ votes for } 1] = \mathcal{P}_1(\vec{d}) = \mathcal{P}(d_{12}^v, \ldots, d_{1P}^v; d_{23}^v, \ldots, d_{P-1P}^v) \tag{7.2}$$

Note that the first $P-1$ arguments have a different influence than the remaining $(P-1)(P-2)/2$ arguments. We require that the multidimensional sigmoidal function $\mathcal{P}$ has the following properties:

(i) $\partial_q \mathcal{P}_p(\vec{d}) = \partial P_p / \partial d_{pq} > 0$ for the first $P-1$ arguments. The probability of voting for party p thus increases with increasing values of d_{pq}, $q \neq p$.

(ii) We require that the voting probabilities are independent of the party labelings. Thus, for any permutation π of $(1, \ldots, P)$ fixing 1, i.e., $\pi(1) = 1$

we have

$$\begin{aligned}&\mathcal{P}(d_{12}, d_{13}, \ldots, d_{1P}; d_{23}, d_{24}, \ldots, d_{2P}, d_{34}, \ldots, d_{P-1,P}) = \\ &\quad \mathcal{P}(d_{\pi(1)\pi(2)}, d_{\pi(1)\pi(3)}, \ldots, d_{\pi(1)\pi(P)}; d_{\pi(2)\pi(3)}, d_{\pi(2)\pi(4)}, \ldots \\ &\qquad \ldots, d_{\pi(2)\pi(P)}, d_{\pi(3)\pi(4)}, \ldots, d_{\pi(P-1),\pi(P)})\end{aligned} \tag{7.3}$$

In the same way $\mathcal{P}_p(\vec{d}) = \mathcal{P}_1(\pi(\vec{d}))$ is constructed for any permutation π mapping 1 to p, where $p \in \{1, 2, \ldots, P\}$. Condition (i) implies that the orbits are bounded within a box $\mathcal{B}^P$, here $\mathcal{B}$ is the I-dimensional box spanned by the extreme voter positions. The proof is analogous to the two and three party models.

The permutation symmetry (ii) implies that we need to compute only the partial derivatives

$$\frac{\partial \mathcal{P}}{\partial y_k^1} \qquad \frac{\partial^2 \mathcal{P}}{\partial y_k^1 \partial y_j^1} \qquad \text{and} \qquad \frac{\partial^2 \mathcal{P}}{\partial y_k^1 \partial y_j^2}\,.$$

All other partial derivatives can be obtained using a permutation π of the indices. In particular, the partial derivatives at the mean voter position $\vec{d} = \vec{o}$ are determined by only three parameters:

$$\mathcal{P}' = \partial_1 \mathcal{P}(\vec{o})\,, \qquad \mathcal{P}'' = \partial_1 \partial_1 \mathcal{P}(\vec{o})\,, \qquad \text{and} \quad \hat{\mathcal{P}} = \partial_1 \partial_2 \mathcal{P}(\vec{o})\,. \tag{7.4}$$

Lemma 1. $\partial_P \mathcal{P}(\vec{o}) = 0$

Proof. Let $\tau : \tau(2) = 3$ be the transposition $(2, 3)$. Then
$\mathcal{P}(d_{12}, d_{13}, \ldots, d_{1P}, d_{23}, d_{24}, \ldots, d_{34}, \ldots, d_{P-1P}) =$
$\mathcal{P}(d_{13}, d_{12}, \ldots, d_{1P}, d_{32}, d_{24}, \ldots, d_{34}, \ldots, d_{P-1P})$, where d_{23} and d_{32}, resp., are at position P. Then $\partial_P \mathcal{P}(d_{12}, d_{13}, \ldots, d_{1P}, d_{23}, \ldots, d_{P-1P}) =$
$-\partial_P \mathcal{P}(d_{13}, d_{12} \ldots, d_{1P}, d_{32} \ldots, d_{P-1P})$. Therefore, $\partial_P \mathcal{P}(\vec{o}) = 0$. ∎

Lemma 2. $\sum_{p=1}^{P-1} \partial_p \partial_P \mathcal{P}(\vec{o}) = 0.$

Proof. Consider $\tau : \tau(2) = 3$. Then $\mathcal{P}(d_{12}, d_{13}, d_{14}, \ldots, d_{1P}, d_{23}, \ldots, d_{P-1,P})$ $= \mathcal{P}(d_{13}, d_{12}, d_{14}, \ldots, d_{1P}, d_{32}, \ldots, d_{P-1,P})$, and d_{23}, resp., d_{32} are on position P. Then

$$
\begin{aligned}
&\partial_P \sum_{p=1}^{P-1} \partial_p \mathcal{P}(d_{12}, d_{13}, d_{14}, \ldots, d_{1P}, d_{23}, \ldots, d_{P-1P}) = \\
&\quad \frac{\partial}{\partial d_{23}} \sum_{p=1}^{P-1} \partial_p \mathcal{P}(d_{12}, d_{13}, d_{14}, \ldots, d_{1P}, d_{23}, \ldots, d_{P-1,P}) = \\
&\quad \frac{\partial}{\partial d_{23}} \sum_{p=1}^{P-1} \partial_p \mathcal{P}(d_{13}, d_{12}, d_{14}, \ldots, d_{1P}, d_{32}, \ldots, d_{P-1,P}) = \\
&\quad - \partial_P \sum_{p=1}^{P-1} \partial_p \mathcal{P}(d_{13}, d_{12}, d_{14}, \ldots, d_{1P}, d_{32}, \ldots, d_{P-1,P})
\end{aligned}
$$

The lemma follows immediately. Clearly, the result holds for every position $q \geq P$ as well. ∎

The dynamics of the multi-party system with discrete voter distribution can thus be written in the form

$$
\dot{y_k}^p = \frac{1}{V} \sum_v \sum_{q \neq p}^{P-1} \partial_q \mathcal{P}_p(\vec{d}) \partial_k u(y^p, x_v) \,. \tag{7.5}
$$

The I-dimensional surface $\mathcal{H}_P \stackrel{\text{def}}{=} \{y | y^1 = y^2 =, \ldots, = y^P\}$ is of course invariant. A fixed point within $\mathcal{H}_P$ will be called *trivial.* Note that by property (ii) in section 7.1 every permutation of the party indices leaves equ.(7.5) unchanged.

7.2. Explicit Example for $\mathcal{P}$

Consider the multidimensional response function

$$\mathcal{P}(d_{12}^v, d_{13}^v, \ldots, d_{1P}^v, d_{23}^v, d_{24}^v, \ldots, d_{2P}^v, \ldots, d_{P-1,P}^v) \stackrel{\text{def}}{=\!=} \frac{1}{A_v} \prod_{p=2}^{P} P_0(d_{1p}^v), \tag{7.6}$$

where the normalization factor is

$$A_v = \sum_{i=1}^{P} \prod_{p \neq i}^{P} P_0(d_{ip}^v) \tag{7.7}$$

Lemma 3. The above response function fulfills the properties (i) and (ii).

Proof.

$$\frac{\partial \mathcal{P}}{\partial d_{12}^v} = \frac{P_0'(d_{12}^v)}{{A_v}^2} \Big[P_0(d_{13}^v) P_0(d_{14}^v) \ldots P_0(d_{1P}^v) [A_v - P_0(d_{12}^v) P_0(d_{13}^v) \ldots P_0(d_{1P}^v)]$$

$$+ P_0(d_{12}^v) P_0(d_{13}^v) \ldots . P_0(d_v 1P) P_0(d_{23}^v) P_0(d_{24}^v) \ldots . P_0(d_v 2P) \Big] > 0$$

Furthermore,

$$\begin{aligned} &\mathcal{P}(d_{\pi(1)\pi(2)}^v, d_{\pi(1)\pi(3)}^v, \ldots, d_{\pi(1)\pi(P)}^v; \\ &\quad \ldots, d_{\pi(2)\pi(3)}^v, d_{\pi(2)\pi(4)}^v, \ldots, d_{\pi(2)\pi(P)}^v d_{\pi(3)\Pi(4)}^v, \ldots, d_{\pi(P-1)\pi(P)}^v) \\ &\quad = \frac{1}{A_v} P_0(d_{\pi(1)\pi(2)}^v) P_0(d_{\pi(1)\pi(3)}^v) \ldots P_0(d_{\pi(1)\pi(P)}^v) \end{aligned}$$

∎

A short calculation yields

$$\mathcal{P}' = \frac{2}{P} P_0'(0), \qquad \mathcal{P}'' = 0, \qquad \hat{\mathcal{P}} = \frac{4(P-1)}{P^2} P_0'(0)^2. \tag{7.8}$$

7.3. Bifurcations at the Trivial Equilibrium

The Jacobian at a point in $\mathcal{H}_P$ has the following entries:

$$\frac{\partial \dot{y_k}^1}{\partial {y_j}^1} = \frac{1}{V}\sum_v \Bigg\{ \Big[\sum_{p=1}^{P-1} {\partial_p}^2 \mathcal{P}_1(\vec{\sigma}) + \sum_{\substack{p,q=1\\ p\neq q}}^{P-1} \partial_p \partial_q \mathcal{P}_1(\vec{\sigma}) \Big] \partial_j u(y,x_v) \partial_k u(y,x_v) + \sum_{p=1}^{P-1} \partial_p \mathcal{P}_1(\vec{\sigma}) \partial_j \partial_k u(y,x_v) \Bigg\}$$

$$\frac{\partial \dot{y_k}^1}{\partial {y_j}^2} = \frac{1}{V}\sum_v \Big[-\partial_1 \sum_{p=1}^{P-1} \partial_p \mathcal{P}_1(\vec{\sigma}) + \sum_{p=1}^{P-1}\sum_{i=P}^{2P-3} \partial_p \partial_i \mathcal{P}_1(\vec{\sigma}) \Big] \partial_j u(y,x_v) \partial_k u(y,x_v)$$

Since for any position $q \geq P$ the expression $\partial_q \sum_{p=1}^{P-1} \partial_p \mathcal{P}_1(\vec{\sigma})$ vanishes, we obtain the following Jacobian at a point in $\mathcal{H}_P$:

$$\begin{aligned}\mathbf{J} = \Big[\mathcal{P}'' + (P-2)\hat{\mathcal{P}}\Big]\, \mathbf{C}(y) \otimes &\begin{pmatrix} P-1 & -1 & -1 & \dots & -1 \\ -1 & P-1 & -1 & \dots & -1 \\ \vdots & \vdots & \vdots & & \vdots \\ -1 & -1 & -1 & \dots & P-1 \end{pmatrix} \\ + \left[(P-1)\mathcal{P}'\right] \mathbf{H}(y) \otimes &\begin{pmatrix} 1 & 0 & 0 & \dots & 0 \\ 0 & 1 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \dots & 1 \end{pmatrix}\end{aligned} \tag{7.9}$$

Note that this expression indeed generalizes equ.(2.11) for $P = 2$ and equ.(4.10) for $P = 3$. The first part of the Jacobian can be written as $P\mathbf{I} - \mathbf{L}$, where $\mathbf{I}$ is the identity matrix and $\mathbf{L}$ is the matrix which has 1 in every entry. It is easy to see that the eigenvalues of $\mathbf{L}$ are P (with multiplicity 1) and 0 (with multiplicity $P-1$). Whenever $\vec{x}$ is an eigenvector of $\mathbf{L}$ with eigenvalue λ, then $P - \lambda$ is an eigenvalue of $P\mathbf{I} - \mathbf{L}$ (counting multiplicities). Therefore, the eigenvalues of $P\mathbf{I} + (-1)\mathbf{L}$ are P (with multiplicity $P-1$) and 0 (with multiplicity 1).

7.4. P-Party Enelow-Hinich Model

From the above calculations, we get the following Jacobian for a model with P parties:

$$\begin{aligned}\mathbf{J} = \frac{4(P-1)(P-2)}{P^2} P_0'(0)^2 \mathbf{C}(y) \otimes &\begin{pmatrix} P-1 & -1 & \dots & -1 \\ -1 & P-1 & \dots & -1 \\ \vdots & \vdots & & \vdots \\ -1 & -1 & \dots & P-1 \end{pmatrix} \\ + \frac{2(P-1)}{P} P_0'(0) \mathbf{H}(y) \otimes &\begin{pmatrix} 1 & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \dots & 1 \end{pmatrix}\end{aligned} \tag{7.10}$$

As in the three-party model, the matrix $\mathbf{C}$ at the trivial fixed point in the Enelow-Hinich model with constant strength $s_{vi} = s$ for all v and i reduces to

$$\mathbf{C}_{ij} = 4s^2 \mathbf{V}_{ij}, \tag{7.11}$$

where $\mathbf{V}$ is the co-variance matrix of the voter distribution in issue space. Let ρ denote the spectral radius (maximum eigenvalue) of the covariance matrix

$$\mathbf{V}_{ij} \stackrel{\text{def}}{=} \frac{1}{V} \sum_v (x_{vi} - \bar{x}_i)(x_{vj} - \bar{x}_j) \tag{7.12}$$

of the voter distribution in issue space. The Hessian $\mathbf{H}$ is diagonal. We have explicitly

$$\mathbf{H} = -2s\mathbf{I}. \tag{7.13}$$

The largest eigenvalue of the Jacobian is therefore

$$\begin{aligned}\lambda_{\max} = &\frac{4P(P-1)(P-2)4s^2\rho}{P^2} P_0'(0)^2 - \frac{4s(P-1)}{P} P_0'(0) \\ = &\frac{4s(P-1)}{P} P_0'(0)\big(4(P-2)s\rho P_0'(0) - 1\big).\end{aligned} \tag{7.14}$$

For $P = 2$ we have $\lambda_{\max} < 0$, and for $P = 3$ we have

$$\lambda_{\max} = \frac{8sP_0'(0)}{3}\big(4s\rho P_0'(0) - 1\big), \tag{7.15}$$

recovering equ.(4.20). The average voter equilibrium becomes unstable when $P_0'(0)$ exceeds the critical value $p^* = \frac{1}{4(P-2)s\rho}$. In the 1-issue case, for $P > 2$, this reduces to

$$p^* = \frac{1}{4(P-2)s\mathrm{var}(x)}, \tag{7.16}$$

where $\mathrm{var}(x)$ is the variance of the one-dimensional voter distribution.

With the response function

$$\mathcal{P}(d_{12}^v, d_{13}^v, \ldots, d_{1P}^v, d_{23}^v, d_{24}^v, \ldots, d_{2P}^v, \ldots, d_{P-1,P}^v) \stackrel{\text{def}}{=\!=} \prod_{p=2}^{P} P_0(d_{1p}^v), \tag{7.17}$$

we get

$$\mathcal{P}' = 2^{2-P}P_0'(0)\,, \qquad \mathcal{P}'' = 0\,, \qquad \text{and} \qquad \hat{\mathcal{P}} = 2^{3-P}P_0'(0)^2\,. \tag{7.18}$$

The largest eigenvalue of the Jacobian at the trivial equilibrium is

$$\lambda_{\max} = 2^{3-P}P_0'(0)^2P(P-2)4s^2\rho - 2^{3-P}P_0'(0)(P-1)s \tag{7.19}$$

A bifurcation at the trivial fixed point occurs at $P_0'(0) = \frac{P-1}{P(P-2)4s\rho}$. In the 1-issue case, for $P > 2$, this reduces to

$$p^* = \frac{P-1}{P(P-2)4s\mathrm{var}(x)}. \tag{7.20}$$

For $P = 2$, we have $\lambda_{\max} < 0$, and for $P = 3$ we have

$$\lambda_{\max} = 2sP_0'(0)(6s\rho P_0'(0) - 1). \tag{7.21}$$

7.5. Two Times Two Parties

Let us consider a model with four parties with pairwise equal platforms. Without losing generality, let us assume $y^1 = y^3$ and $y^2 = y^4$. We want to show that the system reduces to a 2-party model with a modified response function $\mathcal{P}$. With

$$\mathcal{P}_1(\vec{d}) = \mathcal{P}(d_{12}, d_{13}, d_{14}, d_{23}, d_{24}, d_{34}) = \frac{P_0(d_{12})P_0(d_{13})P_0(d_{14})}{\sum_{i=1}^{P}\prod_{p\neq i}^{P} P_0(d_{ip})} \tag{7.22}$$

in the surface $\mathcal{H}_2^4 \overset{\text{def}}{=\!=} \{y \in \mathbb{R}^{4I} | y^1 = y^3, y^2 = y^4\}$ we have the following expressions for the voting probabilities:

$$\begin{aligned}
\mathcal{P}_1(z,-z) &= \frac{P_0(z)^2}{2\big(P_0(z)^2 + P_0(-z)^2\big)} \\
&= \mathcal{P}_3(z,-z) \\
\mathcal{P}_2(z,-z) &= \frac{P_0(-z)^2}{2\big(P_0(z)^2 + P_0(-z)^2\big)} \\
&= \mathcal{P}_4(z,-z)
\end{aligned} \tag{7.23}$$

where $z \overset{\text{def}}{=\!=} d_{12}$, and $-z \overset{\text{def}}{=\!=} d_{21}$.

Using the function $\tilde{\mathcal{P}}(z) \overset{\text{def}}{=\!=} \frac{P_0(z)^2}{P_0(z)^2 + P_0(-z)^2}$ as a response function, we shall see that this model reduces to a two-party model. Of course, the probabilities of voting for each party is $\frac{1}{2}$ if the platforms are equal, i.e. $z = 0$.

The dynamical equations are of the form

$$\begin{aligned}
\dot{y_j}^1 &= \frac{1}{V}\sum_v \frac{2P_0'(z_v)}{[P_0^2(z_v) + P_0^2(-z_v)]^2}[P_0(z_v)P_0(-z_v)]\partial_j u(y^1, x_v) \\
\dot{y_j}^2 &= \frac{1}{V}\sum_v \frac{2P_0'(-z_v)}{[P_0^2(z_v) + P_0^2(-z_v)]^2}[P_0(z_v)P_0(-z_v)]\partial_j u(y^2, x_v)
\end{aligned} \tag{7.24}$$

Remark. If we compare the dynamical equations for the 2×2-party case with those in the 2-party case, we observe that both systems are of the form

$$
\begin{aligned}
\dot{y_j}^1 &= \frac{1}{V} \sum_v \Phi(z_v, -z_v) \partial_j u(y^1, x_v) \\
\dot{y_j}^2 &= \frac{1}{V} \sum_v \Phi(z_v, -z_v) \partial_j u(y^2, x_v).
\end{aligned}
\tag{7.25}
$$

Now we shall show that $\tilde{\mathcal{P}}(z)$ is indeed a sigmoidal function.

Lemma 1. Let $P_0(z)$ be a twice differentiable sigmoidal function with $P_0(z) = 1 - P_0(-z)$ and $P_0(z) \to 0$ for $z \to -\infty$. Then

$$
\tilde{\mathcal{P}}(z) = \frac{P_0(z)^2}{P_0(z)^2 + P_0(-z)^2}
$$

is a twice differentiable function satisfying $\tilde{\mathcal{P}}(z) = 1 - \tilde{\mathcal{P}}(-z)$ and $\tilde{\mathcal{P}}(z) \to 0$ for $z \to -\infty$.

Proof. It is easy to see that $\tilde{\mathcal{P}}(z) + \tilde{\mathcal{P}}(-z) = 1$. Furthermore, $\tilde{\mathcal{P}}(z)$ is strictly monotonically increasing, since

$$
\frac{\partial}{\partial z} \tilde{\mathcal{P}}(z) = \frac{2P_0'(z)P_0(z)P_0(-z)}{[P_0^2(z) + P_0^2(-z)]^2} > 0
$$

It remains to show that $z = 0$ is the only point of inflection of $\tilde{\mathcal{P}}$.

$$
\begin{aligned}
\frac{\partial^2}{\partial z^2} \tilde{\mathcal{P}}(z) = \frac{2}{(1 - 2P_0(z) + 2P_0^2(z))^3} \Big[&(1 - 2P_0(z))(1 + 2P_0(z) - 2P_0^2(z)) \\
&+ P_0''(z)P_0(z)(1 - P_0(z))(1 - 2P_0(z) + 2P_0^2(z)) \Big]
\end{aligned}
$$

The term in the denominator, $(1 - 2P_0(z) + 2P_0^2(z))^3$ is always positive, since it can be written as $[(1 - P_0(z))^2 + P_0(z)^2]^3 > 0$. It occurs also in the second part of the above sum, which reads $P_0''(z)P_0(z)(1-P_0(z))(1-2P_0(z)+2P_0^2(z))$. Of course, we have $P_0''(z)<0$ for all $z>0$ and $P_0''(z)>0$ for all $z<0$, while all other entries are positive independent of z. The term $1+2P_0(z)-2P_0^2(z) = 1+2P_0(z)(1-P_0(z))$ in the first part of the sum is always positive, while $(1-2P_0(z))<0$ for all $z>0$ and $(1-2P_0(z))>0$ for all $z<0$. Therefore, $z=0$ is the only point of inflection of $\tilde{\mathcal{P}}(z)$. ∎

8. Conclusions

8.1. Summary

We have derived a dynamical system describing the adaptation of party platforms in a spatial voting model. Such models are based on the assumption that political issues can be quantified and hence party platforms can be encoded as points in a Euclidean vector space the coordinates of which designate the different political issues.

Voters are characterized by their ideal points and their utility functions that depend monotonically on a measure of the distance between a party's platform and the voter's ideal point. The probability that a given voter votes for a particular party depends on the pairwise differences of the utilities of all the parties for the given voter in such a way that the party yielding the highest utility is the one that is most likely to receive the vote. The corresponding voter response functions can be viewed as multi-dimensional sigmoidal functions. The slope of these sigmoidals, which measures the extent to which the voters are rational (or critical towards the party platforms) has turned out to be the most important parameter.

The active players are the parties. The payoff of a platform is determined as the expected fraction of votes that it receives. Parties change their platforms along the gradient of their payoff functions. The phase space thus has the dimension

$I \times P$ in a model with I issues and P parties. Some general properties of this dynamical system show that it behaves reasonably at least at a global scale: all orbits are eventually bounded within a box that is spanned by the most extreme voter positions. All planes in the phase-space on which two or more platform positions coincide are invariant. In most cases we assume that the voter utility functions are independent of the party labels; then the dynamical system has S_P-permutation symmetry.

In section 2 we have considered a variety of different aspects of the two-party case. Most notably, we find that under a wide variety of circumstances the platforms of all parties converge eventually to the mean voter fixed point. In particular, the mean voter point is globally stable for concave voter utility functions. For a much larger class of models we could at least ensure local stability of the mean voter fixed points: in particular the introduction of non-policy values, which introduce a dependence of the voter utilities on the party labels, does not lead to a bifurcation. It is known, on the other hand, that the mean voter equilibrium can become unstable if the voter utilities are Gaussian [44].

Is section 3 we have considered two party models where voters are allowed to abstain. The probability of abstention was modeled at a heuristic level. In the first case we have assumed that voters will abstain if the utility for both parties is small and/or the parties' platform positions are very similar. The resulting dynamical system maintains its symmetries and it is possible to show that the mean voter equilibrium is stable in the case of convex utility functions, independent of whether the expected fraction or the expected gain of votes is used as a party's payoff function.

As an alternative approach we have assumed that probability for participating in the election decreases with the distance of the platforms from a voter's ideal point. In this case we find that it makes a difference whether the expected fraction of votes for a party or the expected fraction of votes that a party is ahead of its competitor is chosen as the party payoff function. With the expected difference, there is no bifurcation, while with the expected fraction of votes the mean voter fixed point becomes unstable for critical voters and a participation probability that drops sufficiently fast with the distance to the platforms.

In the next chapter the model is generalized to three parties. In contrast to the two-party case, the mean voter equilibrium is unstable for critical voters even in the case of concave voter utility functions. The bifurcation point is determined by the slope of the multidimensional sigmoidal response function that determines how critical the voters are. A number of explicit examples, with both discrete and continuous distributions of voter ideal points in issue space are discussed. We obtain analogous results in all cases. This suggests that the details of the voter distribution have only a minor influence on the qualitative dynamical behavior. The analytical studies are complemented by a numerical survey. We found rather complicated bifurcation diagrams for this type of model. In particular, for sufficiently critical voters there are multiple locally stable fixed points. In other parameter ranges we find stable limit cycles and the absence of stable equilibria.

Perturbations were the theme of chapter 5. After a brief discussion of platform dynamics with ideological parties, i.e., parties whose platform mobility is limited to the surrounding of the parties' preferred position in issue space, we consider the differences between discrete and continuous voter distributions. We show that under fairly general conditions we may approximate a continuous voter distribution by a discrete one in such a way that the discrepancy is a regular perturbation of the dynamical system. This perturbation can become arbitrarily small if the discrete approximation contains a sufficiently large number of voters.

In chapter 6, we have analyzed a two party as well as a three party model with discrete time dynamics and compare the results with those obtained in the continuous time cases. It turns out that even with a simple quadratic voter utility function, a bifurcation occurs at the trivial fixed point in both the two- and the three party model, provided the time between two subsequent elections in the discrete-time model is large enough.

Most of the results obtained for three parties carry over to multi-party systems (chapter 7). In particular the mean voter equilibrium becomes unstable for sufficiently critical voters. Finally we showed that a 4-party system in which two pairs of parties have the same platforms always translates into a two-party system.

In summary, the dynamical model of platform adaptation leads to reasonable behavior. Platform positions converge towards the mean voter position if the voters are not critical towards (or not interested in) the behavior of the parties. On the other hand, a population of critical voters keeps platform positions well separated and in substantial fractions of the parameter space there are no stable equilibria and we observe periodic attractors.

8.2. Directions for Future Research

Several modifications of the payoff function could be considered. An interesting phenomenon described in the book [22] is that the voters are oftentimes more supportive of the incumbent, see also [17, 56]. This could be modeled by an extra payoff contribution to the incumbent's payoff function, (i.e., the party with the largest fraction of votes at the time of the previous election).

Instead of the expected number of votes, a party's probability of winning could be used as its payoff function. The simple-minded model of ideological parties discussed in (5.2) is not entirely satisfactory. It would be interesting to design a more realistic model for this case.

In this work abstention was treated on a very simple phenomenological basis. However, the literature on voter turnout is extensive [2]. While rational choice theory was generally not very successful there are promising hybrids of rational choice modeling and behavioral approaches, see e.g., [29]. Detailed models of this kind could provide more detailed insights into the influence of abstention on platform dynamics.

A severe limitation of all the models presented in this thesis is the fact that voter positions are treated as fixed parameters. In a more realistic setting, voters would respond to the parties' campaigns by modifying their ideal points and/or strength factor. A very simple way making the voters active players would be to allow their ideal points to move in response to a gravitation-like "opinion field" in the vicinities of parties, figure 15. One might set

$$\dot{x}_v = \nabla_{x_v} \sum_p \Psi_p(y^p, x_v) \tag{8.1}$$

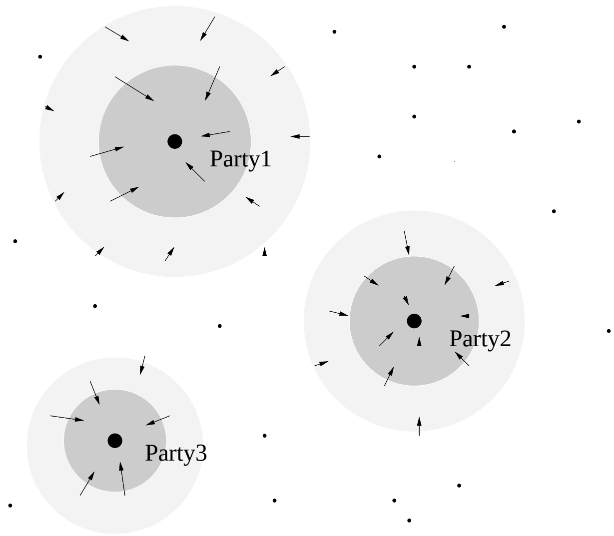

Figure 15: Allowing the voters to change their minds: Voters close to the parties' platform positions are attracted by the campaign, those that are far away remain unaffected. The "sphere of influence" of a party (gray circles) could model the quality of their campaign.

where the field $\Psi_p(y^p, x_v)$ decreases with the distance between voter position x_v and party position y^p. More sophisticated models might also include differential effects on strength factors, or effects that depend on the mutual distances of parties.

The formation of coalitions is a most important effect in multi-party systems that was completely excluded from the models discussed here. It is clear that in models with multi-party systems a party's payoff is given not only by its voters, but also by its possibility to participate in a government coalition. It is by no means clear that this could be formulated within the framework of spatial voting theory.

It might also be worthwhile to study the influence of election procedures [61, 62] (for instance, majority voting versus proportional representation) on platform dynamics. It is very likely that such a model will involve much more complicated versions of the response function $\mathcal{P}$.

Many of the above extensions and modifications of adaptive voting dynamics might not lend themselves to modeling by means of differential equations and most likely can be treated only by computer simulations. In a different vain it would be interesting to investigate applications of the gradient-like dynamics to other models such as firms that are competing with one another over complex goods.

Appendix A: Sigmoidal Functions

Sigmoidal functions play a prominent role in this thesis. In this appendix we review some of their properties in detail. In the following we consider a function $f : \mathbb{R} \to \mathbb{R}$. We shall use X and Y to designate (finite or infinite) intervals.

Definition. A function f is
convex on X if $f(px + qy) \leq pf(x) + qf(y)$;
concave on X if $f(px + qy) \geq pf(x) + qf(y)$;
strictly convex on X if $f(px + qy) < pf(x) + qf(y)$;
strictly concave on X if $f(px + qy) > pf(x) + qf(y)$;
affine on X if $f(px + qy) = pf(x) + qf(y)$
for all $x, y \in X$ such that $x < y$ and $0 < p, q < 1$ such that $p + q = 1$.

Lemma 1. f is strictly convex (concave) on X if and only if f is convex (concave) and there is no interval $Y \subseteq X$ such that f is affine on Y.

Proof. Suppose f is convex and not affine on any interval Y but there are three points $x_1, x_2,$ and $z = (1 - \lambda)x_1 + \lambda x_2$, such that

$$f(z) = (1 - \lambda)f(x_1) + \lambda f(x_2)$$

is fulfilled for some λ with $0 < \lambda < 1$. Then there must be points p and q such that

$$p = (1 - \lambda_p)x_1 + \lambda_p x_2$$

$$q = (1 - \lambda_q)x_1 + \lambda_q x_2,$$

with $0 < \lambda_p < \lambda < \lambda_q < 1$ such that

$$f(p) < (1 - \lambda_p)f(x_1) + \lambda_p f(x_2) \qquad \text{and}$$

$$f(q) < (1 - \lambda_q)f(x_1) + \lambda_q f(x_2).$$

Convexity of f implies that

$$f(z) \leq (1-\alpha)f(p) + \alpha f(q)$$

for some $\alpha \in (0,1)$ and

$$z = (1-\alpha)p + \alpha q,$$

which can be rewritten in the form:

$$z = [(1-\alpha)(1-\lambda_p) + \alpha(1-\lambda_q)]x_1 + [(1-\alpha)\lambda_p + \alpha\lambda_q]x_2.$$

If we define

$$\lambda = (1-\alpha)\lambda_p + \alpha\lambda_q,$$

substituting the strict inequalities for $f(p)$ and $f(q)$ we find

$$\begin{aligned} f(z) &< [(1-\alpha)(1-\lambda_p) + \alpha(1-\lambda_q)]f(x_1) + [(1-\alpha)\lambda_p + \alpha\lambda_q]f(x_2) \\ &= (1-\lambda)f(x_1) + \lambda f(x_2), \end{aligned}$$

and we arrive at a contradiction. Hence, if f is convex and not affine on any interval it must be strictly convex.

The converse is trivial. For concave f the proof is analogous. ∎

Definition. A function $f : \mathbb{R} \to \mathbb{R}$ is *sigmoidal* if it satisfies:

(i) f is bounded.

(ii) f is monotonically increasing on $\mathbb{R}$ or monotonically decreasing on $\mathbb{R}$.

(iii) There is a point $m \in \mathbb{R}$ such that f is concave on $(-\infty, m)$ and convex on $(m, +\infty)$, or *vice versa*. Any point m with this property is called an *inflection point* of f. We say f is *non-trivial* if it is not a constant function.

The function f is *strictly sigmoidal* if (iii) is replaced by (iii') There is a point $m \in \mathbb{R}$ such that f is strictly concave on $(-\infty, m)$ and strictly convex on $(m, +\infty)$, or *vice versa*.

Lemma 2. Let f be a sigmoidal function. Then the set $\mathcal{M}$ of its inflection points is an interval. f is an affine function on $\mathcal{M}$.

Proof. Suppose m_1 and m_2 are points of inflection and let $x \in (m_1, m_2)$. f is concave on $[m_1, \infty)$ and therefore also on $[x, \infty)$, and f is convex on $(-\infty, m_2]$ and therefore also on $(-\infty, x]$. Thus x is a point of inflection, and the set of all inflection points is an interval. Since f is both convex and concave on $[m_1, m_2]$, f is affine on this interval, and hence on $\mathcal{M}$. ∎

Corollary. Consider a sigmoidal function f and let m be an inflection point. Then m is unique if and only if f is strictly sigmoidal in a neighborhood of m.

Proof. Follows immediately from Lemmas 1 and 2. ∎

Definition. A sigmoidal function f is *symmetric* if there exists an $m \in \mathbb{R}$ such that $f(m+x) + f(m-x) = 2f(m)$ for all $x \in \mathbb{R}$. We call m a center of f.

Lemma 3. A center m of a sigmoidal function f is a point of inflection.

Proof. Suppose m is center point of f. Then there exists an $x \in \mathbb{R}$ such that $m - x < m < m + x$ and $m = \frac{1}{2}(m-x) + \frac{1}{2}(m+x)$. If m is no point of inflection, then

$$f(m) = f\left(\frac{1}{2}(m-x) + \frac{1}{2}(m+x)\right) \neq \frac{1}{2}f(m-x) + \frac{1}{2}f(m+x)$$

which is a contradiction to the fact that m is a center point. ∎

Lemma 4. Let f be a non-constant sigmoidal function. Then the center m is unique. If f is constant each $m \in \mathbb{R}$ is a center.

Proof. Suppose there are two center points m_1 and m_2.
Then for every $q \in \mathbb{R}$ there exist $\bar{q} = 2m_2 - q$ and $\tilde{q} = 2m_1 - q$ such that

$$f(q) + f(\bar{q}) = 2f(m_2) \text{ and}$$
$$f(q) + f(\tilde{q}) = 2f(m_1).$$

Particularly, for $\bar{m}_1 \stackrel{\text{def}}{=} 2m_2 - m_1$ and $\tilde{m}_2 \stackrel{\text{def}}{=} 2m_1 - m_2$ the following must hold:

$$f(m_1) + f(\bar{m}_1) = 2f(m_2) \qquad \text{and} \qquad f(m_2) + f(\tilde{m}_2) = 2f(m_1)$$

These expressions can be transformed to:

$$f(m_1) + f(m_2) = f(\bar{m}_1) + f(\tilde{m}_2).$$

Analogously, for $\bar{\tilde{m}}_2 = 2m_2 - \tilde{m}_2$ and $\tilde{\bar{m}}_1 = 2m_1 - \bar{m}_1$
we get

$$f(\bar{\tilde{m}}_2) = 2f(m_2) - f(\tilde{m}_2) \qquad \text{and}$$
$$f(\tilde{\bar{m}}_1) = 2f(m_1) - f(\bar{m}_1).$$

Substituting $f(\bar{m}_1) = 2f(m_2) - f(m_1)$ and $f(\tilde{m}_2) = 2f(m_1) - f(m_2)$, we see that

$$f(\tilde{\bar{m}}_1) + f(\bar{\tilde{m}}_2) = f(m_1) + f(m_2).$$

We now can define $\bar{\tilde{\bar{m}}}_1 = 2m_2 - \tilde{\bar{m}}_1$ and $\tilde{\bar{\tilde{m}}}_2 = 2m_1 - \bar{\tilde{m}}_2$. Then

$$f(\bar{\tilde{\bar{m}}}_1) = 4f(m_2) - 3f(m_1)$$

$$f(\tilde{\bar{\tilde{m}}}_2) = 4f(m_1) - 3f(m_2)$$

and again, we get

$$f(\bar{\bar{\bar{m}}}_1) + f(\bar{\bar{\bar{m}}}_2) = f(m_1) + f(m_2).$$

Continuing this process, we find that the function values of all successive mirror images of m_1 and m_2 that are obtained by iterating the above procedure must lie on a common straight line. On the other hand, f is monotonous and bounded. Thus, f must be constant. ∎

From (i) and (ii) we conclude immediately that the limits

$$a = \lim_{x \to -\infty} f(x) < \lim_{x \to \infty} f(x) = b$$

exist. If f is differentiable, then $f'(x) \geq 0$ for all x. A twice differentiable sigmoidal function satisfies $f''(x)f''(y) \leq 0$ for all x, y such that $x < m < y$. This inequality is strict for strictly sigmoidal functions. Of course we have $f''(m) = 0$.

Theorem 1. A sigmoidal polynomial is constant.

Proof. Let $p(x) = a_n x^n + a_{n-1} x^{n-1} + \ldots + a_0$

$$\lim_{x \to \infty} p(x) = \lim_{x \to \infty} a_n x^n \left(1 + \frac{a_n - 1}{a_n x} + \ldots \frac{a_0}{a_n x^n}\right) = \infty$$

Analogously,

$$\lim_{x \to -\infty} p(x) = \begin{cases} \infty & \text{for} \quad n \text{ even,} \\ -\infty & \text{for} \quad n \text{ odd.} \end{cases}$$

Thus, there exists no sigmoidal polynomial function. ∎

Theorem 2. A sigmoidal rational function is constant.

Proof. Let us assume without loosing generality that
$p(x) = a_n x^n + a_{n-1}x^{n-1} + \ldots + a_0$ and $q(x) = b_m x^m + b_{m-1}x^{m-1} + \ldots + b_0$.
Since $\frac{p(x)}{q(x)}$ has to be bounded, $q(x) \neq 0$ has to be fulfilled. Thus, q must be an even polynomial, i.e. m has to be even. We then have the following rational function:

$$\frac{p(x)}{q(x)} = \frac{a_n x^n + a_{n-1}x^{n-1} + \ldots + a_0}{b_m x^m + b_{m-1}x^{m-1} + \ldots + b_0}$$

For $x \to \infty$,

$$\lim_{x\to\infty} \frac{p(x)}{q(x)} = \begin{cases} \infty & \text{if} \quad n > m \\ 0 & \text{if} \quad n < m \\ \frac{a_n}{b_n} & \text{if} \quad n = m \end{cases}$$

$$\lim_{x\to-\infty} \frac{p(x)}{q(x)} = \begin{cases} -\infty & \text{for } n > m \quad \text{and } n \text{ odd} \\ \infty & \text{for } n > m \quad \text{and } n \text{ even} \\ 0 & \text{for } n < m \\ \frac{a_n}{b_n} & \text{for } n = m \end{cases}$$

There is no possibility for $p(x)/q(x)$ to be monotonically increasing, except if $p(x)/q(x)$ is constant. ∎

Definition. A *normalized* sigmoidal function f has the following additional properties:

(i) $\lim_{x\to-\infty} f(x) = -1$

(ii) $\lim_{x\to\infty} f(x) = 1$

(iii) $f(0) = 0$

We close this section with a table of the most useful sigmoidal functions.

Table 4. Important Normalized Sigmoidal Functions [1].

$f(x)$	$f'(x)$	$f'(0)$	$\int f(x)dx$
$\mathrm{sgn}(x)$	$2\delta(x)$	∞	$\lvert x\rvert$
$\frac{2}{\pi}\arctan(x)$	$\frac{2}{\pi(1+x^2)}$	$2/\pi$	$\frac{2}{\pi}\left[x\arctan x - \frac{1}{2}\ln(1+x^2)\right]$
$\tanh(x)$	$1-\tanh^2(x)$	1	$\ln\cosh(x)$
$\mathrm{erf}\,(x)$	$\frac{2}{\sqrt{\pi}}\exp(-x^2)$	$\frac{2}{\sqrt{\pi}}$	$x\mathrm{erf}\,(x) + \frac{1}{\sqrt{\pi}}\exp(-x^2)$

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